Free Quality School Education

Ministry of Basic and Senior Secondary Education

Pupils' Handbook for

Senior Secondary *Mathematics*

SSS III

Term II

STRICTLY NOT FOR SALE

Foreword

These Lesson Plans and the accompanying Pupils' Handbooks are essential educational resources for the promotion of quality education in senior secondary schools in Sierra Leone. As Minister of Basic and Senior Secondary Education, I am pleased with the professional competencies demonstrated by the writers of these educational materials in English Language and Mathematics.

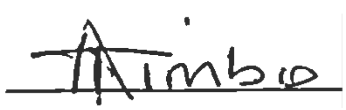
The Lesson Plans give teachers the support they need to cover each element of the national curriculum, as well as prepare pupils for the West African Examinations Council's (WAEC) examinations. The practice activities in the Pupils' Handbooks are designed to support self-study by pupils, and to give them additional opportunities to learn independently. In total, we have produced 516 lesson plans and 516 practice activities – one for each lesson, in each term, in each year, for each class. The production of these materials in a matter of months is a remarkable achievement.

These plans have been written by experienced Sierra Leoneans together with international educators. They have been reviewed by officials of my Ministry to ensure that they meet the specific needs of the Sierra Leonean population. They provide step-by-step guidance for each learning outcome, using a range of recognized techniques to deliver the best teaching.

I call on all teachers and heads of schools across the country to make the best use of these materials. We are supporting our teachers through a detailed training programme designed specifically for these new lesson plans. It is really important that the Lesson Plans and Pupils' Handbooks are used, together with any other materials they may have.

This is just the start of educational transformation in Sierra Leone as pronounced by His Excellency, the President of the Republic of Sierra Leone, Brigadier Rtd Julius Maada Bio. I am committed to continue to strive for the changes that will make our country stronger and better.

I do thank our partners for their continued support. Finally, I also thank the teachers of our country for their hard work in securing our future



Mr. Alpha Osman Timbo

Minister of Basic and Senior Secondary Education

The policy of the Ministry of Basic and Senior Secondary Education, Sierra Leone, on textbooks stipulates that every printed book should have a lifespan of three years.

To achieve thus, DO NOT WRITE IN THE BOOKS.

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Introduction

to the Pupils' Handbook

These practice activities are aligned to the Lesson Plans, and are based on the National Curriculum and the West Africa Examination Council syllabus guidelines. They meet the requirements established by the Ministry of Basic and Senior Secondary Education.

1 The practice activities will not take the whole term, so use any extra time to revise material or re-do activities where you made mistakes.

2 Use other textbooks or resources to help you learn better and practise what you have learned in the lessons.

3 Read the questions carefully before answering them. After completing the practice activities, check your answers using the answer key at the end of the book.

4 Make sure you understand the learning outcomes for the practice activities and check to see that you have achieved them. Each lesson plan shows these using the symbol to the right.

Learning Outcomes

5 Organise yourself so that you have enough time to complete all of the practice activities. If there is time, quickly revise what you learned in the lesson before starting the practice activities. If it is taking you too long to complete the activities, you may need more practice on that particular topic.

6 Seek help from your teacher or your peers if you are having trouble completing the practice activities independently.

7 Make sure you write the answers in your exercise book in a clear and systematic way so that your teacher can check your work and you can refer back to it when you prepare for examinations.

8 Congratulate yourself when you get questions right! Do not worry if you do not get the right answer – ask for help and continue practising!

KEY TAKEAWAYS FROM SIERRA LEONE’S PERFORMANCE IN WEST AFRICAN SENIOR SCHOOL CERTIFICATE EXAMINATION – GENERAL MATHEMATICS[1]

This section, seeks to outline key takeaways from assessing Sierra Leonean pupils’ responses on the West African Senior School Certificate Examination. The common errors pupils make are highlighted below with the intention of giving teachers an insight into areas to focus on, to improve pupil performance on the examination. Suggestions are provided for addressing these issues.

Common errors

1. Errors in applying principles of BODMAS
2. Mistakes in simplifying fractions
3. Errors in application of Maths learned in class to real-life situations, and vis-a-versa.
4. Errors in solving geometric constructions.
5. Mistakes in solving problems on circle theorems.
6. Proofs are often left out from solutions, derivations are often missing from quadratic equations.

Suggested solutions

1. Practice answering questions to the detail requested
2. Practice re-reading questions to make sure all the components are answered.
3. If possible, procure as many geometry sets to practice geometry construction.
4. Check that depth and level of the lesson taught is appropriate for the grade level.

[1] This information is derived from an evaluation of WAEC Examiner Reports, as well as input from WAEC examiners and Sierra Leonean teachers.

Lesson Title: Expression of ratios	**Theme:** Numbers and Numeration
Practice Activity: PHM3-L049	**Class:** SSS 3

Learning Outcomes
By the end of the lesson, you will be able to:
1. Express ratios in their simplest terms.
2. Increase and decrease quantities in a given ratio.

Overview

This lesson provides a review of the basic principles of ratios. Ratios are used to compare quantities of the same type, e.g. length, weight, people, money and more.

Ratios can be described in 2 ways. This is best described with the following example: Suppose there are 24 girls and 36 boys in an SSS 3 class. This can be written as:

24 **:** 36	Use a colon to separate the 2 quantities being compared. Read as "24 to 36"
$\frac{24}{36}$	Write as a fraction

It does not matter which of the 2 ways we write a ratio, we should always simplify it to its lowest terms. This is done by dividing by common factors.

Simplifying the example above gives: 2 : 3 OR $\frac{2}{3}$
This means that for every 2 girls there are 3 boys.

The order in which ratios are written is very important and must be maintained when solving a problem. A ratio written as 2:3 means $\frac{2}{3}$, while a ratio written as 3:2 means $\frac{3}{2}$. The fractions are different and give different answers.

We can only simplify ratios when the quantities are in the same units. If the quantities are not in the same unit, we must convert one to the other before we simplify.
Example:

Simplify 5 cm to 5 m

5 m	=	500 cm	Convert m to cm
5 cm	to	500 cm	
5	**:**	500	Same units, so can be omitted
1	**:**	100	For every 1 cm of one quantity there are 100 cm of another.

It is not always the case that we get straightforward ratios to simplify. We may be required to simplify and also to solve for missing parts of equivalent ratios. We will show this in question 1 in the Solved Examples below.

Ratio problems have to be interpreted in different ways depending on what we are required to find. For example, we are sometimes required to increase or decrease quantities by a given ratio to find the new amounts.

The calculation to increase or decrease a quantity Q by a ratio $m : n$ is given by: $\frac{m}{n} \times Q$. An example showing how to increase a quantity in a given ratio is shown in question 2. below.

Solved Examples

1. Express $24 : 14$ as a fraction in its lowest terms. Set your result equal to $x: 35$ and find x.

   Solution:

   Step 1. Assess and extract the given information from the problem.

   given: $24: 14; x : 35$

   Step 2. Change the ratio to its fraction form.

   $\frac{24}{14} = \frac{12}{7}$ Original ratio in the lowest terms

   Step 3. Make the ratios equal (given)

   $\frac{12}{7} = \frac{x}{35}$

   Step 4. Make x the subject of the equality.

   $\frac{35\times 12}{7} = x$

   Step 5. Solve for x.

   $x = 60$

2. Increase Le 60,000.00 in the ratio 8 **:** 5.

   Solution:

   Given: Increase Le 60,000.00 in the ratio 8 **:** 5

   - This means that every Le 5 is increased to Le 8.
   - We know that it is an increase because the first part of the ratio is larger than the second part of the ratio.

   New amount $= \frac{8}{5} \times 60{,}000$ Change the ratios to their fraction forms.

   $= 96{,}000$ This is a reasonable result as we know the amount increased from before.

   The new amount is Le 96,000.00.

3. The length and width of a rectangular plot 12 m by 8 m are decreased in the ratio 4:3. Find the ratio by which the area is decreased.

Solution:

Original length	=	12 m
Original width	=	8 m
Area of original plot	=	$12\ m \times 8\ m$
	=	$96\ m^2$
New length	=	$\frac{3}{4} \times 12\ m$
	=	$9\ m$
New width	=	$\frac{3}{4} \times 8\ m$
	=	$6\ m$
Area of new plot	=	$9\ m \times 6\ m$
	=	$54\ m^2$
Ratio by which the area is decreased	=	$96\ m^2 : 54\ m^2$
	=	$\frac{96\ m^2}{54\ m^2}$
	=	$16:9$

Practice

1. A rectangle 20 cm by 10 cm has its length decreased in the ratio 5:4 and its width increased in the ratio 2:3. Find:
   a. The new length and width
   b. The ratio in which the area has changed
2. Increase the following amounts according to the ratios given.
   a. A mass of 60 kg in the ratio 3:7
   b. The price of an article costing Le 2,750 in the ratio 5:8
   c. A length of 2.52 km in the ratio 7:12
   d. A crop of 70 mangoes in the ratio $\frac{1}{2} : 1\frac{1}{2}$
3. Decrease the following amounts according to the ratios given.
   a. 120 m in the ratio 5:2
   b. 1 hour 30 minutes in the ratio 9:4
   c. 6.5 litres in the ratio 13:7
   d. A class of 75 pupils in the ratio $\frac{10}{9} : \frac{2}{3}$

Lesson Title: Comparison of ratios	Theme: Numbers and Numeration
Practice Activity: PHM3-L050	**Class:** SSS 3

Learning Outcome
By the end of the lesson, you will be able to compare and simplify ratios.

Overview

We are often asked to compare 2 or more ratios to find out which is the biggest or smallest relative to the others. One way we can compare ratios is by writing them as unit ratios.

If we have a ratio in the form $m : n$, we can write it either as $m : 1$ or $1 : n$.

- To write as $m : 1$, we divide both ratios by n.
- To write as $1 : n$, we divide both ratios by m

Once we have converted the given ratios to unit fractions, we can then determine which ratio is greatest or smallest in relation to the others.

A second way to compare ratios is to use LCM to convert each ratio into an equivalent fraction. Both fractions will then have the same denominator. As before, inspect the numerators and determine which ratio is greatest or smallest in relation to the others.

Solved Examples

1. Express the ratios $3 : 8$ and $4 : 15$ in the form $m : 1$. Which ratio is greater? Use LCM to confirm your result.

   Solution:

   Step 1. Assess and extract the given information from the problem.

   Step 2. Change the ratios to their fraction form.

   Method 1.

   Simplify each ratio independently to a unit fraction.

   $\frac{3}{8} = \frac{0.375}{1}$ divide the numerator and denominator by 8

   $\frac{4}{15} = \frac{0.267}{1}$ divide the numerator and denominator by 15

   Now, compare the 2 ratios.

   $3 : 8 = 0.375 : 1$

   $4 : 15 = 0.267 : 1$

   $3 : 8 > 4 : 15$ since $0.375 > 0.267$

   Method 2.

   Find the LCM of the 2 fractions.

   $\frac{3}{8} = \frac{45}{120}$ since the LCM of 8 and 15 is 120

$$\frac{4}{15} = \frac{32}{120}$$

Now, compare the 2 ratios.

$$3 : 8 = 45 : 120$$
$$4 : 15 = 32 : 120$$
$$3 : 8 > 4 : 15 \quad \text{since } 45 > 32$$

Step 3. Write the answer.

$\therefore$ $3 : 8$ is the greater ratio.

2. Express each of these following ratios first in the form 1 : r, then in the form r : 1.

a. 200 : 150

Solution:

$$200 : 150$$
$$\frac{200}{200} : \frac{150}{200}$$
$$1 : 0.75$$

$$200 : 150$$
$$\frac{200}{150} : \frac{150}{150}$$
$$1.33 : 1$$

b. 48 : 60

Solution:

$$48 : 60$$
$$\frac{48}{48} : \frac{60}{48}$$
$$1 : 1.25$$

$$48 : 60$$
$$\frac{48}{60} : \frac{60}{60}$$
$$0.8 : 1$$

c. 1.8 : 4.5

Solution:

$$1.8 : 4.5$$
$$\frac{1.8}{1.8} : \frac{4.5}{1.8}$$
$$1 : 2.5$$

$$1.8 : 4.5$$
$$\frac{1.8}{4.5} : \frac{4.5}{4.5}$$
$$0.4 : 1$$

3. Find out which of the pair of ratios is greater.

a. 2 : 5 or 4 : 7 b. 9 :13 or 3 : 4

Solutions:

a.

$$2 : 5 = \frac{2}{5}$$
$$\frac{2}{5} = \frac{14}{35}$$

$$4 : 7 = \frac{4}{7}$$
$$\frac{4}{7} = \frac{20}{35}$$

Since $\frac{20}{35} > \frac{14}{35}$

Therefore $4 : 7 > 2 : 5$

b.

$$9 : 13 = \frac{9}{13}$$
$$\frac{9}{13} = \frac{36}{52}$$

$$3 : 4 = \frac{3}{4}$$

$$\frac{3}{4} = \frac{39}{52}$$

Since $\frac{39}{52} > \frac{36}{52}$

Therefore $3 : 4 > 9 : 13$

Practice

1. Express the following ratios in the form r : 1

   a. 3 : 8 b. 6 m : 8 m c. Le 425 : Le 250 d. 8 kg : 20 kg

2. Express the following ratios in the form 1 : r

   a. 9 : 13 b. 2.5 m : 4 m c. 1.5 g : 48 g d. 5 cm : 1 km

3. Find which of the following pairs of ratios is greater.

   a. 7 : 15 or 8 : 17 b. 11 : 6 or 13 : 7 c. 11 m : 13 m or 7 g : 8 g

   d. Le 170 : Le 90 or Le 300 : Le 160

Lesson Title: Rate	Theme: Numbers and Numeration
Practice Activity: PHM3-L051	Class: SSS 3

Learning Outcome
By the end of the lesson, you will be able to use rates to connect quantities of different kinds.

Overview

We use ratios to compare two or more "like" quantities. This means that they are of the same kind, e.g. height, temperature, mass, or weight. The quantities must be expressed in the same unit for them to be compared.

We use rates when we want to compare quantities of different kinds, e.g. how far a motor bike travels in kilometres for a particular length of time in hours or how much money someone is paid per month at their job.

The quantities in the ratio are measured with one unit. When we write the ratio as a fraction, the units in the **ratio** cancel each other out because they are the same, e.g. $\frac{\text{area of square}}{\text{area of triangle}} = \frac{\text{cm}^2}{\text{cm}^2}$.

The quantities in the **rate** are measured with 2 units. The units in a rate take on the unit from the numerator and the unit from the denominator, e.g. $\frac{\text{distance}}{\text{time}} = \frac{\text{km}}{\text{hr}}$.

Look at Solved Example 1. The method used is called the unitary method. The basic procedure is given below.

- We first find the unit rate. This is the amount of the first quantity for every 1 of the second.
- In Example 1, this is "80" kilometres for every "1" hour, i.e. 80 km/hr.
- We then use the unit rate to find all other amounts of one quantity given the other.
- A unit rate is a ratio that tells us how many units of one quantity there are for every one unit of the second quantity.
- Rates use words and symbols such as "per" (/), "each" (ea) and "at" (@).

Solved Examples

1. A car travels a distance of 240 km in 3 hours.
   a. What is the average speed in kilometres per hour (km/hr)?
   b. How far will it travel in 5 hours?

Solution:

Step 1. Assess and extract the given information from the problem.

Given: A car travels 240 km for every 3 hours

i. **Step 2.** Convert to a ratio and simplify.

240 km	**:**	3 hrs	Write as a ratio
rate	$=$	$\frac{240 \text{ km}}{3 \text{ hrs}}$	Write as a fraction
	$=$	$\frac{80 \text{ km}}{1 \text{ hr}}$	Write in the form $m : 1$ by dividing the numerator and denominator by 3
	$=$	80 km/hr	Write as a rate in km/hr.

Step 3. Write the answer.

The average speed is 80 km/hr.

ii.

speed	$=$	80km/hr	
	$=$	$\frac{80 \text{ km}}{1 \text{ hr}}$	80 km in 1 hr
Distance travelled	$=$	$\frac{80 \text{ km}}{1 \text{ hr}} \times 5$ hrs	Multiply by 5 for distance travelled in 5 hours
	$=$	400 km	The hours cancel each other

The distance in 5 hours is 400 km.

2. Each week a person works from 8.00 am to 12.30 pm on two days and from 2.00 pm to 5.30 pm on four days. The rate of pay is Le 4,800.00 per hour. What is the person‘s total weekly wages?

Solution:

Calculate the number of hours the person works in 1 week:

$$\begin{aligned} 8.00 \text{ am to } 12.30 \text{ pm} &= 4 \text{ hours } 30 \text{ min} \\ 2.00 \text{ pm to } 5.30 \text{ pm} &= 3 \text{ hours } 30 \text{ min} \\ 4 \text{ hours } 30 \text{ min} \times 2 &= 9 \text{ hours} \\ 3 \text{ hours } 30 \text{ min} \times 4 &= 14 \text{ hours} \\ 9 \text{ hours} + 14 \text{ hours} &= 23 \text{ hours} \end{aligned}$$

Calculate the person’s weekly wages:

$$\begin{aligned} \text{Total weekly wages} &= \text{Le } 4{,}800 \times 23 \\ &= \text{Le } 110{,}400.00 \end{aligned}$$

3. Three boys can weed a piece of land in 4 hours. How long would it take 18 boys to weed the piece of land, if they weed at the same rate?

Solution:

Calculate the time it would take 1 boy to weed the land:

$$\begin{aligned} 3 \text{ boys} &= 4 \text{ hours} \\ 1 \text{ boy} &= 3 \times 4 \text{ hours} \end{aligned}$$

$$= 12 \text{ hours}$$

Calculate the time it would take 18 boys to weed the land:

$$18 \text{ boys} = \frac{12}{18} \text{ hours}$$

$$= \frac{2}{3} \text{ hours}$$

$$= \frac{2}{3} \times 60 \text{ min}$$

$$= 40 \text{ min}$$

Practice

1. A vehicle consumes 180 litres of fuel for a distance of 650 km. How many litres of fuel will be consumed for a distance of 1,625 km?
2. A piece of work takes 20 labourers to complete in 10 days. How many labourers will complete the same job in 25 days if they work at the same rate?
3. A car uses petrol at the rate of 1 litre for every 11 km. If the price of petrol is Le 6,000.00 per litre, find the cost of the petrol for a journey of 891 km.
4. In the year 2000, a factory produced 9,324 bicycles. Allowing 2 weeks for holidays and a further 100 days for weekends, find the rate of production in bicycles per day.
5. A workman is paid Le 152,000.00 for 40 hours of work. Calculate his hourly rate of pay.
6. A car travels 153 km in $2\frac{1}{4}h$. Calculate its average speed in km/h.

Lesson Title: Proportional division	**Theme:** Numbers and Numeration
Practice Activity: PHM3-L052	**Class:** SSS 3

Learning Outcome
By the end of the lesson, you will be able to divide quantities into given proportions.

Overview

This lesson is a review of proportional division.

We are familiar with doing calculations where we share quantities, for example money, equally among several people. Suppose we are asked instead to share the same amount of money according to a given ratio. We perform a **proportional division** according to the given ratio.

A quantity shared equally will result in the same amount per share.

A quantity shared in different proportions will result in different amounts per share according to the given ratio.

Solved Examples

1. Share Le 750,000.00 between 2 children in the ratio 8 : 7. How much will each child receive?

   Solution:

   Step 1. Assess and extract the given information from the problem.

   Step 2. Find the total number of parts to the ratio.

   Total number of parts $= 8 + 7 = 15$

   This ratio means that for every Le15.00 of the amount to be shared, Le 8.00 will go to Child 1 and Le 7.00 will go to Child 2.

   Step 3. Find what proportion (fraction) of the total is given to each part

   Child 1 receives: $\frac{8}{15} \times 750{,}000 = 400{,}000$

   Child 2 receives: $\frac{7}{15} \times 750{,}000 = 350{,}000$

   Step 4. Write the answer.

   Child 1 receives Le 400,000.00. Child 2 receives Le 350,000.00.
   This answer is reasonable as it adds up to Le 750,000.00

2. The weights of two pets are in the ratio 3: 5. The heavier pet is 10 kg. What is the weight of the lighter pet?

Solution:

Let k = weight of the two pets. Find the value of k:

Total number of parts = $3 + 5$ = 8

Use the weight of the heavier pet to set up the equation:

$$\frac{5}{8} \times k = 10 \text{ kg}$$

$$\frac{5k}{8} = 10 \text{ kg}$$

$$5k = 8 \times 10 \text{ kg}$$

$$\frac{5k}{5} = \frac{80}{5}$$

$$k = 16 \text{ kg}$$

Use k to find the weight of the lighter pet:

weight of lighter pet $= \frac{3}{8} \times 16$ kg

$= 6$ kg

3. If 3 boys shared x oranges among themselves in the ratio 3 : 5 : 8 and the smallest share was 45 oranges, find the value of x.

Solution:

Total number of parts = $3 + 5 + 8$ = 16

$\frac{3}{16} \times x$ = 45 oranges   smallest share

$$\frac{3x}{16} = 45$$

$$3x = 16 \times 45$$

$$3x = 720$$

$$\frac{3x}{3} = \frac{720}{3}$$

$$x = 240$$

4. The expenditure on education, health and water supply are in the ratio 7 : 11: 5. If the expenditure on health is Le 22,000,000.00:

   a. What is the budget for these three projects?

   Find the expenditure on:

   b. Education   c. Water supply

Solution:

total number of parts $= 7 + 11 + 5$ = 23

Let h = budget of the 3 projects

Expenditure on health $\frac{11}{23} \times h$ = Le 22,000,000.00

$$\frac{11h}{23} = \text{Le } 22{,}000{,}000.00$$

$$11h = 23 \times 22{,}000{,}000$$

$$11h = \text{Le } 506{,}000{,}000.00$$

$$\frac{11h}{11} = \frac{\text{Le}506{,}000{,}000}{11}$$

$$h = \text{Le } 46{,}000{,}000.00$$

budget for the 3 projects $= \text{Le } 46{,}000{,}000.00$

b. Expenditure on education $= \frac{7}{23} \times 46{,}000{,}000$

$= \text{Le } 14{,}000{,}000.00$

c. Expenditure on water supplies $= \frac{5}{23} \times 46{,}000{,}000$

$= \text{Le } 10{,}000{,}000.00$

Practice

1. In a bag of oranges, the ratio of the good ones to the bad ones is 5:4. If the number of bad oranges in the bag is 36, how many oranges are there altogether?

2. The ratio of men to women in a church is 5:7. If there are 1,200 people in the church, how many men are there?

3. If 3 boys shared x oranges among themselves in the ratio 3: 5: 8, and the smallest share was 45 oranges, find the value of x.

4. The ratio of pocket money received by three friends, Momodu, Musa and Amadu is 5 : 2 : 7. If Amadu gets Le 1,400.00 pocket money, find out how much Momodu and Musa have received.

5. The cost Le 232,000.00 of producing a machine arises from the cost of materials, labour and overhead in the ratio 7 : 9 : 2. Calculate the cost of labour for producing 32 such machines.

Lesson Title: Scales – Part 1	Theme: Numbers and Numeration	
Lesson Number: PHM3-L053	**Class:** SSS 3	**Time:** 40 minutes
Learning Outcome By the end of the lesson, you will be able to interpret scales used in drawing plans and maps.		

Overview

Plans and maps are diagrams of real-life objects and places. Scales are used to reduce the size of the objects in the plans and maps in order to make them fit on to a piece of paper. Scales allow the diagrams to be drawn in proportion to their original size. They are examples of ratio and are usually written in the form $1 : n$.

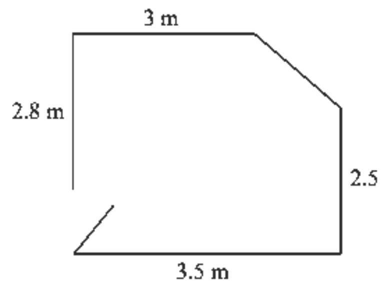
For example, a scale of 1 : 50 on a plan, means that 1 cm on the plan is 50 cm in real-life.

On a map, scales are usually of the order 1 : 50,000. This means 1 cm on the map represents an actual distance of 50,000 cm (500 m or 0.5 km) on the ground.
We can read a plan or map and be able to deduce the actual sizes or distances of the objects or places they shop.

Solved Examples

1. The diagram shows the sketch of a bedroom (not to scale).
   a. Copy the table below. Use a scale of $1 : 50$ to complete the table. The doorway is 0.7 m wide.

Actual size m	cm	Size in plan cm		
3	300	$300 \div 50$	=	6
2.5			=	
1.5			=	
2.8			=	
0.7			=	



   b. Draw an accurate plan of the bedroom using the measurements from your table.
   c. What is the actual measurement in m of the unmarked side?

Solution:

Step 1. Assess and extract the given information from the problem.
Given: table to copy and complete, sketch of plan of bedroom

a. **Step 2.** Complete the table following given procedure.

- Using the unitary method, the measurement is given by:

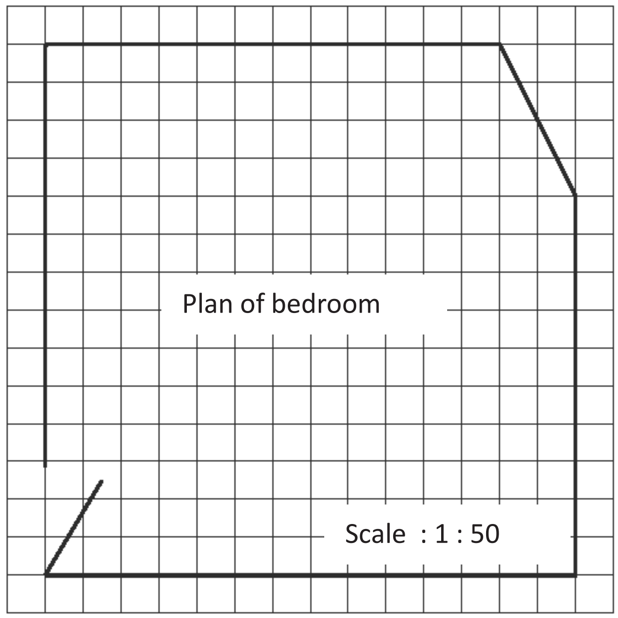
$$\frac{1}{50} = \frac{\text{size in plan}}{\text{actual size}}$$

$$\text{size in plan} = \frac{\text{actual size}}{50}$$

Take care to match the ratio of size in plan : actual size
Use the same units for both sizes

Actual size m	cm	Size in plan cm		
3	300	300 ÷ 50	=	6
2.5	250	250 ÷ 50	=	5
3.5	350	350 ÷ 50	=	7
2.8	280	280 ÷ 50	=	5.6
0.7	70	70 ÷ 50	=	1.4

b. **Step 3.** Draw the plan
Plan (drawn to scale).



c. **Step 4.** Measure the unmarked length and calculate its actual size.

Depending on accuracy of the drawing the unmarked length measures between 2.2 and 2.4 cm.

$$\text{size in plan} = 2.2 \text{ cm}$$

$$\frac{1}{50} = \frac{\text{size in plan}}{\text{actual size}}$$

$$\text{actual size} = \text{size in plan} \times 50$$

$$= 2.2 \times 50 = 110 \text{ cm}$$

$$= 1.1 \text{ m}$$

The actual size of the unmarked length is 1.1 m.
(acceptable lengths between 1.1 and 1.2 m)

4. A hall measures 10 m wide by 15 m long. Give the dimensions of the hall on plans with the scales below.

a. 1 : 100 b. 1 : 200
c. 1 : 50 d. 1 : 20

Solution:

Given: Hall measuring 10 m wide by 15 m long

width: $10 \text{ m} = 1{,}000 \text{ cm}$

length: $15 \text{ m} = 1{,}500 \text{ cm}$

$$\text{scale} = 1 : n$$

$$\frac{1}{n} = \frac{\text{size in plan}}{\text{actual size}}$$

$$\text{size in plan} = \frac{\text{actual size}}{n}$$

a. $\text{scale} = 1 : 100$

$$\text{size in plan} = \frac{\text{actual size}}{100}$$

b. $\text{scale} = 1 : 200$

$$\text{size in plan} = \frac{\text{actual size}}{200}$$

width: 1,000 cm $= \frac{1,000}{100}$
$= 10$ cm
length: 1,500 cm $= \frac{1,500}{100}$
$= 15$ cm
On 1 : 100 scale:
width = 10 cm, length = 15 cm

width: 1,000 cm $= \frac{1,000}{200}$
$= 5$ cm
length: 1,500 cm $= \frac{1,500}{200}$
$= 7.5$ cm
On 1 : 200 scale:
width = 5 cm, length = 7.5 cm

c. scale = 1 : 50
size in plan $= \frac{\text{actual size}}{50}$
width: 1,000 cm $= \frac{1,000}{50}$
$= 20$ cm
length: 1,500 cm $= \frac{1500}{50}$
$= 30$ cm
On 1 : 50 scale:
width = 20 cm, length = 30 cm

d. scale = 1 : 20
size in plan $= \frac{\text{actual size}}{20}$
width: 1,000 cm $= \frac{1,000}{20}$
$= 50$ cm
length: 1,500 cm $= \frac{1,500}{20}$
$= 75$ cm
On 1 : 20 scale:
width = 50 cm, length = 75 cm

Practice

1. a. A line on a map joining two villages is 25 cm long. If the towns are actually 120 km apart, what is the scale of the map?
   b. What is the actual distance between two villages 15 cm apart on the map?

2. A dining hall measures 16 m wide by 22 m long. Give the dimensions of the dining hall on plans with the scales below.

   a. 1 : 40 b. 1 : 50
   c. 1 : 100 d. 1 : 200

3. On a plan the actual distance of 12 m is represented by 40 cm. What is the scale of the plan?

4. The distance from city A to city B is 238 km. What would be the distance between these 2 cities on a map with a scale of:

   a. 1 : 400,000 b. 1 : 500,000
   c. 1 : 119,000 d. 1 : 1,000,000

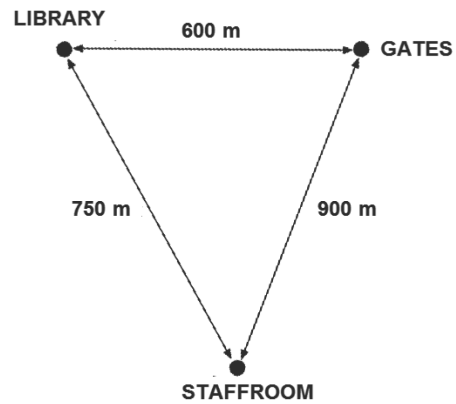
Lesson Title: Scales – Part 2	**Theme:** Numbers and Numeration
Practice Activity: PHM3-L054	**Class:** SSS 3

Learning Outcome
By the end of the lesson, you will be able to use scales to calculate the distance between two points.

Overview

This lesson allows you to practise using different types of scales to calculate the distance between two points.

Solved Examples



1. A pupil measures the distance between various points in her school compound. The various points are shown in the diagram which is not drawn to scale.
   a. Draw a map to show this information, using a scale of 1 : 10 000.
   b. A pupil is exactly halfway between the gates and the staffroom. How far are they from the library?
   c. Another pupil stands at the gates looking towards the library. They turn anti-clockwise so that they are looking at the staffroom. What angle does the pupil turn through?

Solutions:

a. Given: library to gates $= 600$ m, gates to staffroom $= 900$ m, library to staffroom $= 750$ m , scale: 1: 10,000

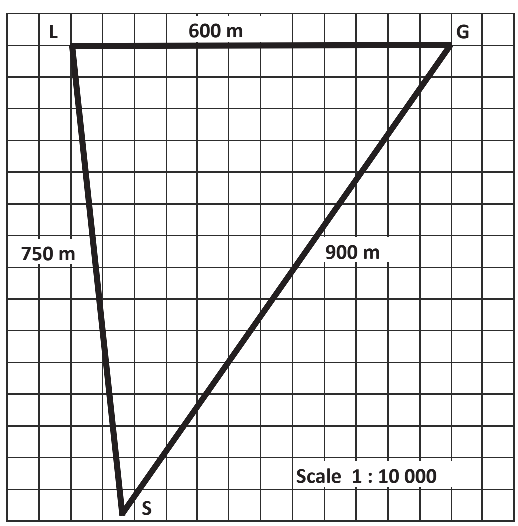
$$\frac{1}{10{,}000} = \frac{\text{distance on map}}{\text{actual distance}}$$

$$\text{distance on map} = \frac{\text{actual distance}}{10{,}000}$$

$$\text{library to gates} = 600 \text{ m}$$
$$= 60{,}000 \text{ cm}$$
$$\text{distance on map} = \frac{60{,}000}{10{,}000}$$
$$= 6 \text{ cm}$$

$$\text{gates to staffroom} = 900 \text{ m}$$
$$= 90{,}000 \text{ cm}$$
$$\text{distance on map} = \frac{90{,}000}{10{,}000}$$
$$= 9 \text{ cm}$$

$$\text{library to staffroom} = 750 \text{ m}$$
$$= 75{,}000 \text{ cm}$$



$$\text{distance on map} = \frac{75{,}000}{10{,}000}$$
$$= 7.5 \text{ cm}$$

b. measured: distance on map = 5.1 cm, scale: 1: 10,000

$$\frac{1}{10{,}000} = \frac{\text{distance on map}}{\text{actual distance}}$$
$$\text{actual distance} = \text{distance on map} \times 10{,}000$$
$$= 5.1 \times 10{,}000$$
$$= 510 \text{ m}$$

c. The angle is measured on map at 55°

2. On a map, two towns *P* and *Q* are 15.5 cm apart. The scale of the map is 1 cm : 4 km. Calculate the actual distance between *P* and *Q.*

Solution:

$$1 \text{ cm} : 4 \text{ km}$$
$$1 \text{ cm} = 400{,}000 \text{ cm}$$
$$1 : 400{,}000$$
$$\frac{1}{400{,}000} = \frac{15.5 \text{ cm}}{\text{actual distance}}$$

$$\text{actual distance} = 400{,}000 \times 15.5 \text{ cm}$$
$$= 6{,}200{,}000 \text{ cm}$$
$$= \frac{6{,}200{,}000}{100{,}000}$$
$$= 62 \text{ km}$$

3. A map is drawn to a scale of 1 : 50,000. Find:

a. The distance on the map between two places that is 10 km apart

b. The actual distance between two places that are 8.5 cm apart on the map.

Solutions:

a.

$$10 \text{ km} = 1{,}000{,}000 \text{ cm}$$
$$\frac{1}{50{,}000} = \frac{\text{distance on the map}}{1{,}000{,}000}$$

$$\text{distance on the map} = \frac{1{,}000{,}000}{50{,}000}$$
$$\text{distance on the map} = 20 \text{ cm}$$

b.

$$\frac{1}{50{,}000} = \frac{8.5 \text{ cm}}{\text{actual distance}}$$

$$
\begin{aligned}
\text{actual distance} &= 50{,}000 \times 8.5 \text{ cm} \\
&= 425{,}000 \text{ cm} \\
&= \frac{425{,}000}{100{,}000} \\
&= 4.25 \text{ km}
\end{aligned}
$$

4. The distance between two towns 48 m long is represented on a map by a line 40 cm long. What is the scale of the map?

   Solution:

$$
\begin{aligned}
40 \text{ cm} &: 48 \text{ m} \\
48 \text{ m} &= 4{,}800 \text{ cm} \\
40 \text{ cm} &: 4{,}800 \text{ cm} \\
\frac{40 \text{ cm}}{40 \text{ cm}} &: \frac{4{,}800 \text{ cm}}{40 \text{ cm}} \\
1 &: 120
\end{aligned}
$$

Practice

1. A model farm is made using a scale of 1 : 24.
   a. The height of a model cow is 4 cm, what is the height of the real cow?
   b. A real horse is 168 cm long. How long is the model horse?
2. A map is drawn to a scale of 1 : 40,000. Find:
   a. The distance on the map between two places that is 20 km apart.
   b. The actual distance between the two places that is 18.5 cm apart on the map.
3. A map is drawn on a scale of 1 cm to 5 km.
   a. Find the scale of the map in the form 1: n.
   b. Find, in cm, the distance on the map between Lafia and Makurdi (94 km).
4. A plan is made of a hotel. The length of the swimming pool, 18.48 m, is represented on the plan by a line 10.5 cm long. Find the scale of the plan in the form 1 : n.

Lesson Title: Speed – Part 1	**Theme:** Numbers and Numeration
Practice Activity: PHM3-L055	**Class:** SSS 3

Learning Outcome
By the end of the lesson, you will be able to solve problems involving speed.

Overview

This lesson reviews work done on speed from previous years.

We know that formula connecting speed, distance and time is given by:

$d = st$ where d is the distance travelled, s is the speed and t is the time taken to cover the distance.

The other 2 formulas are easily derived from the above:

$s = \frac{d}{t}$ and $t = \frac{d}{s}$, with variables having the same meaning as above.

Use these formulas whenever a problem asks "how fast", "how far", or "how long".

The speed s can be defined either as a constant speed over a particular distance or the average speed for a journey.

If it is average speed it is given by: average speed $= \frac{\text{total distance travelled}}{\text{total time taken}}$

Solved Examples

1. An Okada driver covered half the distance between two towns in 2 hr 30 min. After that he increased his speed by 2 km/hr. He covered the second half of the distance in 2 hr 20 min. Find:
   a. the initial speed of the driver b. the distance between the two towns.

Solutions:

a. **Step 1.** Assess and extract the given information from the problem

Given: 2 part journey: 1st part: 2 hr 30 min at speed x km/hr

2nd part: 2 hr 20 min at speed $x + 2$ km/hr

total distance travelled = $2y$ km

where x is the initial speed and y is half the distance between the towns.

Step 2. Substitute into the appropriate formula.

$$\text{distance} = \text{speed} \times \text{time}$$

1st part: $y = x \times 2\frac{1}{2}$ since 2 hr 30 min $= 2\frac{1}{2}$ hr

$= \frac{5}{2}x$

2nd part: $y = (x+2) \times 2\frac{1}{3}$ since 2 hr 20 min $= 2\frac{1}{3}$ hr

$= \frac{7}{3}(x+2)$

since the distances are equal:

$$\frac{5}{2}x = \frac{7}{3}(x+2)$$
$$= \frac{7}{3}x + \frac{14}{3}$$
$$\left(\frac{5}{2} - \frac{7}{3}\right)x = \frac{14}{3}$$
$$\frac{1}{6}x = \frac{14}{3}$$
$$x = \frac{14 \times 6}{3} = 28 \text{ km/hr}$$

The initial speed of the driver = 28 km/hr

b. distance $y = \frac{5}{2}x$

$= \frac{5}{2} \times 28$

$= 70$ km

The distance between the 2 towns is 70 km.

2. Mr. Kotey left his office at Kaneshie in his car at 5:30 pm to his house at Abeka at a steady speed of 72 km/h. If he arrived at his house at 5:35 pm, find the distance between his office and the house.

Solution:

$$5.35\ pm - 5.30\ pm = 5\ minutes$$
$$time = \frac{1}{12} hours$$
$$speed = 72\ km/h$$
$$distance = speed \times time$$
$$= 72 \times \frac{1}{12}$$
$$= 6\ km$$

The distance between his office and the house is 6 km.

3. A motorist drove from town P to town Q, a distance of 80 km, in 30 minutes.
a. What is his average speed in km/h?
b. What is his average speed in m/s?

Solutions:

a. $distance = 80\ km$

$time = 30\ minutes$

$= \frac{1}{2}\ hour$

$speed = \frac{distance}{time}$

b. $distance = 80\ km$

$= 80 \times 1{,}000$

$= 80{,}000$

$time = 30\ minutes$

$$= \frac{80}{\frac{1}{2}}$$

$$= 80 \times \frac{2}{1}$$

$$= 160\ km/h$$

$$= 30 \times 60$$

$$= 1{,}800\ sec$$

$$speed = \frac{distance}{time}$$

$$= \frac{80{,}000}{1{,}800}$$

$$= 44.4\ m/s$$

Practice

1. Find the distance in kilometres travelled by an airplane moving at 100 m/s for 15 minutes.
2. A motorist covers a distance of 600 m in 10 minutes; find his average speed in kilometres per hour.
3. A man covered a distance of 10 kilometres in 45 minutes on his bicycle. Find his speed in km per hour. Give your answer to 3 significant figures.
4. A boy cycles 15 kilometres to school in 55 minutes. Find his average speed in meters per second. Give your answer to 1 decimal place.
5. An athlete ran 1.5 kilometres in 4 minutes 10 seconds. What was his average speed in metres per second? Give your answer to 1 decimal place.
6. A car is travelling at an average speed of 80 km/h. What is its speed in metres per second (m/s) Give your answer to 2 decimal places.

Lesson Title: Speed – Part 2	**Theme:** Numbers and Numeration	
Lesson Number: PHM3-L056	**Class:** SSS 3	**Time:** 40 minutes
Learning Outcome By the end of the lesson, you will be able to solve more complex problems involving speed.		

Overview

There are times when the connection between distance, speed and time leads to more complex equations.

We will look at 2 instances of these types of situations. One leads to solving simultaneous linear equations. The other leads to solving a quadratic equation.

Solved Examples

1. A bus and a poda-poda both left the bus terminal at the same time heading in the same direction. The average speed of the poda-poda is 30 km/hr less than twice the speed of the bus. In two hours, the poda-poda is 20 kilometres ahead of the bus. Find:

   a. The speed of the bus b. The speed of the poda-poda.

   Solution:

   a. **Step 1.** Assess and extract the given information from the problem.

   Step 2. Assign variables to the unknown quantities

   distance of the bus $= d$ distance of the poda-poda $= d + 20$

   speed of the bus $= s$ speed of the poda-poda $= 2s - 30$

   Step 3. Set up the equations.

   $$\begin{aligned} \text{distance} &= \text{speed} \times \text{time} \\ d &= 2s \quad (1) \quad t = 2 \text{ hours} \\ d + 20 &= 2(2s - 30) \quad (2) \quad \text{same time} \end{aligned}$$

   We now have 2 linear equations in d and s

   Solve simultaneously by substitution

   Step 4. Substitute equation (1) into equation (2) and simplify

   $$\begin{aligned} 2s + 20 &= 2(2s - 30) \\ &= 4s - 60 \\ 80 &= 2s \\ s &= 40 \text{ km/hr} \end{aligned}$$

   Step 5. Write the speed of the bus.

   The speed of the bus is 40 km/hr.

   b. **Step 6.** Find the speed of the poda-poda

   speed of poda-poda $= 2s - 30$

$= (2 \times 40) - 30$

$= 80 - 30 = 50$ km/hr

The speed of the poda-poda is 50 km/hr.

2. A cyclist covers a distance of 60 km in x hours. Another cyclist who is 2 km/h slower than the first cyclist spends 1 hour more to cover the same distance. Calculate:
   a. The value of x.
   b. The speed of the first cyclist.

   Solution:

   a. Distance of 1st cyclist = 60 km

   Time of the 1st cyclist = x hour

   Speed of the 1st cyclist = $\frac{60}{x}\text{km/h}$

   Speed of the 2nd cyclist = $\left(\frac{60}{x} - 2\right)\text{km/h}$

   $= (\frac{60-2x}{x})\text{km/h}$

   Time of the 2nd cyclist = $(x+1)$ hour

   Distance of the 2nd cyclist = speed × time

   $= \left(\frac{60-2x}{x}\right) \times (x+1)$

   $= \frac{60x+60-2x^2-2x}{x}$

   $= \frac{58x+60-2x^2}{x}$

   Distance of the 1st cyclist = Distance of 2nd cyclist

   $60 = \frac{58x+60-2x^2}{x}$

   $60x = 58x + 60 - 2x^2$

   $2x^2 + 2x - 60 = 0$

   $x^2 + x - 30 = 0$

   $(x+6)(x-5) = 0$

   $x = -6$ or $x = 5$

   Therefore x=5 kilometres since distance cannot be negative.

   b. Speed of the 1st cyclist = $\frac{60}{x}$ km/h

   $= \frac{60}{5}$ km/h

   $= 12$ km/h

Practice

1. A cyclist covers a distance of 90 km in y hours. Another cyclist who is 5km/h faster than the first cyclist spends 3 hours less to cover the same distance. Find:
   a. The value of y.
   b. The speed of the first cyclist.

2. Mr. Kotey left his office at Kaneshie in his car at 5.30 pm to his house at Abeka at a steady speed of 72 Km/h. If he arrived at his house at 5.35 pm, find the distance between his office and the house.

3. A driver covers a distance of 120 km in k hours. Another driver who is 4km/h slower than the first driver spends 1 hour more to cover the same distance. Find:
    a. The value of k.
    b. The speed of the second driver.

4. A driver covers a distance of 140 km in p hours. Another driver who is 8km/h faster than the first driver spends 2 hour less to cover the same distance. Find:
    a. The value of k.
    b. The speed of the first driver.
    c. The speed of the second driver.

Lesson Title: Travel graphs	**Theme:** Numbers and Numeration	
Lesson Number: PHM3-L057	**Class:** SSS 3	**Time:** 40 minutes
Learning Outcome By the end of the lesson, you will be able to draw and interpret travel graphs.		

Overview

A travel graph shows the relationship between distance and time of a moving object for a journey. Travel graphs are also referred to as distance-time graphs.

Look at the travel graph in question 1.
The vertical scale shows the distance from the starting or reference point.
The horizontal graph shows the time taken.

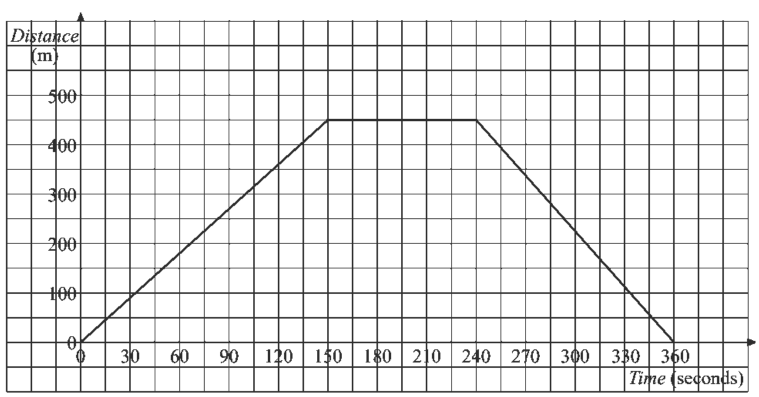
Travel graphs can have 1, 2, 3 or more parts which represent different parts of the journey. The example shown in question 1 has 3 parts.

- The first part of the journey shows a straight line sloping upwards (positive gradient). This shows the object travelling away from the starting point at a constant speed.
- The horizontal line (zero gradient) shows that for a period of time there was no increase or decrease in distance from the starting point.
- The straight line sloping downwards (negative gradient) shows the object coming back towards its starting point at a constant speed.
- The steeper the slope of the line, the faster the object is travelling for a given time. This slope is given by speed $= \frac{\text{distance}}{\text{time}}$.
- The average speed can also be calculated for the whole journey.

$$\text{average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

Solved Examples

1. The graph shows Fatu's journey from home.
   a. Describe how Fatu moves on each part of her journey.
   b. Calculate her speed on each part.
   c. What is her average speed for the whole journey?



Solutions:

Step 1. Assess and extract the given information from the problem.

Given: travel graph showing Fatu's journey

Step 2. Describe each part of her journey.

a. 1st part: Fatu moves away from the house at a constant speed.

2nd part: Fatu remains at the same place for 90 seconds.

3rd part: Fatu returns home at a constant speed.

Step 3. Substitute into the appropriate formula.

b.
$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

1st part: Fatu travels 450 m in 150 seconds

$$\text{speed} = \frac{450 \text{ m}}{150 \text{ s}} = 3 \text{ m/s}$$

Fatu's speed in the first part of her journey is 3 m/s.

2nd part: Fatu's speed is 0 (horizontal line has 0 gradient)

3rd part: Fatu travels 450 m in 120 seconds

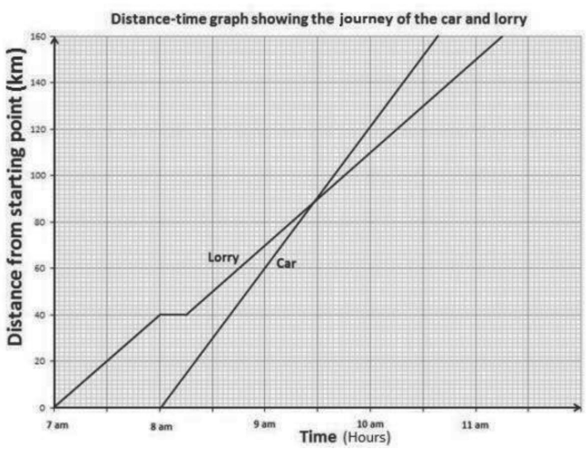
$$\text{speed} = \frac{450 \text{ m}}{120 \text{ s}} = 3.75 \text{ m/s}$$

Fatu's speed in the second part of her journey is 3.75 m/s.

c.
$$\text{average speed} = \frac{\text{total distance travelled}}{\text{total time taken}} = \frac{900 \text{ m}}{360 \text{ s}} = 2.5 \text{ m/s}$$

Fatu's average speed is 2.5 m/s.

2. A lorry and a car set out from the same point towards a common destination 160 km away. The lorry sets out at 7 am and maintains a speed of 40 km/h. The car sets out at 8 am maintaining a speed of 60 km/h. After 1 hour, the lorry stops for 15 minutes and then continues at its former speed.
   a. Using the same axes, draw distance-time graphs for the car and the lorry.
   b. At what time did they cross each other?



Solutions:

a. See graph.

b. Approximately 9.27 am.

3. A car leaves town A for town B, a distance of 200 km away, at 8 am travelling at 60 km/h. At 9 am, a jeep leaves B for A, travelling at 80 km/h. After one and a half hours, the car stops at a trade fair for half an hour. It then continues at its original speed.

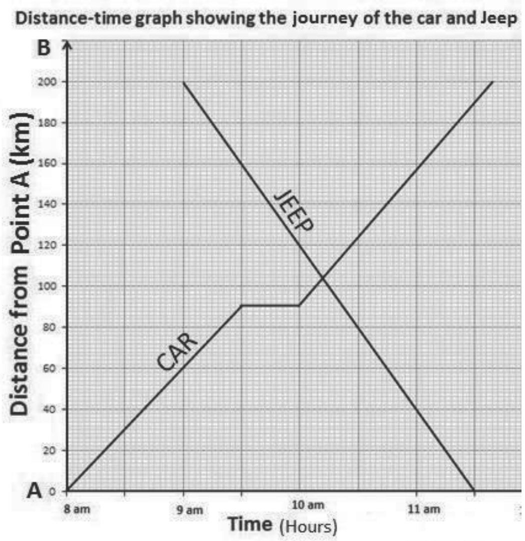
   a. Draw suitable distance-time graphs on the same axes.

   b. At what time did they cross each other?

   Solutions:

   a. See graph.

   b. Approximately 10.12 am.



Practice

1. Two aircraft fly between Lagos and Accra, a distance of about 400 km. They both fly at a constant speed. Their times of arrival and departure are given in this timetable:

   Aircraft: A
   Depart Accra 10.00 GMT
   Arrive Lagos 11.00 GMT

   Aircraft B:
   Depart Lagos 10.20 GMT
   Arrive Accra 11.40 GMT

   a. Draw a graph of the two flights.
   b. What are the speeds of the two aircrafts?
   c. When and where did they pass each other?

2. Morie cycles from Mattru to see his father who lives 12 km away. He travels at 8 km/h. At the same instant as Morie leaves, his father heads for Mattru walking at 4 km/h. Half an hour later, Morie had a puncture. He tried to mend the puncture but after half an hour, he gave up and continued on foot at 4 km/h.

   a. Represent the two journeys on the same distance-time graph.
   b. Find when and where Morie meets his father.

Lesson Title: Density	Theme: Numbers and Numeration
Practice Activity: PHM3-L058	**Class:** SSS 3

Learning Outcome
By the end of the lesson, you will be able to calculate the density of a population or an object using ratio and proportion.

Overview

Density is a measure of how closely packed together a mass of objects or people are in a given amount of space.

This lesson looks at 2 different types of density. We are more familiar with the formula for the density of an object.

This is a measure of the mass of an object per cm^3 of volume.

$$\text{density} = \frac{\text{mass}}{\text{volume}}$$

The mass of an object is how much matter it has. The volume is the amount of space that the object takes up.

We can also measure population density. This is a measure of how close people or organisms are to each other.

$$\text{population density} = \frac{\text{number of people in area}}{\text{area}}$$

It is the ratio of the number of people or organisms per unit area of available space.

We will look at both the density of an object and population density problems.

Solved Examples

1. A piece of silver has a mass of 84 g and a volume of 8 cm^3. Work out the density of the silver.
   Solution:

   Step 1. Assess and extract the given information from the problem.
   Given: piece of silver with a mass of 84 g and a volume of 8 cm^3

   Step 2. Substitute into the appropriate formula.

$$\begin{aligned}\text{density} &= \frac{\text{mass}}{\text{volume}}\\ &= \frac{84}{8}\\ &= 10.5 \text{ g/cm}^3\end{aligned}$$

Step 3. Write the answer. The piece of silver has a density of 10.5 g/cm^3.

2. A village has an area of 70 km^2. It has a population of 6,200 people. Calculate the population density in people/ km^2. Give your answer to 3 significant figures.
Solution:
Given: village with an area of 70 km^2 and population of 6,200 people

$$\begin{aligned} \text{population density} &= \frac{\text{number of people in area}}{\text{area}} \\ &= \frac{6{,}200}{70} \\ &= 88.571 \\ &= 88.6 \end{aligned}$$

The population density of the village is 88.6 people/km^2.

3. A container has a capacity of 20 litres. It is filled with wine whose density is 0.8 kg/litre. What is the mass of the wine?
Solution:

$$\begin{aligned} density &= \frac{mass}{volume} \\ mass &= density \times volume \\ &= 0.8 \times 20 \\ &= 16\ kg \end{aligned}$$

4. A village is roughly the shape of a rectangle 1⅓ km by 1¼ km. What is its total population if the average density is 570 persons per km^2?
Solution:

$$\begin{aligned} area\ of\ the\ village &= 1\frac{1}{3} \times 1\frac{1}{4} \\ &= \frac{4}{3} \times \frac{5}{4} \\ &= \frac{5}{3}\ km^2 \\ total\ population &= population\ density \times area \\ &= 570 \times \frac{5}{3} \\ &= 950 \end{aligned}$$

5. A town has an area of 3,040 hectares and a population of 90,500 people. Calculate its population density (in persons/hectare) correct to three significant figures.
Solution:

$$\begin{aligned} Population\ density &= \frac{number\ of\ people}{area} \\ &= \frac{90{,}500}{3{,}040} \\ &= 29.8\ persons/hectare \end{aligned}$$

Practice

1. A village is roughly a square in shape. Its perimeter is about 8 km. If the population density of the village is 1,200 people/km^2, find the approximate population of the village.
2. A town has an area of 7,500 square kilometres and a population of 690,000. Calculate its population density (in persons/km^2).
3. A community has a population of 1,350,000 people and a population density of 730 persons/km^2, find the area of the community correct to 3 significant figures.
4. A town with a population density of 870 persons/km^2 has an area of 2,345 km^2. Find the population of the town, correct to 3 significant figures.
5. A town has an area of 24 km^2 and a population of 31,000. Calculate the population density of the town per km^2, correct to 2 significant figures.
6. If 42 cm^3 of sea water has a mass of 43.26 g, find its density in g/cm^3.

Lesson Title: Rates of pay	Theme: Numbers and Numeration
Lesson Number: PHM3-L059	**Class:** SSS 3
Learning Outcome By the end of the lesson, you will be able to calculate rates of pay using ratio and proportion and data given.	

Overview

To calculate pay we are usually given the pay rate for a period and asked to find how much was earned by a worker.

We are also sometimes asked to find the new pay after a percentage increase or decrease in salary.

Calculations on pay rates are best done by working through examples which show the different types of problems and methods.

Solved Examples

1. Each week Sia works from 8:30 a.m. to 1:00 p.m. for 5 days and from 2:00 p.m. to 5:30 p.m. for 2 days. Her rate of pay is Le 5,000.00 per hour. What is her total weekly wage?

   Solution:

   Step 1. Assess and extract the given information from the problem.
   Given: Sia works from 8:30 a.m. to 1:00 p.m. for 5 days and from 2:00 p.m. to 5:30 p.m. for 2 days; rate of pay is Le 5,000.00 per hour.

   Step 2. Use Sia's daily wage to calculate how much she earns.

   hours worked per day from 8:30 a.m. to 1:00 p.m. $= 4.5$ hrs
   hours worked for 5 days $= 5 \times 4.5 = 22.5$ hrs
   hours worked per day from 2:00 p.m. to 5:30 p.m. $= 3.5$ hrs
   hours worked for 2 days $= 2 \times 3.5 = 7$ hrs
   total hours worked $= 22.5 + 7 = 29.5$ hrs
   weekly wage $= 29.5 \times 5000 = \text{Le } 147{,}500.00$

   Step 3. Write the answer.
   Sia's total weekly wage was Le 147,500.00

2. The table shows the salaries of two workers. If each worker receives a 4.2% salary increase, what is the new salary of each worker?

Fatu	Le 720,000.00
Mohammed	Le 960,000.00

   Solution:
   Two different methods of finding the new salaries are shown.

Given: Fatu's salary = Le 720,000.00; Mohammed's salary = Le 960,000.00

Method 1 (Fatu's new salary)

Step 1. Find increase in pay.

increase in pay $= \frac{4.2}{100} \times 720{,}000 =$ Le $30{,}240.00$

Step 2. Add increase in pay to original amount.

Fatu's new salary $= 720{,}000 + 30{,}240 =$ Le $750{,}240.00$

Method 2 (Mohammed's new salary)

Step 1. Find the multiplier for the increase in pay

multiplier $= 100\% + 4.2\% = 104.2\%$

[NOTE]: 100% refers to the original salary before the increase.

Step 2. Use the multiplier to calculate the new salary

Mohammed's new salary $= \frac{104.2}{100} \times 960{,}000 =$ Le $1{,}000{,}032.00$

We can use either of the above method as appropriate, to increase or decrease a quantity by a given percentage.

3. Ibrahim received a 3% cut to his pay. If his new salary is Le 727,500.00, what was his old salary?

Solution:

Given: Ibrahim's pay cut = 3% , new salary= $727{,}500$

Let Ibrahim's initial salary $= x$

multiplier $= 100\% - 3\% = 97\%$

$\frac{97}{100}$ of $x = 727{,}500$

$\frac{97x}{100} = 727{,}500$

$0.97x = 727{,}500$

$x = \frac{727{,}500}{0.97} = 750{,}000$

Ibrahim's initial salary = Le 750,000.00

Practice

1. Thomas earns Le 6,000.00 per hour working Mondays to Fridays. He earns 15% extra if he works on Saturdays and Sundays. In a week that Momoh Thomas worked for 45 hours from Monday to Friday and a further 8 hours on Saturday, how much did he earn in total?

2. The salaries of Malcolm and James are in the ratio 5:2. James' salary is Le 750,000.00, what is Malcolm's salary?

3. A company has a sliding scale they use to pay their senior staff as follows:

Manager	Le 4,000,000.00
Finance Director	85% of the Manager's pay
Assistant Manager	70% of the Manager's pay

What is the difference in pay between the Finance Director and the Assistant Manager?

4. Each week, Thorlu works from 7:00 a.m. to 12:00 noon for 4 days and from 1:00 p.m. to 4:00 p.m. for 3 days. Her rate of pay is Le 5,500.00 per hour. What is his total weekly wage?

5. The table shows the salaries of four workers. If each worker receives a 5.5% salary increase, what is the new salary of each worker?

Margaret	Le 570,000.00
Kabba	Le 690,000.00
Tommy	Le 750,000.00
Christiana	Le 940,000.00

6. Samuel received a 4% cut to his pay. If his new salary is Le 950,000.00, what was his old salary?

Lesson Title: Commission	**Theme:** Numbers and Numeration
Lesson Number: PHM3-L060	**Class:** SSS 3
Learning Outcome By the end of the lesson, you will be able to calculate commission on a transaction by applying percentage.	

Overview

Some employees, particularly sales people, are given commission on top of (or instead of) their wages or salaries. The value of the commission is usually worked out as a percentage of the amount they sold during the month or year.

The value of the amount sold is taken as 100%
To calculate x commission on a particular sales amount, use the formula:

$$\text{commission} = \frac{x}{100} \times \text{sales amount}$$

Solved Examples

1. A newspaper vendor makes a commission of 12% on his sales. Calculate his commission on the following sales:
   a. Le $2,000$ b. Le $6,000$
   c. Le $340,000$ d. Le $18,000$

Solution:

Step 1. Assess and extract the given information from the problem.
Given: commission received by sales vendor = 10% of sales

Step 2. Calculate commission for each sales amount.

$$\text{commission} = \frac{12}{100} \times \text{sales amount} = 0.12 \times \text{sales amount}$$

a. Le 2,000 sales: commission $= 0.12 \times 2,000$ = Le 240
b. Le 6,000 sales: commission $= 0.12 \times 6,000$ = Le 720
c. Le 340,000 sales: commission $= 0.12 \times 340,000$ = Le $40,800$
d. Le 18,000 sales: commission $= 0.12 \times 18,000$ = Le $2,160$

2. Jenneh gets a commission of 10% on bread sold. In one week, Jenneh's commission was Le $45,000.00$. How much bread did she sell during that week?

Solution:

Given: 10% commission on bread sold; Jenneh's commission was Le $45,000.00$

$$\text{Let amount of sales} = x$$
$$\text{commission} = \frac{10}{100} \times x = 0.1x$$
$$45,000 = 0.1x$$
$$x = \frac{45,000}{0.1}$$

$$x = \text{Le } 450{,}000.00$$

Jenneh sold Le $450{,}000.00$ worth of bread.

3. Every month, a sales agent selling electrical goods makes commission of 3% on the first Le 2 million of sales, 4% on the next Le 3 million of sales and 5% on any sales over Le 5 million. How much commission does he make on sales of Le 16 million in December?

   Solution:

   Given: 3% commission on the first Le 2 million sales, 4% commission on the next Le 3 million and 5% on any sales above Le 5 million;

   sales in December = Le 16 million

   commission on the first Le $2{,}000{,}000$ $= \frac{3}{100} \times 2{,}000{,}000 =$ Le $60{,}000.00$

   commission on the next Le $3{,}000{,}000$ $= \frac{4}{100} \times 3{,}000{,}000 =$ Le $120{,}000.00$

   remainder $= 16{,}000{,}000 - 5{,}000{,}000$
   $=$ Le$11{,}000{,}000.00$

   commission on remainder $= \frac{5}{100} \times 11{,}000{,}000 =$ Le $550{,}000.00$

   total commission $= 60{,}000 + 120{,}000 + 550{,}000 =$ Le $730{,}000.00$

   Total commission of sales agent = Le $730{,}000.00$

4. Dahlina receives a salary of Le$700{,}000.00$ and a 2% commission on all sales for the month. If her total income in a particular month was Le$1{,}000{,}000.00$, what was the amount of her sales for the month?

   Solution:

   Given: Dahlina'salary= Le $700{,}000$ with 2% commission on all sales, monthly income is Le$1{,}000{,}000.00$

   commission $= 1{,}000{,}000 - 700{,}000 =$ Le $300{,}000.00$

   Let amount of sales $= x$

   commission $= \frac{2}{100} \times x = 0.02x$

   $300{,}000 = 0.02x$

   $x = \frac{300{,}000}{0.02} =$ Le $15{,}000{,}000.00$

   Dahlina made sales of Le $15{,}000{,}000.00$ for the month.

Practice

1. How much commission will an agent receive from the sales of Le 2,125,000.00 at a rate of 14%?

2. A sales agent makes a commission of 15% on his sales. Calculate his commission on the following sales:

   a. Le $3{,}000.00$ b Le $8{,}000.00$ c. Le $24{,}000.00$ d. Le $590{,}000.00$

3. Umaru gets a commission of 10% on books sold. In one week, Umaru's commission was Le $86{,}000.00$. What was the total cost of the books sold during that week?

4. An agent receives 3% commission for transacting a sales business for his principal.
   a. Calculate the commission if the amount involved is Le 623,000.00.
   b. Find the total sales if the agent receives a commission of Le 1,860.00.

5. Find the price of an article if the Le 6,050.00 commission allowed is 5%.

6. An agent collected Le 40,000.00 commission on sales made on an article marked Le 1,600,000.00. What is the rate of the commission in percent?

Lesson Title: Income taxes	**Theme:** Numbers and Numeration
Lesson Number: PHM3-L061	**Class:** SSS 3

Learning Outcome
By the end of the lesson, you will be able to calculate the amount of income tax to be paid using percentages.

Overview

Tax is deducted every month by the government from the money people earn.
This tax is called **Income Tax** and it is used to provide services to the country such as education, health, police, military and social welfare.

Employee taxes are deducted from their salaries by their employers using a method called PAYE. PAYE stands for Pay As You Earn and the 2017 rates are shown on the table on the board.

Sierra Leone PAYE Tax Rate	
Not over Le 500,000 per month	Nil
Next Le 500,000 per month	15%
Next Le 500,000 per month	20%
Next Le 500,000 per month	30%
Above Le 2 million per month	35%

Every employee has a tax-free income. This is the income below which you do not have to pay any income tax. The current tax-free income is Le 500,000.00.

The net income an employee earns is the income after tax has been deducted.

Solved Examples

1. Use the table to calculate how much tax is paid on the salaries below each month:
   a. Le 850,000.00 b. Le 1,700,000.00
   Solutions:
   Step 1. Assess and extract the given information from the problem.
   Given: Income tax for salaries of Le 850,000 and Le 1,700,000
   Le500,000.00 is tax- free
   a. **Step 2.** Calculate the income tax paid per month on Le 850,000.00 salary
   taxable income $= 850{,}000 - 500{,}000 =$ Le $350{,}000.00$
   income tax $= \frac{15}{100} \times 350{,}000 =$ Le $52{,}500.00$
   Step 3. Write the answer.
   The income tax is Le$52{,}500.00$ per month
   b. **Step 4.** Calculate the income tax paid per month on Le 1,700,000.00 salary
   taxable income $= 1{,}700{,}000 - 500{,}000 =$ Le $1{,}200{,}000.00$
   A table is used to aid the calculation.
   The remaining income is the income left after the taxation at each stage.

Remaining income (Le)	Amount to be taxed (Le)	Rate of tax (%)	Income Tax (Rate × Amount)
$1{,}200{,}000$	$500{,}000$	15	$\frac{15}{100} \times 500{,}000 = 75{,}000$
$1{,}200{,}000 - 500{,}000 = 700{,}000$	$500{,}000$	20	$\frac{20}{100} \times 500{,}000 = 100{,}000$
$700{,}000 - 500{,}000 = 200{,}000$	$200{,}000$	30	$\frac{30}{100} \times 200{,}000 = 60{,}000$

income tax $= 75{,}000 + 100{,}000 + 60{,}000 =$ Le $235{,}000.00$

Step 5. Write the answer.

The income tax is Le $235{,}000.00$ per month.

2. Sama pays income tax of Le 187,000 each month. How much does he earn per month?

Solution:

Given: income tax of Le 187,000.00 each month

We do a reverse calculation to question 1.

Remaining tax (Le)	Amount to be taxed (Le)	Rate of tax (%)	Income Tax
$187{,}000$	$500{,}000$	15	$\frac{15}{100} \times 500{,}000 = 75{,}000$
$187{,}000 - 75{,}000 = 112{,}000$	$500{,}000$	20	$\frac{20}{100} \times 500{,}000 = 100{,}000$
$112{,}000 - 100{,}000 = 12{,}000$	x	30	$\frac{30}{100} \times x = 0.3x$

From the amount of tax remaining in the last line, we know that the amount to be paid is less than Le 500,000.00. Let the amount $= x$, such that:

$0.3x = 12{,}000$ income tax = remaining tax

$x = \frac{12{,}000}{0.3} =$ Le $40{,}000.00$

taxable income $= 500{,}000 + 500{,}000 + x$ from the table

$= 500{,}000 + 500{,}000 + 40{,}000 = 1{,}040{,}000$

income = tax-free income + taxable income

$= 500{,}000 + 1{,}040{,}000 = 1{,}540{,}000$

Sama earns Le $1{,}540{,}000.00$.

3. Adama earns Le 1,200,000.00 per month. In addition to her tax-free income, she can claim Le 50,000.00 for every dependent child. She has 3 children. Calculate:
   a. Her total tax-free income
   b. Her taxable income
   c. Her total tax per month
   d. Her net income per month

Solution:

Given: Adama earns Le 1,200,000.00 per month, has 3 children for whom she can claim Le 50,000.00 per child.

a. total tax-free income $= 500{,}000 + (3 \times 50{,}000) =$ Le $650{,}000.00$

b. taxable income $= 1{,}200{,}000 - 650{,}000 =$ Le $550{,}000.00$

c.

Remaining income (Le)	Amount to be taxed (Le)	Rate of tax (%)	Income Tax (Rate × Amount)
$550{,}000$	$500{,}000$	15	$\frac{15}{100} \times 500{,}000 = 75{,}000$
$550{,}000 - 500{,}000 = 50{,}000$	$50{,}000$	20	$\frac{20}{100} \times 50{,}000 = 10{,}000$

total income tax $= 75{,}000 + 10{,}000 =$ Le $85{,}000.00$

d. net income $= 1{,}200{,}000 - 85{,}000 =$ Le $1{,}115{,}000.00$

Practice

1. Calculate how much tax is paid on the salaries below each month:
   a. Le 960,000.00 b. Le 1,850,000.00

2. Doris pays income tax of Le 205,000 each month. How much does she earn per month?

3. Kaprie earns Le 1,400,000.00 per month. In addition to his tax-free income, he claims Le 50,000.00 for every dependent child. He has 3 children. Calculate Kaprie's:
   a. Total tax-free income
   b. Taxable income
   c. Total tax per month
   d. Yearly tax
   e. Net income per month
   f. Yearly net income

Lesson Title: Simple interest	**Theme:** Numbers and Numeration
Lesson Number: PHM3-L062	**Class:** SSS 3
Learning Outcome By the end of the lesson, you will be able to calculate simple interest rates and time.	

Overview

When someone deposits money in a bank, the bank pays them interest on the money deposited. When a bank lends money to its customers, it charges them interest on the money borrowed.

There are two types of interest earned or charged on money – **simple interest and compound interest.**

We will look at simple interest in this lesson, compound interest in the next.

The simple interest, I, is the amount earned or charged on the initial amount or principal, P, at a given rate, R, and for a given period of time, T (in years).

$$I = \frac{PRT}{100}$$

It is in effect, the percentage of the principal that is earned or charged for the use of the money.

The amount, A, at the end of the period is given by Principal + Interest.

That is:

$$A = P + I$$

Solved Examples

1. Alusine deposits Le50 0,000.00 in the bank at the rate of 4% per annum for 2 years. How much interest does he receive?

   Solution:

   Step 1. Assess and extract the given information from the problem.

   Given: Le 500,000.00 deposited by Alusine 4% interest rate per annum for 2 years

   Step 2. Calculate the interest.

   $$I = \frac{PRT}{100}$$

   $$I = \frac{500{,}000 \times 4 \times 2}{100} = \text{Le } 40{,}000.00$$

   Step 3. Write the answer.

   The interest received by Alusine is Le $40{,}000.00$.

2. The simple interest on Le 725,000.00 for 4 years is Le 87,000.00. How much per annum is the interest rate?

Solution:

Given: simple interest of Le 87,000.00 on Le 725,000.0 for 4 years

$I = \frac{PRT}{100}$

$R = \frac{I \times 100}{PT}$ make R the subject of the formula

$R = \frac{87,000 \times 100}{725,000 \times 4} = 3\%$

The interest rate per annum is 3%.

3. How much money should be invested if interest of Le 90,000.00 is to be paid after 3 years at 5% per annum? What is the amount after 3 years?

Solution:

Given: interest of Le 90,000.00, after 3 years at 5% per annum

$P = \frac{I \times 100}{RT}$ make P the subject of the formula

$P = \frac{90,000 \times 100}{5 \times 3}$

$= \text{Le } 600,000.00$

amount after 3 years $= 600,000 + 90,000 = \text{Le } 690,000.00$

The amount after 3 years = Le $690,000.00$.

4. Find the time period in which the interest on Le 300,000.00 at 3% interest rate is Le 45,000.00.

Solution:

Given: interest on Le 300,000.00 at 3% interest rate = Le 45,000.00.

$T = \frac{I \times 100}{PR}$ make T the subject of the formula

$T = \frac{45,000 \times 100}{300,000 \times 3} = 5$ years

The time period $= 5$ years.

5. A man has Le 1,000,000.00. He invests ⅖ of this amount at 9% per annum and the rest at 3½% per annum. How long will it be before he earns total interest of Le 171,000.00?

Solution:

$P = \frac{2}{5} \times \text{Le } 1,000,000.00$

$P = \text{Le } 400,000.00$

$\text{Le } 400,000.00$ was invested at a rate of 9%.

$P = \text{Le } 400,000.00 \; R = 9\%$

$I = \frac{\text{Le } 400,000.00 \times 9 \times T}{100}$

$I = \text{Le } 36,000.00T$

The balance of his money

$P = \text{Le } 1,000,000.00 - \text{Le } 400,000.00$

$= \text{Le} 600,000.00$

$\text{Le } 600,000$ was invested at a rate of $3\frac{1}{2}\%$.

$P = \text{Le } 600{,}000.00 \; R = \; 3\frac{1}{2}\% = \frac{7}{2}$

$$I = \frac{\text{Le } 600{,}000.00 \times 7 \times T}{100 \times 2}$$

$$I = \text{Le } 21{,}000.00T$$

$$\text{Le } 36{,}000.00T = \text{Le } 21{,}000.00T \quad = \quad \text{Le } 171{,}000.00$$

$$\text{Le } 57{,}000.00T = \text{Le } 171{,}000.00$$

$$T = \frac{\text{Le } 171{,}000.00}{\text{Le } 57{,}000.00}$$

$$T = 3 \text{ years}$$

It will take 3 years.

Practice

1. Mrs. Conteh lends Le 15,400.00 for 5 years. If she receives a total of Le 20,790.00 at the end of this time, at what rate did she lend the money?

2. At the end of 4 years 6 months, a borrower has to pay interest of Le 20,700.00. How much money did he borrow, if interest was paid at 11½% per annum?

3. Unisa invests Le 360,000.00 for 3 years at 8% per annum. How long will it take Mr. Johnson, who invested Le 200,000.00 at 12% per annum, to earn the same interest as Unisa?

4. Lamina invests Le 140,000.00 for 2 years, at the end of which time he receives a total of Le 165,200.00. What amount must Abdulai invest for 3½ years at the same rate in order to earn the same interest as Lamina?

5. Mabinty borrows Le 500,000 from her uncle, at an annual interest rate of 5%.She immediately lends this money to Alimamy at the rate of 8% per annum. How much does she gain over three years?

Lesson Title: Compound interest – Part 1	Theme: Numbers and Numeration
Lesson Number: PHM3-L063	**Class:** SSS 3
Learning Outcome By the end of the lesson, you will be able to calculate compound interest using successive addition.	

Overview

When we calculate simple interest, we are finding the percentage of the principal that is added to the investment or loan over the whole period at a given interest rate. The principal remains unchanged for the entire period of the loan. However, investments and loans are not usually calculated using simple interest.

Compound interest is the interest calculated at given intervals over the loan period and added to the principal. This new amount becomes the principal and changes every time the interest is calculated. Each time we do the calculation, we compound the principal by adding the interest calculated for a given period to the previous principal.

We are in effect earning or paying interest on the interest

Each period is called a **compounding period** and can be at intervals of 1 year, 6 months ($\frac{1}{2}$ year), 3 months ($\frac{1}{4}$ year) or any other agreed time period.

The compound interest, CI, is given by:

$CI = A - P$ where $A =$ Amount at the end of the period
$P =$ Principal

We will now do an example to show 2 different methods of calculating compound interest. We will concentrate on calculating the compound interest annually.

Solved Examples

1. Find the simple interest on a loan of Le 500,000 for 4 years at 5% per annum.
   Solution:
   Given: loan of Le 500,000 for 4 years at a simple interest rate of 5% per annum.
   $$I = \frac{PRT}{100}$$
   $$= \frac{500{,}000 \times 5 \times 4}{100} = \text{Le } 100{,}000.00$$
   The simple interest = Le $100{,}000.00$.

2. Find the interest on a loan of Le 500,000.00 for 4 years at a compound interest rate of 5% per annum. What additional interest is earned using compound as compared to simple interest rate?

Solution:

Step 1. Assess and extract the given information from the problem.

Given: loan of Le 500,000.00 for 4 years at a compound interest rate of 5% per annum.

Step 2. Calculate the amount at the end of the period.

Method 1. Using successive addition

Year	Principal at start of year (Le)	Interest (Le)	Amount at end of year (Le)
1	500,000	$\frac{5}{100} \times 500000 = 25,000$	$500,000 + 25,000 = 525,000$
2	525,000	$\frac{5}{100} \times 525000 = 26,250$	$525,000 + 26,250 = 551,250$
3	551,250	$\frac{5}{100} \times 551250 = 27,563$	$551,250 + 27,563 = 578,813$
4	578,813	$\frac{5}{100} \times 578,813 = 28941$	$578813 + 28941 = 607,754$

Method 2. Using a multiplier

Explain:

- A multiplier is used whenever we wish to increase or decrease an amount by a given percentage.
- The original amount is 100% or 1. $\left(\frac{100}{100}\right)$
- We add to increase the amount by the given percentage – successive addition is embedded in the calculation.

Multiplier $= 1 + \frac{5}{100} = 1.05$

Year	Principal at start of year (Le)	Amount at end of year (Le)
1	500,000	$500,000 \times 1.05 = 525,000$
2	525,000	$525,000 \times 1.05 = 551,250$
3	551,250	$551,250 \times 1.05 = 578,813$
4	578,813	$578,813 \times 1.05 = 607,754$

Step 3. Calculate the compound interest

$$CI = A - P$$
$$= 607,754 - 500,000 = \text{Le } 107,754.00$$

Step 4. Write the answer.

The compound interest at the end of 4 years = Le $107,754.00$.

Comparing the answer to question 1, the additional interest earned is $107,754 - 100,000 =$ Le $7,754.00$.

3. A businesswoman deposited Le 3,000,000.00 in her bank account at 7% compound interest rate per annum for 5 years. At the end of the third year, she withdrew Le 1,000,000.00. Calculate the amount she has in her account after 5 years. Give your answer to the nearest cent.

Solution:

Given: Businesswoman deposits Le 3,000,000.00 at 7% rate per annum for 5 years.

Multiplier $= 1 + \frac{7}{100} = 1.07$

Year	Principal at start of year (Le)	Amount at end of year (Le)
1	$3,000,000$	$3,000,000 \times 1.07 \quad = \quad 3,210,000$
2	$3,210,000$	$3,210,000 \times 1.07 \quad = \quad 3,434,700$
3	$3,434,700$	$3,434,700 \times 1.07 \quad = \quad 3,675,129$
	withdrawal of Le 1,000,000.00: new principal	$= \quad 3,675,129 - 1,000,000$
		$=$ Le $2,675,129.00$
4	$2,675,129$	$2,675,129 \times 1.07 \quad = \quad 2,862,388.03$
5	$2,862,388.03$	$2,862,388.03 \times 1.07 \quad = \quad 3,062,755.19$
After 5 years, the businesswoman has Le $3,062,755.19$ to the nearest cent in her account.		

Practice

1. Find the interest on Le 250,000.00 which was deposited in a bank for 4 years at a compound interest rate of 5% per annum.

2. Find the interest on a loan of Le 400,000.00 for 3 years at a compound interest rate of 4% per annum. What additional interest is earned using compound as compared to a simple interest rate?

3. A business man deposited Le 500,000.00 in his bank account at 5% compound interest rate per annum for 6 years. At the end of the fourth year, he withdrew Le 150,000.00. Calculate the amount he has in his account after 5 years.

4. Morlai borrowed Le 600,000.00 for 4 years at 10% compound interest rate.
   a. What was the amount at the end of the 4 years?
   b. How much was the compound interest?

Lesson Title: Compound interest – Part 2	Theme: Numbers and Numeration
Lesson Number: PHM3-L064	**Class:** SSS 3

Learning Outcome
By the end of the lesson, you will be able to calculate compound interest using the formula.

Overview

This lesson shows how the formula to calculate compound interest is derived and used.

Consider question 1 below. The table shows how the multiplier method from the last lesson is expanded to calculate the amount at the end of each year. The year for which the interest is being calculated gives the index for the multiplier.

Suppose the loan was extended to 4 years, the amount would equal $500{,}000 \times 1.04^4$.

From the table, the amount for a particular year is calculated using the formula:

$$\text{amount at end of } n \text{ years} = 500{,}000 \times (1.04)^n$$

$$= 500{,}000 \times \left(1 + \frac{4}{100}\right)^n$$

From this we can write a general formula to find the amount at the end of any period as:

$$A = P\left(1 + \frac{R}{100}\right)^n$$

where A = Amount at the end of the period, P = Principal, R = Rate, n = Period

The compound interest, CI, is given as before by:

$$CI = A - P$$

where A = Amount at the end of the period, P = Principal

In this lesson, we also consider how to calculate the compound interest when the compounding period is not per year.

Solved Examples

1. A sum of Le 500,000.00 is to be invested for 3 years. Use the multiplier method to find the final value of the investment if the annual compound interest rate is 4%.

   Solution:

   Given: Le 500,000.00 for 3 years at a rate of 4% per annum.

   $$\text{Multiplier} = 1 + \frac{4}{100} = 1.04$$

Year (n)	Principal (Le)	Amount at end of year (Le)		
1	$500{,}000$	$500{,}000 \times 1.04 = 500{,}000 \times 1.04^1$	$=$	$520{,}000$

2	$520{,}000$	$520{,}200 \times 1.04 = 500{,}000 \times 1.04^2$	$=$	$540{,}800$
3	$540{,}800$	$540{,}800 \times 1.04 = 500{,}000 \times 1.04^3$	$=$	$562{,}432$

The compound interest = $562{,}432 - 500{,}000 = 62{,}432$
The compound interest at the end of the period = Le $62{,}432.00$.

2. Alice borrowed Le 400,000.00 for 3 years at 10% compound interest rate.
   a. What was the amount at the end of the 3 years?
   b. How much was the compound interest?

Solutions:

Step 1. Assess and extract the given information from the problem.
given: Le400,000.00 borrowed by Alice for 3 years at 10%

a. **Step 2.** Calculate the amount at the end of the loan period.

$$A = P\left(1 + \frac{R}{100}\right)^n$$
$$= 400{,}000\left(1 + \frac{10}{100}\right)^3 = 400{,}000 \times (1.1)^3$$
$$= \textbf{Le } 532{,}400$$

Step 3. Write the answer for a.
The amount = Le$532{,}400.00$

b. **Step 4.** Calculate the compound interest

$$CI = A - P$$
$$= 532{,}400 - 400{,}000 = \text{Le } 132{,}400.00$$

Step 5. Write the answer for b.
The compound interest = Le $132{,}400.00$

3. A market trader deposited Le 250,000.00 into his account in a bank at a compound interest rate of 4% per annum. Interest is compounded half-yearly.
   a. How much does he have in his account after 3 years?
   b. How much compound interest did he earn?

Give your answers to the nearest cent.

Solutions:

Given: Le 250,000.00 deposited by market trader for 3 years at 4% per annum compounded half-yearly

a. $$A = P\left(1 + \frac{R}{100}\right)^n$$

Since the loan is compounded half-yearly,

- The rate R is equivalent to $\frac{4}{2}$% or 2% per half-year
- There are 6 half-yearly periods in 3 years, so $n = 6$

$$\therefore \quad A = 250{,}000\left(1 + \frac{2}{100}\right)^6 = 250{,}000 \times (1.02)^6$$
$$= \text{Le } 281{,}540.60.00$$

The amount at the end of 2 years = Le $281{,}540.60$ to the nearest cent.

b. $$CI = A - P$$
$$= 281{,}540.60 - 250{,}000 = \text{Le}31{,}540.60$$

The compound interest = Le $31{,}540.60$ to the nearest cent.

Practice

1. Find the interest on Le 250,000.00 which was deposited in a bank for 4 years at a compound interest rate of 5% per annum.

2. Find the interest on a loan of Le 300,000.00 for 5 years at a compound interest rate of 3% per annum. What additional interest is earned using compound as compared to simple interest rate?

3. Abu borrowed Le 500,000.00 for 3 years at 8% compound interest rate.
    a. What was the amount at the end of the 3 years?
    b. How much was the compound interest?

4. A market trader deposited Le 450,000.00 into his account in a bank at a compound interest rate of 3% per annum. If interest is compounded half-yearly,
    a. How much does he have in his account after 2 years?
    b. How much compound interest did he earn?

Lesson Title: Profit and Loss – Part 1	**Theme:** Numbers and Numeration
Lesson Number: PHM3-L065	**Class:** SSS 3
Learning Outcome By the end of the lesson, you will be able to calculate profit and loss on transactions by applying percentage.	

Overview

An item is sold at a profit when the selling price is greater than the cost price of the item. If, however the cost price of the item is greater than the selling price, the item is sold at a loss. The profit or loss is calculated by taking the difference between the cost price (*CP)* and selling price (*SP).*

Note that as difference is always positive,

$$\text{profit} = SP - CP$$

$$\text{loss} = CP - SP$$

Percentage profit or loss based on the cost price is given by:

$$\text{Percentage profit } = \frac{SP-CP}{CP} \times 100$$

Percentage loss based on the cost price is given by:

$$\text{Percentage loss } = \frac{CP-SP}{CP} \times 100$$

Solved Examples

1. John buys a set of bicycle pumps for Le 40,000 and sells them for Le 50,000. Find his percentage profit.

   Solution:

   Step 1. Assess and extract the given information from the problem.

   Given: John buys a set of bicycle pumps for Le 40,000.00 and sells them for Le 50,000.00.

   Step 2. Calculate the percentage profit.

   $$\begin{aligned} \textbf{\% profit} &= \frac{SP-CP}{CP} \times 100 \\ &= \frac{50,000-40,000}{40,000} \times 100 \\ &= \frac{10,000}{40,000} \times 100 \\ &= 25\% \end{aligned}$$

   Step 3. Write the answer.

   John made a 25% profit.

2. A man bought a car for Le 15,000,000.00. He later sold it for Le 12,000,000.00. What was his percentage loss on the sale of the car?

   Solution:

   Given: cost price of Le 15,000,000.00, selling price of Le 12,000,000.00

$$\text{percentage loss} = \frac{SP-CP}{CP}\times 100$$

$$\text{percentage loss} = \frac{15{,}000{,}000-12{,}000{,}000}{15{,}000{,}000}\times 100$$

$$= \frac{3{,}000{,}000}{15{,}000{,}000}\times 100$$

$$= 20\%$$

The percentage loss was 20%.

3. Akin bought a television set at the second-hand shop. He sold it for Le 2,500,000.00. If he made a profit of 25%, what price did he pay for the television?

Solution:

Given: television set bought by Akin, sold for Le 2,500,000.00, percentage profit 25%

Method 1. Use the formula for percentage profit.

$$\text{percentage profit} = \frac{SP-CP}{CP}\times 100$$

$$25 = \frac{2{,}500{,}000-CP}{CP}\times 100$$

$$25 = \frac{(2{,}500{,}000-CP)\times 100}{CP}$$

Multiply throughout by the cost price, CP

$$25CP = (2{,}500{,}000 - CP)\times 100$$

$$\frac{25}{100}CP = 2{,}500{,}000 - CP$$

$$0.25CP + CP = 2{,}500{,}000$$

$$1.25CP = 2{,}500.000$$

$$CP = \frac{2{,}500{,}000}{1.25}$$

$$= \text{Le } 2{,}000{,}000.00$$

Method 2. Use a multiplier

$$SP = CP + \frac{25}{100}CP$$

since Akin made a profit, we add the percentage profit to 100% of the cost price

$$= CP\left(1+\frac{25}{100}\right)$$

$$\text{multiplier} = 1+\frac{25}{100} = 1.25$$

$$SP = 1.25\times CP$$

$$2{,}500{,}000 = 1.25CP$$

$$CP = \frac{2{,}500{,}000}{1.25}$$

$$= \text{Le } 2{,}000{,}000.00$$

Akin bought the television set for Le 2,000,000.00.

Practice

1. A house bought for Le 250,000,000.00 was later sold for Le 230,000,000.00. Find the percentage loss.

2. A trader makes a loss of 15% when selling an article he bought for Le 35,000.00. Find the selling price of the article.

3. A woman sold a television at a profit of $22\frac{1}{2}\%$. If the selling price was Le 450,000.00, find out the cost price of the television.

4. A bicycle was sold for Le 400,000.00 at a loss of 25%. Find out the cost price.

5. A factory produces water tanks. The cost of making a tank is Le 1,500,000.00. If they want to make a profit of 10%, at what price should the factory sell a tank?

6. A trader bought mangoes for Le 10,000.00 per dozen. He sold them at 1 for Le 1,500.00. Calculate his percentage profit.

7. A trader bought a radio for Le 180,000.00. If he made a loss of 4.5%, what was the selling price for the radio?

Lesson Title: Profit and loss – Part 2	Theme: Numbers and Numeration
Lesson Number: PHM3-L066	**Class:** SSS 3
Learning Outcome By the end of the lesson, you will be able to calculate profit and loss on transactions by applying percentage.	

Overview

This lesson focuses on more complex profit and loss questions.

Solved Examples

1. A shop imported frozen chicken at a cost of Le 7,500,000.00. They paid import duty of 10% of the cost. They also paid a sales tax of 15% of the total cost of the goods including the import duty. If they sold the chicken for Le 11,000,000.00, calculate the percentage profit to the nearest whole number.

   Solution:

   Step 1. Assess and extract the given information from the problem.
   Given: frozen chickens imported for Le7,500,000.00, sold for Le11,000,000.00, import duty $= 10\%$, sales tax $= 15\%$

   Step 2. Calculate the cost price.

   cost price including import duty $= 7{,}500{,}000\left(1+\frac{10}{100}\right)$
   $= 7{,}500{,}000 \times 1.1 \quad = \quad$ Le $8{,}250{,}000.00$

   cost price including sales tax $= 8{,}250{,}000\left(1+\frac{15}{100}\right)$
   $= 8{,}250{,}000 \times 1.15 \quad = \quad$ Le $9{,}487{,}500.00$

   Step 3. Calculate the percentage profit

   percentage profit $= \frac{SP-CP}{CP} \times 100$
   percentage profit $= \frac{11{,}000{,}000-9{,}487{,}500}{9{,}487{,}500} \times 100$
   $= \frac{1{,}512{,}500}{9{,}487{,}500} \times 100$
   $= 16\%$

   Step 4. Write the answer.
   The percentage profit is 16%.

2. A fishmonger bought m fishes for Le 480,000.00. She found that 4 of them were rotten. She then sold all the remaining fishes. The selling price of one fish was Le 10,000.00 more than the cost price. Find in terms of m:
   a. The cost price of one fish.
   b. The number of fishes that she sold.
   c. The selling price of one fish.
   d. An expression for the total sum that she received from the sale.

If she made a profit of Le 120,000.00 from the sales, find:

e. The number of fishes she originally bought.

f. The cost price of one fish.

Solution:

Given: fishmonger bought m fish for Le 480,000.00, number rotten $= 4$, selling price = Le 10,000.00 more than the cost price

a. Let cost of one fish $= y$

$$y = \frac{480,000}{m}$$

b. number of fishes sold $= m - 4$ since 4 of the m fishes were rotten

c. selling price of one fish $= y + 10,000$

$$= \frac{480,000}{m} + 10,000$$

d. total sum from sales $= (m-4) \times \left(\frac{480,000}{m} + 10,000\right)$ (1)

e. profit from sales = Le120,000.00

total sum from sales $= 480,000 + 120,000 = 600,000$ (2)

Equation (1) = Equation (2)

$$(m-4) \times \left(\frac{480,000}{m} + 10,000\right) = 600,000$$

$$480,000 + 10,000m - \frac{1,920,000}{m} - 40,000 = 600,000$$

$$10,000m - \frac{1,920,000}{m} = 600,000 - 440000$$

$$10,000m - \frac{1920000}{m} = 160,000$$

Multiply throughout by m

$$10,000m^2 - 1,920,000 = 160,000m$$

Divide throughout by 10,000

$$m^2 - 192 = 16m$$

$$m^2 - 16m - 192 = 0$$

$$(m-16)(m+12) = 0$$

So, $(m-16) = 0 \implies m = 16$

$(m+12) = 0 \implies m = -12$

We ignore $m = -12$ as quantities cannot be negative

$\therefore$ number of fish bought $= 16$

f. cost of one fish $= \frac{480,000}{m}$

$$= \frac{480,000}{16} = \text{Le } 30,000.00$$

The cost of one fish is Le $30,000.00$

3. A trader bought 10 boxes of fruit at Le 20,000.00 each. She sold 4 boxes for Le 25,000.00 each, 3 boxes for Le 30,000.00 and the remainder for Le 18,000.00 each.

   a. How much profit or loss did the trader make on the boxes of fruit?

b. What was the average selling price per box?

Solution:

Given: 10 boxes of fruit at Le 20,000.00 each, sold 4 boxes for Le 25,000.00 each, 3 boxes for Le 30,000.00 and the remainder for Le 18,000.00 each

a. total cost $= 10 \times 20{,}000 =$ Le $200{,}000.00$

number of boxes sold for Le $18{,}000 = 10 - (4 + 3)) = 10 - 7 = 3$

total income $= 4(25{,}000) + 3(30{,}000) + 3(18{,}000) =$ Le $244{,}000.00$

profit $= 244{,}000 - 200{,}000 =$ Le $44{,}000.00$

The trader made Le $44{,}000.00$ profit.

b. average selling price $= \frac{244{,}000}{10} =$ Le $24{,}400.00$

The average selling price per box $=$ Le $24{,}400.00$.

Practice

1. A bookshop had 650 copies of a book for sale. The books were marked at Le 7,500.00 per copy in order to make a profit of 30%. A bookseller bought 300 copies at a 5% discount. If the remaining copies sold at Le 7,500.00 each, calculate the percentage profit the bookshop will make on the whole.
2. A woman buys eggs at Le 75,000 per carton of 60 eggs. He finds out that 15% are broken but sells the rest at Le 1,500.00 each. Find the percentage profit.
3. A man sold an article for Le 900,000.00 and made a profit of 20%. Find:
   a. The cost price.
   b. The profit.
4. A television set was sold for Le 300,000.00 at a loss of 25%. Find:
   a. The cost price.
   b. If a similar television set was sold at Le 345,000.00 at a profit of 15%. Find the cost price.

Lesson Title: Hire purchase	Theme: Numbers and Numeration
Lesson Number: PHM3-L067	**Class:** SSS 3
Learning Outcome By the end of the lesson, you will be able to calculate hire purchase based on percentages.	

Overview

There are instances when an item is bought and the full amount is paid for in regular instalments over several months or years. Since the item is being paid for over time, it usually costs more than the cash price when bought outright. This is because interest is usually added to the price of the item being sold.

In many cases, a deposit is paid for the item so that the buyer can make use of it right away. However, the item does not belong to the buyer until it has been completely paid for. The interest charged can be calculated using the simple interest rate based on the length of the loan. However, more complicated formulas are used to calculate the interest on hire purchase loans.

We will use the average time for the loan in the simple interest formula as it gives a good approximation for the actual interest charged.

Solved Examples

1. Mr. Kargbo wants to buy a car on sale at Le 25,000,000.00 cash. He paid Le 5,000,000.00 deposit and 15% simple interest charged on the remainder for 2 years. How much interest did he pay?

   Solution:

   Step 1. Assess and extract the given information from the problem.
   Given: Le 25,000,000.00, simple interest rate 15% per annum for 2 years.

   Step 2. Calculate the interest.

   remainder to be paid $= 25{,}000{,}000 - 5{,}000{,}000 = 20{,}000{,}000$

   $$I = \frac{PRT}{100}$$

   $$I = \frac{20{,}000{,}000 \times 15 \times 2}{100}$$ use $T =$ length of loan

   $=$ Le $6{,}000{,}000.00$

   Step 3. Write the answer.
   The interest paid by Mr. Kargbo is Le $6{,}000{,}000.00$.

2. A retailer offers the following hire purchase terms on generators:
   Deposit 30% of the cash price, then 4 monthly instalments charged at a simple interest rate of 20% on the remainder. If the cash price is Le 2,250,000.00, find:

a. The remainder on which interest is charged.
b. The monthly instalments.

Solution:

Given: deposit 30% of the cash price, 4 monthly instalments charged at a simple interest rate of 20% on the remainder, cash price is Le 2,250,000.00

i. remainder $= \left(1-\frac{30}{100}\right)\times 2{,}250{,}000 =$ Le $1{,}575{,}000.00$

ii. average time, $T = \frac{1+4}{2}$ average time is used as explained before

$= \frac{5}{2} =$ 2.5 months $= \frac{2.5}{12}$ years

$I = \frac{PRT}{100} = \frac{1{,}575{,}000\times 20\times 2.5}{100\times 12}$

$=$ Le $65{,}625.00$

total cost over 4 months $= 1{,}575{,}000 + 65{,}625$

monthly instalment $= \frac{1{,}640{,}625}{4}$

$=$ Le $410{,}156.25$

Interest is charged on Le $1{,}575{,}000.00$ with 4 monthly instalments of Le $410{,}156.25$.

3. Mrs. Mansaray bought an oven on hire purchase for Le 1,687,500.00. She paid 12.5% more than if she had paid cash for the oven. If she made an initial deposit of 20% of the cash price and then paid the rest in 6 monthly instalments, find:
   c. The initial deposit.
   d. The amount of each instalment.
   e. The approximate rate of interest to 1 decimal place.

Solutions:

Given: cost of oven = Le 1,687,500.00 = 12.5% more than cash price, initial deposit = 20% of cash price, rest in 6 monthly instalments

a. $\left(1+\frac{12.5}{100}\right)\times$ cash price of oven $= 1{,}687{,}500$

$1.125\times$ cash price of oven $= 1{,}687{,}500$

cash price of oven $= \frac{1{,}687{,}500}{1.125} =$ Le $1{,}500{,}000.00$

initial deposit $= \frac{20}{100}\times 1{,}500{,}000 =$ Le $300{,}000.00$

b. remainder to be paid $= 1{,}500{,}000 - 300{,}000 =$ Le $1{,}200{,}000.00$

amount per instalment $= \frac{1{,}200{,}000}{6} =$ Le $200{,}000.00$

c. average time, $T = \frac{1+6}{2}$

$= \frac{7}{2} =$ 3.5 months $= \frac{3.5}{12}$ years

interest, $I = 1{,}687{,}500 - 1{,}500{,}000 =$ Le $187{,}500.00$

$R = \frac{I\times 100}{PT}$

$R = \frac{187{,}500\times 100\times 12}{1{,}200{,}000\times 3.5}$

$= 53.57\%$

Mrs. Mansaray paid an initial deposit of Le $300{,}000.00$; monthly instalment is Le $200{,}000.00$ and approximate interest rate is 53.6% to 1 d.p.

Practice

1. Amadu wants to buy a motorbike on sale at Le 4,500,000.00 cash. He paid a Le 1,500,000.00 deposit and 12% simple interest charged on the remainder for 3 years. How much interest did he pay?

2. A merchant offers the following hire purchase terms on printing machines: Deposit 40% of the cash price, then 6 monthly instalments charged at a simple interest rate of 30% on the remainder. If the cash price is Le 12,000,000.00, find:
   a. The remainder on which interest is charged.
   b. The monthly instalments.

3. Jane wants to buy a sewing machine on sale at Le 600,000.00 cash. He paid a Le 250,000.00 deposit and 15% simple interest charged on the remainder for 12 months. How much interest did he pay?

4. Adama put a vehicle for sale at Le 8,900,000.00. She agreed to sell it to Josephine under the following hire purchase terms: An initial payment of 20% of the price and the balance paid at 15% simple interest per annum in twelve monthly equal instalments. Calculate:
   a. The initial deposit paid by Josephine for the car.
   b. The amount paid every month.

Lesson Title: Discount	**Theme:** Numbers and Numeration
Lesson Number: PHM3-L068	**Class:** SSS 3
Learning Outcome By the end of the lesson, you will be able to calculate discount on a transaction by applying percentage.	

Overview

Discount is given in shops when customers buy in bulk or when there is a special offer. The discount is usually given as a percentage of the original price.

The original price is 100% or 1 $\left(\frac{100}{100}\right)$.

We use a multiplier which is given by $1 - \frac{R}{100}$ where R is the percentage discount.

Solved Examples

1. A gas cooker costs Le 1,250,000.00. If the shop offers a customer a 20% discount, how much will the customer pay for the cooker?
   Solution:
   Step 1. Assess and extract the given information from the problem.
   Given: cost for gas cooker is Le 1,250,000.00 with a 20% discount offer.
   Step 2. Calculate how much the customer pays.
   multiplier $= 1 - \frac{20}{100} = 1 - 0.2$
   $= 0.8$
   amount paid $= 0.8 \times 1{,}250{,}000 =$ Le $1{,}000{,}000.00$
   Step 3. Write the answer.
   The customer pays Le $1{,}000{,}000.00$

2. What percentage discount was given on an item reduced from Le 250,000.00 to Le 212,500.00?
   Solution:
   Given: cost price Le 250,000.00, discounted price Le 212,500.00
   discount $= 250{,}000 - 212{,}500 =$ Le $37{,}500.00$
   percentage discount $= \frac{37500}{250{,}000} \times 100$
   $= 15\%$
   The percentage discount is 15%.

3. A motorbike costs Le 4,500,000.00. The shop gives a discount of $33\frac{1}{3}\%$ for cash.

a. How much will a buyer save by paying cash for the motorbike?
b. How much will the buyer pay for the motor bike?

Solution:

Given: cost price Le 4,500,000.00, percentage discount $33\frac{1}{3}\%$

i. percentage discount $= 33\frac{1}{3}\%$

$= \frac{1}{3}$ since $33\frac{1}{3}\% = \frac{100}{3}\% = \frac{\frac{100}{3}}{100} = \frac{100}{3\times100} = \frac{1}{3}$

discount $= \frac{1}{3} \times 4{,}500{,}000$

$=$ Le $1{,}500{,}000.00$

ii. amount paid $= 4{,}500{,}000 - 1{,}500{,}000$

$=$ Le $300{,}000.00$

The buyer pays Le$300{,}000.00$.

4. A school buys exercise books from a supplier. He gives the school a 15% discount if they buy more than 500 books and a 20% discount for buying over 1,000 books. If each book costs Le 1,500.00, how much will they save if they buy:
   a. 750 books b. 1,250 books?

Solutions:

Given: cost per book is Le 1,500.00, 15% discount for more than 500 books, 20% discount for more than 1,000 books.

a. 750 books: discount per book $= \frac{15}{100} \times 1{,}500$ $=$ Le 225.00

amount saved $= 225 \times 750$

$=$ Le $168{,}750.00$

ii. 1,250 books discount per book $= \frac{20}{100} \times 1{,}500$ $=$ Le 300.00

amount saved $= 300 \times 1{,}250$

$=$ Le $375{,}000.00$

The school saved Le $168{,}750.00$ when they bought 750 books and Le $375{,}000.00$ when they bought 1,250 books.

5. A retailer discounted her prices by 15% for a month. She then gave a further 10% off the discounted price.
   a. How much will an item originally costing Le 48,000.00 now cost?
   b. How much percentage profit will she lose by selling at this price?

Solutions:

Given: original discount is 15%, further 10% additional discount, cost of item is Le 48,000.00

a. multiplier $= 1 - \frac{15}{100}$ $=$ $1 - 0.15$

$= 0.85$

discounted price $= 0.85 \times 48{,}000$ $=$ Le $40{,}800.00$

multiplier $= 1 - \frac{10}{100} = 1 - 0.1$

$= 0.9$

new price $= 0.9 \times 40{,}800$

$=$ Le $36{,}720.00$

b. % profit lost $= \frac{48{,}000 - 36{,}720}{48000} \times 100$

$= \frac{11{,}280}{48{,}000} \times 100$

$= 23.5\%$

Practice

1. An electric fan costs Le 550,000.00. If the shop offers a customer 25% discount, how much will the customer pay for the electric fan?

2. What percentage discount was given on an item reduced from Le 800,000.00 to Le 650,000.00?

3. A computer costs Le 6,000,000.00. The shop gives a discount of $26\frac{1}{2}\%$ for cash.
   a. How much will the buyer pay for the computer?
   b. How much will a buyer save by paying cash for the computer?

4. A retailer discounted his prices by 8% for a month. He then gave a further 5% off the discounted price.
   a. How much will an item originally costing Le 380,000.00 now cost?
   b. How much will the retailer lose by selling at this price?

5. A mobile phone costs Le 880,000.00. If the shop offers a customer 30% discount, how much will the customer pay for the mobile phone?

6. What percentage discount was given on an item reduced from Le 350,000.00 to Le 200,000.00? Give your answer to 3 significant figures.

Lesson Title: Depreciation	**Theme:** Numbers and Numeration
Lesson Number: PHM3-L069	**Class:** SSS 3

Learning Outcome
By the end of the lesson, you will be able to calculate depreciation using percentages.

Overview

Many goods lose their value over time as they get older and are no longer in prime condition. Examples are cars, computers, mobile phones and most electrical appliances.

This decrease in value is called **depreciation**.

In a previous lesson, we looked at calculating compound interest using a formula:

$$A = P\left(1 + \frac{R}{100}\right)^n)$$

We can use a similar formula to calculate depreciation.

With compound interest the value appreciates or increases over time, but with depreciation, it depreciates or decreases over time. For depreciation, the percentage rate will be subtracted.

The value at the end of a particular time period is given by:

$$V = P\left(1 - \frac{R}{100}\right)^n$$

where V = Value at the end of the period
R = rate of depreciation
P = Original price
n = Period

The rate of depreciation can be found using the formula:

$$R = \frac{P-V}{P} \times 100$$

Solved Examples

1. A car costs Le 25,000,000.00. It depreciates at 20% per annum. Find its value after:
   a. 1 year b. 3 years

Solutions:

Step 1. Assess and extract the given information from the problem.
Given: original cost of car = Le 25,000,000.00, depreciates at 20% per annum.

Step 2. Calculate the value at the end of each period.

$$V = P\left(1 - \frac{R}{100}\right)^n$$

1 year: $V = 25{,}000{,}000\left(1 - \frac{20}{100}\right)^1$
$= \text{Le } 20{,}000{,}000.00$

3 years: $V = 25{,}000{,}000\left(1 - \frac{20}{100}\right)^3$

$= \text{Le } 12{,}800{,}000.00$

2. A motor bike costs Le 4,500,000.00. Its value depreciates by 18% the first year and 15% the second and subsequent years.
   a. What is its value at the end of 5 years? Give your answer to 2 decimal places.
   b. What was the average rate of depreciation over the 5 years?

Solutions:

Given: motor bike costs Le 4,500,000.00, depreciates 18% the first year and 15% the second and subsequent years

$$V = P\left(1 - \frac{R}{100}\right)^n$$

a. After 1 year, $V = 4{,}500{,}000\left(1 - \frac{18}{100}\right)^1 = \text{Le } 3{,}690{,}000.00$

After 4 more years, $V = 3{,}690{,}000\left(1 - \frac{15}{100}\right)^4 = \text{Le } 1{,}926{,}203.06$

b. average rate $= \frac{18+15}{2} = 16.5\%$

After 5 years, the motor bike is worth Le $1{,}926{,}203.06$.
The average rate of depreciation is 16.5%.

3. A gas cooker depreciates at a rate of 15% per annum. If its value after 2 years is Le 614,125.00, what was its original price?

Solution:

Given: gas cooker depreciates 15% per annum, value after 2 years is Le 614,125.00

$$V = P\left(1 - \frac{R}{100}\right)^n$$

$$614{,}125 = P\left(1 - \frac{15}{100}\right)^2$$

$$= P \times 0.85^2$$

$$614{,}125 = 0.7225P$$

$$P = \frac{614{,}125}{0.7225}$$

$$= \text{Le } 850{,}000.00$$

The original price of the gas cooker was Le $850{,}000.00$.

4. A computer costs Le 2,500,000.00. Its value depreciates by 20% the first year, 15% the second year and 12% the third year.
   a. What is its value at the end of the third year?
   b. If the owner decides to sell it at that price, what was the percentage loss on the original price to the nearest whole number?

Solutions:

Given: computer costs Le 2,500,000.00, depreciates by 20% in the first year, 15% the second year and 12% the third year

$$V = P\left(1 - \frac{R}{100}\right)^n$$

a. After 1 year, $V = 2{,}500{,}000\left(1 - \frac{20}{100}\right)^1 =$ Le $2{,}000{,}000.00$

After 2 years, $V = 2{,}000{,}000\left(1 - \frac{15}{100}\right)^1 =$ Le $1{,}700{,}000.00$

After 3 years, $V = 1{,}700{,}000\left(1 - \frac{12}{100}\right)^1 =$ Le $1{,}496{,}000.00$

The computer's value after 3 years is Le $1{,}496{,}000.00$

b. percentage loss $= \frac{CP - SP}{CP} \times 100$

percentage loss $= \frac{2{,}500{,}000 - 1{,}496{,}000}{2{,}500{,}000} \times 100$

$= \frac{1{,}004{,}000}{2{,}500{,}000} \times 100$

$= 40.16\%$

The computer sold at a loss of 40%.

Practice

1. The value of a tractor depreciates at 18% per annum. A man keeps the tractor for 4 years and then sells it. If the tractor initially costs Le 60,000,000 find:
   a. Its value after 4 years.
   b. The selling price as a percentage of the original value to 1 decimal place.
2. A refrigerator costs Le 1,500,000.00. Its value depreciates by 15% the first year, 12% the second year and 10% the third year.
   a. What is its value at the end of the third year?
   b. If the owner decides to sell it at that price, what was the percentage loss on the original price, to 2 significant figures?
3. A lorry costs Le 40,000,000.00. Its value depreciates by 20% the first year and 18% the second and subsequent years.
   a. What is its value at the end of 5 years?
   b. What was the average rate of depreciation over the 5 years?
4. An oven depreciates at a rate of 20% per annum. If its value after 4 years is Le 800,000.00, what was its original price?
5. A generator costs Le 5,000,000.00. It depreciates at 10% per annum. Find its value after:
   a. 1 year　　　　b. 4 years

Lesson Title: Financial partnerships	Theme: Numbers and Numeration
Lesson Number: PHM3-L070	**Class:** SSS 3

Learning Outcome
By the end of the lesson, you will be able to calculate financial partnership using percentage.

Overview

When 2 or more people come together and invest money for the purpose of providing goods or services at a profit, it is called a financial or business partnership. Partnerships are usually formed by professionals such as lawyers, doctors, architects and engineers who wish to pool their resources together. In many instances, the partners pay out profit in proportion to the money or capital invested.

Solved Examples

1. Two sisters, Kemi and Yemi, entered into a business partnerships. Kemi contributed Le 5,600,000.00 and Yemi contributed Le 2,400,000.00. At the end of the year, they made a profit of 70% of their total contribution. A total of 20% of the profit was reserved for re-investment and 2.5% of the remaining profit was paid into a trust fund for their children. If they shared the remaining profit in the ratio of their contributions, find:
   a. The amount reserved for re-investment.
   b. The amount paid into the trust fund.
   c. The amount received by each partner as her share of the profit.
   d. Each sister's share as a percentage of her contribution.

Solution:

Step 1. Assess and extract the given information from the problem.
Given: Kemi contributed Le 5,600,000.00, Yemi contributed Le 2,400,000.00, profit = 70% of their total contribution

Step 2. Calculate the amount reserved for re-investment.

a.
total contribution $= 5,600,000 + 2,400,000 =$ Le $8,000,000.00$
profit $= \frac{70}{100} \times 8,000,000 =$ Le $5,600,000.00$
amount reserved for re-investment $= \frac{20}{100} \times 5,600,000 =$ Le $1,120,000.00$

Step 3. Calculate the amount paid into a trust fund.

b.
remaining profit $= 5,600,000 - 1,120,000 =$ Le $4,480,000.00$
amount paid into a trust fund $= \frac{2.5}{100} \times 4,480,000 =$ Le $112,000.00$

Step 4. Calculate each partner's share.

c.
remaining profit $= 4,480,000 - 112,000 =$ Le $4,368,000.00$
ratio of contribution $= \frac{5,600,000}{2,400,000} = \frac{7}{3}$
total number of parts $= 7 + 3 = 10$

Kemi's share $= \frac{7}{10} \times 4{,}368{,}000$ = Le $3{,}057{,}600.00$

Yemi's share $= \frac{3}{10} \times 4{,}368{,}000$ = Le $1{,}310{,}400.00$

Step 5. Calculate each partner's percentage share.

d. percentage share for Kemi $= \frac{3{,}057{,}600}{5{,}600{,}000} \times 100 = 54.6\%$

percentage share for Femi $= \frac{1{,}310{,}400}{2{,}400{,}000} \times 100 = 54.6\%$

Step 6. Write the answers.

The amount reserved for re-investment is Le $1{,}120{,}000.00$.
The amount paid into a trust fund is Le $112{,}000.00$.
Kemi's share of the profit is Le $3{,}057{,}600.00$ which is 54.6% of her contribution.
Yemi's share of the profit is Le $1{,}310{,}400.00$ which is 54.6% of her contribution.

2. Mr. Koroma and Mr. Kamara entered into a financial partnership with a total capital of Le 45,000,000.00. They agreed to contribute the capital in the ratio 2 : 1 respectively. The profit was shared as follows: Mr. Koroma was paid 6% of the total profit for his services as a manager. Each partner was paid 4% of the capital he invested. The remainder of the profit was then shared in the ratio of the capital invested. If Mr. Koroma's share of the total profits was Le4,000,000.00, find:
   c. The total profit for the year to the nearest thousand Leones.
   d. Mr. Kamara's share of the total profits.

Solution:

Given: total capital invested of Le 45,000,000.00 by Mr. Koroma and Mr. Kamara in the ratio 2 : 1 respectively

a. total number of parts $= 2 + 1 = 3$

Mr. Koroma's contribution $= \frac{2}{3} \times 45{,}000{,}000$ = Le $30{,}000{,}000.00$

Mr. Kamara's contribution $= \frac{1}{3} \times 45{,}000{,}000$ = Le $15{,}000{,}000.00$

Let total profit $= x$

Mr. Koroma's share of profit

6% as manager $= \frac{6}{100} \times x = 0.06x$

4% of investment $= \frac{4}{100} \times 30{,}000{,}000$ = Le $1{,}200{,}000.00$

Mr. Kamara's share of profit

4% of investment $= \frac{4}{100} \times 15{,}000{,}000$ = Le $600{,}000.00$

profit shared so far $= 0.06x + 1{,}200{,}000 + 600{,}000$

$= 0.06x + 1{,}800{,}000$

remaining profit $= x - (0.06x + 1{,}800{,}000)$

$= 0.94x - 1{,}800{,}000$

The remaining profit is shared in the ratio of the partners' investments.

Mr. Koroma's share $= \frac{2}{3}(0.94x - 1{,}800{,}000) = 0.626x - 1{,}200{,}000$

Mr. Kamara's share $= \frac{1}{3}(0.94x - 1{,}800{,}000) = 0.313x - 600{,}000$

Mr. Koroma's share of the total profit $=$ Le $4{,}000{,}000.00$

$$4{,}000{,}000 = 0.06x + 1{,}200{,}000 + 0.626x - 1{,}200{,}000$$

$$4{,}000{,}000 = 0.686x$$

$$x = \frac{4{,}000{,}000}{0.686} = \text{Le } 5{,}830{,}903.79$$

The total profit to the nearest thousand Leones is Le $5{,}831{,}000.00$

b. Mr. Kamara's share of the total profit $= 5{,}831{,}000 - 4{,}000{,}000$
$=$ Le $1{,}831{,}000.00$

Practice

1. Umaru and Victor entered into a business partnership. The capital for the business is made of Le 1,875,000.00 from Umaru and Le 525,000.00 from Victor. They agreed to share the yearly profit in the following manner: Victor as Managing Director is paid Le 75,000.00 and additional 7.5% of the total profit. Each partner is paid a sum equal to 3% of the capital he invested. The remainder of the profit is shared between the partners in the ratio of their contribution to the capital. If the profit at the end of a certain year was Le 480,000.00, calculate the total amount each partner received from the profit.
2. Victoria and Hannah entered into a partnership with capital of Le 504,000.00 and Le 768,000.00 respectively. After three months, they were joined by Janet with capital of Le 648,000.00. It was agreed that the profit should be shared in proportion to their capital. During the first three months of the year, the business made a profit which was 24% of the working capital, and the remaining nine months the profit was 32% of the working capital.
    a. Find the amount received by each partner as a share of the profits for the year.
    b. Express Victoria's share of the profit as a percentage of her investment.
3. Shola and Ade entered into a business partnership in January 2015. The capital was Le 2,700,000 which they agreed to contribute in the ratio 2:1 respectively. The annual profit for 2015 was shared as follows: Shola was paid $5\frac{1}{2}\%$ of the total profit for his services as a manager. The remainder of the profit was then shared between them in the ratio of their contribution to the capital. If Shola received a sum of Le 685,000.00 out of the profit, calculate:
    a. The total profit for the year.
    b. Ade's share of the profit as a percentage of his initial contribution to the capital.
    c. If Ade had to pay tax at 30% on the amount he received, how much did he pay?

Lesson Title: Foreign exchange	**Theme:** Numbers and Numeration
Lesson Number: PHM3-L071	**Class:** SSS 3

Learning Outcome
By the end of the lesson, you will be able to convert one type of currency to another based on given rates using ratio and proportion.

Overview

Every country has its own currency which it uses for its money. The table on the right gives some countries and currencies. The exchange rate is the rate at which one unit of a particular currency is converted to another currency. There are usually two rates – the **buying** and the **selling** rate.

Country	Currency	Symbol
Ghana	Cedi	GH¢
Gambia	Dalasi	D
Germany	Euro	€
Great Britain	Pounds	£
Nigeria	Naira	₦
Sierra Leone	Leones	Le
United Sates	Dollars	$

The table shows the buying and selling rates in a bank for various currencies on a particular day. The bank buys from customers at the buying rate and sells at the selling rate. The selling rate is higher than the buying rate. This allows the bank to make a profit in trading in the currency.

Currency	Buying	Selling
€ 1.00	Le 8,600	Le 8,900
GH¢ 1.00	Le 1,500	Le 1,560
GMD 1.00	Le 150	Le 156
₦ 1.00	Le 20	Le20.80
£ 1.00	Le 9,500	Le 9,800
$ 1.00	Le 7,600	Le 7,900

Solved Examples

For each question, decide whether the bank is buying foreign currency from you or selling foreign currency to you. Then use the appropriate rate.

Where appropriate give your answer to 2 decimal places.

1. How much will Le 5,000,000.00 give you in the following currencies?
   Use the selling rate.
   a. US $ b. GB £ c. Gambia D

   Solution:

   a. **Step 1.** Assess and extract the given information from the problem.
   Given: Le5,000,000.00 to buy US $, GB £ and Gambia D

   Step 2. Use the unitary method and conversion rate to calculate the amount in the required currency.

   i.
$$\text{Le } 7{,}900.00 = \$\ 1.00$$
$$\text{Le } 1.00 = \$\ \frac{1.00}{7{,}900}$$
$$\text{Le } 5{,}000{,}000.00 = \$\ \frac{1.00}{7{,}900} \times 5{,}000{,}000$$

$= \$632.91$

ii. Le $9,800.00 = £\ 1.00$

Le $5,000,000.00 = £\ \frac{1.00}{9,800} \times 5,000,000$

$= £510.20$

iii. Le $156.00 =$ Gambian D1.00

Le $5,000,000.00 = \frac{1.00}{156} \times 5,000,000$

$=$ Gambian D$32,051.28$

Step 3. Write the answer.

Le $5,000,000.00$ buys $\$632.91$, £ 510.20 and Gambian D$32,051.28$.

2. How much in Leones will you get for the following amounts? Use the buying rate.
   a. GH¢5,000 b. €200.00 c. ₦1,500.00

Solution:

Given: sell various amounts of foreign currency (bank buys from customer)

a. GH¢ $1.00 =$ Le $1,500.00$

GH¢$5,000$: GH¢$5,000 = 5,000 \times 1,500 = Le\ 7,500,000.00$

b. €$1.00 =$ Le $8,600.00$

€200.00: €$200.00 = 200 \times 8,600 = Le\ 1,720,000.00$

₦$1.00 =$ Le 20.00

c. ₦1,500.00: ₦$1,500.00 = 1,500 \times 20 = Le\ 30,000.00$

The customer gets Le$7,500,000.00$ for GH¢$5,000$, $Le\ 1,720,000.00\ for$ €200.00 $and\ Le\ 30,000.00\ for$ ₦1500.00.

3. How much profit will a bank make if they buy, then sell $500.00?

Solution:

Given: bank buys then sells $500.00

$\$1.00 =$ Le $7,600.00$ buying rate

$\$500.00 = 7,600 \times 500 =$ Le$3,800,000.00$

$\$1.00 =$ Le $7,900.00$ selling rate

$\$500.00 = 7,900 \times 500 =$ Le $3,950,000.00$

profit $= 3,950,000 - 3,800,000$

$=$ Le$150,000.00$

The bank makes Le$150,000.00$ profit.

4. How much will you lose if you sell then buy $200.00?

Solution:

g. Given: sell then buy $\$200.00$ – bank buys from customer and sells to customer

$\$1.00 =$ Le$7,600.00$ buying rate

$\$200.00 = 7,600 \times 200 = Le\ 1,520,000.00$

$\$1.00 =$ Le$7,900.00$ selling rate

$\$200.00 = 7,900 \times 200 = Le\ 1,580,000.00$

$amount\ lost = 1,580,000 - 1,520,000 =$ Le $60,000.00$

You will lose Le$60,000.00$ from selling then buying $\$200.00$.

5. Mrs. Sesay buys goods from all over the world for her shop. She wants to order Le 10,000,000.00 worth of goods each from Nigeria, Great Britain and Germany. How much of each country's currency will she need?

   Solution:

   Given: Mrs. Sesay spends Le 10,000,000.00 to buy foreign currency

   $₦1.00 = \text{Le } 20.80$

   $\text{Le } 10,000,000.00 = \frac{10,000,000}{20.80} = ₦480,769.23$

   $£1.00 = \text{Le } 9,800.00$

   $Le\ 10,000,000.00 = \frac{10,000,000}{9,800} = £1,020.41$

   $€1.00 = \text{Le } 8,900.00$

   $\text{Le } 10,000,000.00 = \frac{10,000,000}{8,900} = €1,123.60$

   She needs ₦480,769.23 for Nigeria, £1,020.41 for GB and €1,123.60 for Germany.

Practice

1. How much will Le 7,000,000.00 give you in the following currencies? Use the selling rate.

   a. GH ¢ b. € c. US $

2. How much in Leones will you get for the following amounts? Use the buying rate

   a. Gambia D4,000.00 b. GB £300.00 c. ₦2,500.00

3. How much profit will a bank make if they buy then sell $800.00?

4. How much will you lose if you sell then buy GB £500.00?

5. A businessman wants to buy Le 15,000,000.00 worth of goods from the United States of America and Le 8,000,000.00 worth of goods from Ghana. How much of each country's currency will he need?

Lesson Title: Additional Practice with Applications of Percentage	**Theme:** Numbers and Numeration
Lesson Number: TGM3-L072	**Class:** SSS 3

Learning Outcomes
By the end of the lesson, you will be able to:
1. Calculate value added tax using percentages.
2. Calculate the amount to be paid for employer health insurance based on percentages.

Overview

Value Added Tax (VAT) is the tax charged on goods and services in some West African countries such as Ghana. It is similar to the Goods and Services (GST) tax charged in Sierra Leone.

If the VAT / GST is given as x%, then:

$$\text{VAT / GST} = \text{basic cost} \times \frac{x}{100}$$
$$\text{Cost of goods/services} = \text{basic cost} \times \left(1 + \frac{x}{100}\right)$$

Basic cost is **exclusive** of VAT / GST. This is the cost before the tax is added.

Some countries in West Africa also add a levy called the National Health Insurance Levy (NHIL) to goods and services to cover National Health Insurance. It is added to the VAT and charged on the basic cost of goods and services.

In the questions below, take VAT to also mean GST.

Solved Examples

1. An electric oven is sold at Le 1,250,000 + VAT. How much will the oven cost inclusive of VAT charged at 15%?

   Solution:

   Step 1. Assess and extract the given information from the problem.
   Given: electric oven sold at Le 1,250,000 + VAT, VAT charged at 15%

   Step 2. Calculate the cost of the oven.

   $$\text{cost of oven} = \text{basic cost} \times \left(1 + \frac{x}{100}\right)$$
   $$\text{cost of oven} = 1{,}250{,}000 \times \left(1 + \frac{15}{100}\right) = 1{,}250{,}000 \times 1.15$$
   $$= \text{Le } 1{,}437{,}500.00$$

   Step 3. Write the answer.
   The electric oven cost Le $1{,}437{,}500.00$.

2. After eating in a restaurant, a woman's bill came to Le 69,000.00 inclusive of VAT. If VAT is charged at 15%, how much did she pay for VAT?

   Solution:

   Given: woman's restaurant bill of Le 69,000.00, VAT charged at 15%

   $$\begin{aligned} \text{cost of service} &= \text{basic cost} \times \left(1 + \frac{x}{100}\right) \\ \text{Let basic cost} &= y \\ 69{,}000 &= y \times \left(1 + \frac{15}{100}\right) \\ 69{,}000 &= 1.15y \\ y &= \frac{69{,}000}{1.15} \quad = \text{Le } 60{,}000.00 \\ \text{VAT} &= 69{,}000 - 60{,}000 \quad (\text{or } 0.15 \times 60{,}000) \\ &= \text{Le } 9{,}000.00 \end{aligned}$$

   The woman paid VAT of Le 9,000.00.

3. Goods sold exclusive of VAT cost Le 600,000.00. When VAT is added they cost Le 705,000.00. How much is the VAT rate?

   Solution:

   Given: basic cost Le 600,000.00. VAT inclusive cost Le 705,000.00

   $$\begin{aligned} \text{VAT} &= 705{,}000 - 600{,}000 \quad = \text{Le } 105{,}000.00 \\ \text{VAT} &= \text{basic cost} \times \frac{x}{100} \\ 105{,}000 &= 600{,}000 \times \frac{x}{100} \\ 105{,}000 &= 6{,}000x \\ x &= \frac{105{,}000}{6{,}000} \quad = 17.5\% \end{aligned}$$

   The VAT rate is 17.5%.

4. The VAT rate of a country is $12\frac{1}{2}$% and the NHIL rate is 2%. The basic cost of an item was Le 675,000.00. Find the full cost of the item.

   Solution:

   Given: VAT rate is $12\frac{1}{2}$%, NHIL rate is 2%, basic cost of item was Le 675,000.00

   $$\begin{aligned} \text{total rate charged} &= 12\tfrac{1}{2} + 2 \quad = 14\tfrac{1}{2}\% \\ \text{cost of item} &= \text{basic cost} \times \left(1 + \frac{14.5}{100}\right) \\ &= 675{,}000 \times 1.145 \quad = \text{Le } 772{,}875.00 \end{aligned}$$

   The cost of the item was Le 772,875.00 inclusive of VAT and NHIL.

5. The VAT and NHIL marked inclusive price of a computer is Le 5,170,000.00. The VAT is charged at 15% and the NHIL is charged at 2.5%. Find:
   a. The cost of the computer (VAT and NHIL exclusive).
   b. The NHIL charged.
   c. The VAT charged.

   Solution:

   Given: VAT and NHIL inclusive price of computer is Le 5,170,000.00. VAT charged at 15%, NHIL charged at 2.5%.

   $$\text{total rate} = 15 + 2.5 \quad = 17.5$$

a. cost of computer $= \text{basic cost} \times \left(1 + \frac{17.5}{100}\right)$

$5{,}170{,}000 = \text{basic cost} \times (1.175)$

$\text{basic cost} = \frac{5{,}170{,}000}{1.175} = \text{Le } 4{,}400{,}000.00$

b. $\text{NHIL} = \text{basic cost} \times \frac{\text{NHIL rate}}{100}$

$= 4{,}400{,}000 \times \frac{2.5}{100} = \text{Le } 110{,}000.00$

c. $\text{VAT} = \text{basic cost} \times \frac{\text{VAT rate}}{100}$

$= 4{,}400{,}000 \times \frac{15}{100} = \text{Le } 660{,}000.00$

The basic cost of the computer is Le 4,400,000.00, NHIL is Le 110,000.00 and VAT is Le660,000.00.

Practice

1. A musical set was sold for Le 1,200,000.00 + VAT. How much will the set cost inclusive of VAT charged at 13%?

2. After eating in a restaurant, a man's bill came to Le 118,650.00 inclusive of VAT. If VAT is charged at 13%, how much did he pay for VAT?

3. Goods sold exclusive of VAT cost Le 900,000.00. When VAT is added they cost Le 1,000,000.00. How much is the VAT rate?

4. The VAT rate of a country is 11% and the NHIL rate is 3%. The basic cost of an item is Le 750,000.00. Find the full cost of the item.

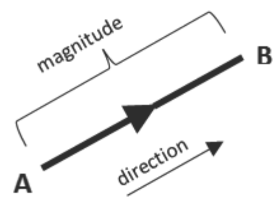
5. The VAT and NHIL marked inclusive price of a freezer is Le 4,060,000.00. The VAT is charged at 14% and the NHIL is charged at 2%. Find:
   a. The cost of the freezer (VAT and NHIL exclusive).
   b. The NHIL charged.
   c. The VAT charged.

Lesson Title: Introduction to vectors and scalars	**Theme:** Vectors and Transformations
Practice Activity: PHM3-L073	**Class:** SSS 3

Learning Outcomes
By the end of the lesson, you will be able to:
1. Define and describe vectors and scalars and their uses.
2. Use correct notation and representation for vectors.

Overview

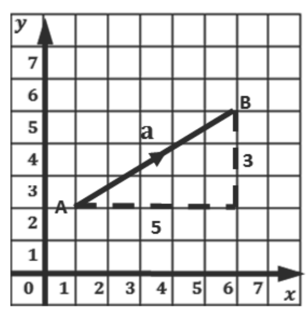
A **vector** is any quantity which has both magnitude and direction. Examples of vectors are displacement (translation), velocity, and force.
A **scalar** is any quantity which has only magnitude but no direction. Examples of scalars are distance, speed, and time.
Vectors are represented in various ways. The simplest representation is as a line segment with length equal to the magnitude of the vector and an arrow indicating its direction. The vector on the right shows a displacement of a point from position A to position B. It can be written as:



$$\overrightarrow{AB},\ \overline{AB},\ \mathbf{AB},\ \vec{a},\quad \bar{a},\quad \underline{a},\quad \mathbf{a}$$

Hand-written vectors can be represented using arrows, and over- or under-bars.
Vectors written in lowercase letters are called position vectors.
We will learn more about position vectors in a later lesson.

Vectors can be represented on a Cartesian plane as shown at right.
Consider the vector $\overrightarrow{AB}$: It can be written as a column matrix or column vector:



$$\overrightarrow{AB} = \mathbf{a} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

The vector is drawn by starting at point A, moving 5 units to the right and 3 units up.
In general, any vector $\overrightarrow{AB} = \begin{pmatrix} a \\ b \end{pmatrix}$ has 2 components: the horizontal component a measured along the x-axis, and the vertical component, b measured along the y-axis from point A to point B.
Any move to the left or downwards is movement in the negative direction.

There is one important exception to vectors having magnitude and direction. The **zero vector**, denoted by $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, is the vector of zero length or magnitude. It has no length, and does not point in any particular direction. We call it the zero vector since there is only one vector of zero length.

Solved Examples

1. The line segments shown represents column vectors $\overrightarrow{AB}$, $\overrightarrow{HI}$, $\overrightarrow{IJ}$ and $\overrightarrow{JK}$.

   Write these as vectors in the form $\begin{pmatrix} x \\ y \end{pmatrix}$.

   Solution:

   Step 1. Assess and extract the given information from the problem.

   Given: line segments $\overrightarrow{AB}$, $\overrightarrow{HI}$, $\overrightarrow{IJ}$ and $\overrightarrow{JK}$

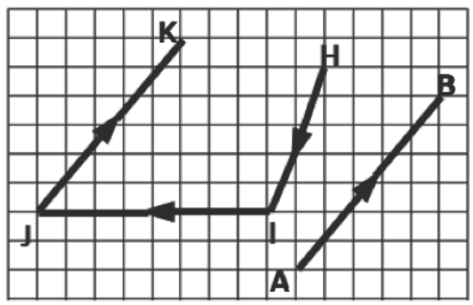
   Step 2. Write each line segment as a column vector.

$$\overrightarrow{AB} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

$$\overrightarrow{HI} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$$

$$\overrightarrow{IJ} = \begin{pmatrix} -8 \\ 0 \end{pmatrix}$$

$$\overrightarrow{JK} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

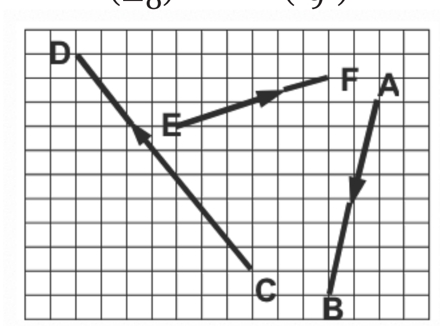


2. Draw line segments on a graph paper to represent the following column vectors:

$$\overrightarrow{AB} = \begin{pmatrix} -2 \\ -8 \end{pmatrix}, \overrightarrow{CD} = \begin{pmatrix} -7 \\ 9 \end{pmatrix}, \overrightarrow{EF} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

   Solution:

   Given: column vectors: $\overrightarrow{AB} = \begin{pmatrix} -2 \\ -8 \end{pmatrix}, \overrightarrow{CD} = \begin{pmatrix} -7 \\ 9 \end{pmatrix}, \overrightarrow{EF} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$



3. Draw line segments with respect to *O* to represent the following column vectors:

   a. $\overrightarrow{OA} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$

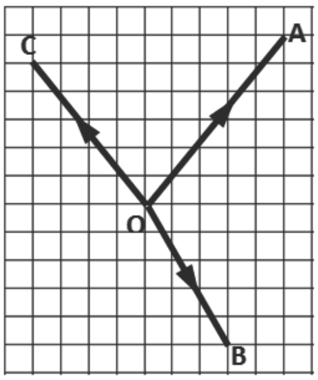
   b. Hence give the vector $\overrightarrow{AC}$ in the form $\begin{pmatrix} a \\ b \end{pmatrix}$.

   c. What is $\overrightarrow{CA}$?

   Solutions:

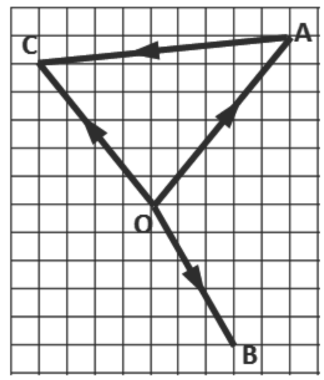
   Given: column vectors: $\overrightarrow{OA} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$

a.



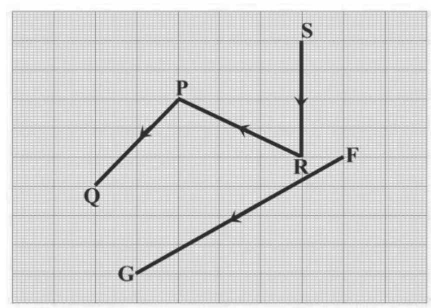
b. $\overrightarrow{AC} = \begin{pmatrix} -9 \\ -1 \end{pmatrix}$

c. $\overrightarrow{CA} = \begin{pmatrix} 9 \\ 1 \end{pmatrix}$



Practice

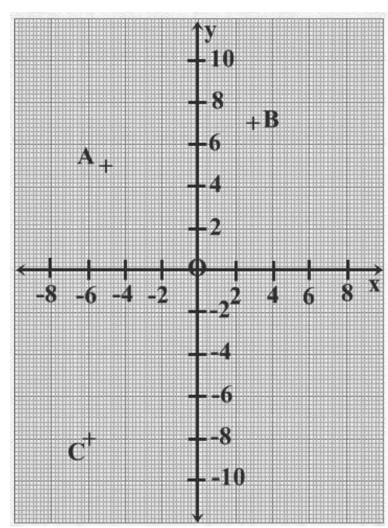
1. The line segments shown represent column vectors $\overrightarrow{PQ}$, $\overrightarrow{RP}$, $\overrightarrow{SR}$ and $\overrightarrow{FG}$. Write these as vectors in the form$\begin{pmatrix} x \\ y \end{pmatrix}$.



2. Draw line segments on a graph paper to represent the following column vectors:

$\overrightarrow{AB} = \begin{pmatrix} -4 \\ -6 \end{pmatrix}, \overrightarrow{CD} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}, \overrightarrow{EF} = \begin{pmatrix} -8 \\ 4 \end{pmatrix}$

3. Draw line segments with respect to *O* to represent the following column vectors:
   a. $\overrightarrow{OA} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} -5 \\ 6 \end{pmatrix}$
   b. Hence give the vector $\overrightarrow{AC}$ in the form $\begin{pmatrix} a \\ b \end{pmatrix}$.
   c. What is $\overrightarrow{CA}$?
4. The points $A(-5,5)$,$B(3,7)$ and $C(-6,-8)$ are shown on a grid. Write each of the following vectors in the form $\begin{pmatrix} a \\ b \end{pmatrix}$. (Hint: Draw the lines joining the points.)
   a. $\overrightarrow{AB}$ b. $\overrightarrow{BC}$ c. $\overrightarrow{AC}$

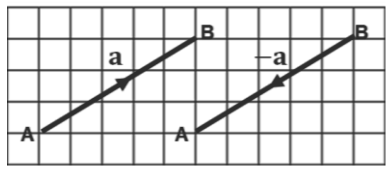


Lesson Title: Basic Vector Properties	**Theme:** Vectors and Transformations
Lesson Number: PHM3-L074	**Class:** SSS 3
Learning Outcome By the end of the lesson, you will be able identify and use basic properties of vectors.	

Overview

In this lesson, we will look at inverse, zero and equal vectors.

Inverse Vectors



Consider the vector $\overrightarrow{AB}$ shown in the diagram

$$\begin{aligned}\overrightarrow{AB} &= \mathbf{a}\\ &= \binom{5}{3}\end{aligned}$$

Then the vector:

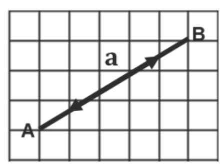
$$\begin{aligned}\overrightarrow{BA} &= -\overrightarrow{AB}\\ &= -\mathbf{a}\\ &= -\binom{5}{3}\\ &= \binom{-5}{-3}\end{aligned}$$

is the inverse vector of $\mathbf{a}$

The inverse vector, $-\mathbf{a}$ is equal in magnitude (or length) to $\mathbf{a}$, but opposite in direction. The direction changes from $A \to B$ to $B \to A$.

Every vector has an **inverse vector which is equal in magnitude and opposite in direction** to it.

Zero Vectors

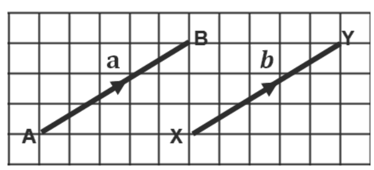


The diagram shows a vector and its inverse. When a point moves along a vector $\overrightarrow{AB}$ and then along its inverse $\overrightarrow{BA}$, the effect is that of zero movement. The end result is a vector of zero magnitude and no direction. This is an example of the zero vector.

Equal Vectors

We can also have vectors which are **equal in both magnitude and direction**.

We say that if $\mathbf{a} = \mathbf{b}$, then



- $|\mathbf{a}| = |\mathbf{b}|$
  That is the absolute value of $\mathbf{a}$ = absolute value of $\mathbf{b}$
- $\mathbf{a} \parallel \mathbf{b}$ ($\mathbf{a}$ is parallel to $\mathbf{b}$)

Also, if $\mathbf{a} = \binom{x_1}{y_1}$ and $\mathbf{b} = \binom{x_2}{y_2}$, then

- $x_1 = x_2$, and $y_1 = y_2$

Solved Examples

1. a. If $\overrightarrow{PQ} = \binom{3}{4}$, find $\overrightarrow{QP}$. b. If $\overrightarrow{RS} = \binom{6}{-2}$, find $\overrightarrow{SR}$.

Solution:

a. **Step 1.** Assess and extract the given information from the problem.

Given: vector $\overrightarrow{PQ} = \binom{3}{4}$

Step 2. Find the inverse vector.

$$\begin{aligned} \overrightarrow{QP} &= -\overrightarrow{PQ} \\ &= -\binom{3}{4} \\ &= \binom{-3}{-4} \end{aligned}$$

Step 3. Write the answer.

The inverse vector of $\overrightarrow{PQ} = \binom{3}{4}$ is $\overrightarrow{QP} = \binom{-3}{-4}$.

b. Given: vector $\overrightarrow{RS} = \binom{6}{-2}$, find $\overrightarrow{SR}$.

$$\begin{aligned} \overrightarrow{SR} &= -\overrightarrow{RS} \\ &= -\binom{6}{-2} \\ &= \binom{-6}{2} \end{aligned}$$

The inverse vector of $\overrightarrow{RS} = \binom{6}{-2}$ is $\overrightarrow{SR} = \binom{-6}{2}$

2. Find a and b given that $\mathbf{p} = \binom{a+6}{2-b}$, $\mathbf{q} = \binom{8}{4}$ and $\mathbf{p} = \mathbf{q}$.

Solution:

Given: $\mathbf{p} = \binom{a+6}{2-b}$, $\mathbf{q} = \binom{8}{4}$, $\mathbf{p} = \mathbf{q}$

$\mathbf{p} = \mathbf{q} \quad \Longrightarrow \quad \binom{a+6}{2-b} = \binom{8}{4}$

Equate corresponding components

$$\begin{aligned} a + 6 &= 8 \\ a &= 8 - 6 \\ &= 2 \\ 2 - b &= 4 \\ b &= 2 - 4 \\ &= -2 \end{aligned}$$

$\therefore \quad a = 2,\ b = -2$

3. Find x and y given that $\overrightarrow{OA} = \binom{6}{6}$ and $\overrightarrow{OB} = \overrightarrow{AO} = \binom{x+1}{y-3}$

Solution:

Given: that $\overrightarrow{OA} = \binom{6}{6}$ and $\overrightarrow{OB} = \overrightarrow{AO} = \binom{x+1}{y-3}$

$$\begin{aligned} \overrightarrow{OB} = \overrightarrow{AO} &= -\overrightarrow{OA} \\ &= -\binom{6}{6} \\ &= \binom{-6}{-6} \end{aligned}$$

$$\binom{x+1}{y-3} = \binom{-6}{-6}$$

$$x + 1 = -6$$

$$
\begin{aligned}
x &= -6 - 1 \\
&= -7 \\
y &= -6 + 3 \\
&= -3
\end{aligned}
$$

$\therefore \quad x = -7, y = -3$

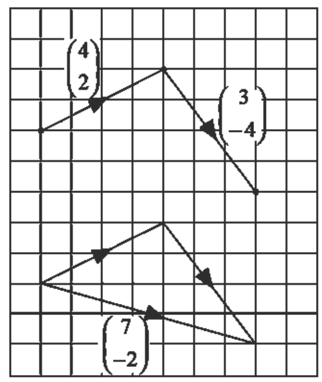
Practice

1. a. If $\overrightarrow{PQ} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, find $\overrightarrow{QP}$ b. If $\overrightarrow{RS} = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$, find $\overrightarrow{SR}$
   c. If $\overrightarrow{XY} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$**,** find$\overrightarrow{YX}$ d. If $\overrightarrow{AB} = \begin{pmatrix} -9 \\ -6 \end{pmatrix}$**,** find$\overrightarrow{BA}$
2. Find a and b given that $\mathbf{p} = \begin{pmatrix} a+7 \\ 3-b \end{pmatrix}$**,** $\mathbf{q} = \begin{pmatrix} 9 \\ 5 \end{pmatrix}$ and $\mathbf{p} = \mathbf{q}$
3. Find x and y given that $\mathbf{a} = \begin{pmatrix} 4x \\ x+y \end{pmatrix}$**,** $\mathbf{b} = \begin{pmatrix} 12 \\ 8 \end{pmatrix}$ and $\mathbf{a} = \mathbf{b}$
4. a. If $\overrightarrow{AB} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$**,** find $\overrightarrow{BA}$ b. If $\overrightarrow{XY} = \begin{pmatrix} -9 \\ 7 \end{pmatrix}$**,** find $\overrightarrow{YX}$
   c. If $\overrightarrow{RS} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, find $\overrightarrow{SR}$
5. Find a and b given that $\mathbf{p} = \begin{pmatrix} 3a+4 \\ -3 \end{pmatrix}$**,** $\mathbf{q} = \begin{pmatrix} 7 \\ b+4 \end{pmatrix}$ and $\mathbf{p} = \mathbf{q}$
6. Find x and y given that $\overrightarrow{OA} = \begin{pmatrix} 8 \\ 8 \end{pmatrix}$ and $\overrightarrow{OB} = \overrightarrow{AO} = \begin{pmatrix} \text{x}+1 \\ \text{y}-3 \end{pmatrix}$

Lesson Title: Addition and Subtraction of Vectors	Theme: Vectors and Transformations
Lesson Number: PHM3-L075	**Class:** SSS 3

Learning Outcome
By the end of the lesson, you will be able to add or subtract vectors based on information given.

Overview



Consider the vectors $\mathbf{a} = \binom{4}{2}$ and $\mathbf{b} = \binom{3}{-4}$ shown in the first diagram on the right.
The result of adding **a** and **b** is then shown further below.

$$\mathbf{a}+\mathbf{b} = \binom{4}{2}+\binom{3}{-4}$$
$$= \binom{7}{-2} \quad \text{from the diagram}$$

This is the same as adding the corresponding x and y components together

$$\mathbf{a}+\mathbf{b} = \binom{4+3}{2+(-4)}$$
$$= \binom{7}{-2} \quad \text{by calculation}$$

In general, if $\mathbf{a} = \binom{x_1}{y_1}$ and $\mathbf{b} = \binom{x_2}{y_2}$, then

$$\mathbf{a}+\mathbf{b} = \binom{x_1+x_2}{y_1+y_2}$$

Similarly,

$$\mathbf{a}-\mathbf{b} = \binom{x_1-x_2}{y_1-y_2}$$

Solved Examples

1. If **a**= $\binom{4}{7}$, **b**= $\binom{3}{-5}$ and **c**= $\binom{0}{4}$, find:

   a. **a + b** b. **b + c** c. **a − c**
   d. **a − b** e. **a + b− c**

   Solutions:

   Step 1. Assess and extract the given information from the problem.
   Given**: a**= $\binom{4}{7}$, **b**= $\binom{3}{-5}$, and **c**= $\binom{0}{4}$

   Step 2. Complete the vector addition/subtraction.

   Step 3. Write the answer.

   a. $$\mathbf{a}+\mathbf{b} = \binom{4}{7}+\binom{3}{-5} = \binom{4+3}{7+(-5)} = \binom{7}{2}$$

   b. $$\mathbf{b}+\mathbf{c} = \binom{3}{-5}+\binom{0}{4} = \binom{3+0}{-5+4} = \binom{3}{-1}$$

c. $\mathbf{a} - \mathbf{c} = \begin{pmatrix}4\\7\end{pmatrix} - \begin{pmatrix}0\\4\end{pmatrix} = \begin{pmatrix}4-0\\7-4\end{pmatrix} = \begin{pmatrix}4\\3\end{pmatrix}$

d. $\mathbf{a} - \mathbf{b} = \begin{pmatrix}4\\7\end{pmatrix} - \begin{pmatrix}3\\-5\end{pmatrix} = \begin{pmatrix}4-3\\7-(-5)\end{pmatrix} = \begin{pmatrix}1\\12\end{pmatrix}$

e. $\mathbf{a} + \mathbf{b} - \mathbf{c} = \begin{pmatrix}4\\7\end{pmatrix} + \begin{pmatrix}3\\-5\end{pmatrix} - \begin{pmatrix}0\\4\end{pmatrix} = \begin{pmatrix}4+3-0\\7-5-4\end{pmatrix} = \begin{pmatrix}7\\-2\end{pmatrix}$

2. If **a**= $\begin{pmatrix}4\\2\end{pmatrix}$, **b**= $\begin{pmatrix}3\\-1\end{pmatrix}$., and **c**= $\begin{pmatrix}-2\\1\end{pmatrix}$, solve the equations below to find the column vector **x**.
   a. $\mathbf{a} + \mathbf{x} = \mathbf{b}$ b. $\mathbf{x} - \mathbf{c} = \mathbf{a}$
   c. $\mathbf{x} + \mathbf{b} = \mathbf{c}$ d. $\mathbf{b} + \mathbf{x} = \mathbf{a}$

Solution:

Given: **a**= $\begin{pmatrix}4\\2\end{pmatrix}$, **b**= $\begin{pmatrix}3\\-1\end{pmatrix}$., and **c**= $\begin{pmatrix}-2\\1\end{pmatrix}$

a.
$$\begin{aligned}\mathbf{a} + \mathbf{x} &= \mathbf{b}\\ \begin{pmatrix}4\\2\end{pmatrix} + \mathbf{x} &= \begin{pmatrix}3\\-1\end{pmatrix}\\ \mathbf{x} &= \begin{pmatrix}3\\-1\end{pmatrix} - \begin{pmatrix}4\\2\end{pmatrix}\\ &= \begin{pmatrix}3-4\\-1-2\end{pmatrix}\\ &= \begin{pmatrix}-1\\-3\end{pmatrix}\end{aligned}$$

b.
$$\begin{aligned}\mathbf{x} - \mathbf{c} &= \mathbf{a}\\ \mathbf{x} - \begin{pmatrix}-2\\1\end{pmatrix} &= \begin{pmatrix}4\\2\end{pmatrix}\\ \mathbf{x} &= \begin{pmatrix}4\\2\end{pmatrix} + \begin{pmatrix}-2\\1\end{pmatrix}\\ &= \begin{pmatrix}4+(-2)\\2+1\end{pmatrix}\\ &= \begin{pmatrix}2\\3\end{pmatrix}\end{aligned}$$

c.
$$\begin{aligned}\mathbf{x} + \mathbf{b} &= \mathbf{c}\\ \mathbf{x} + \begin{pmatrix}3\\-1\end{pmatrix} &= \begin{pmatrix}-2\\1\end{pmatrix}\\ \mathbf{x} &= \begin{pmatrix}-2\\1\end{pmatrix} - \begin{pmatrix}3\\-1\end{pmatrix}\\ &= \begin{pmatrix}-2-3\\1-(-1)\end{pmatrix}\\ &= \begin{pmatrix}-5\\2\end{pmatrix}\end{aligned}$$

d.
$$\begin{aligned}\mathbf{b} + \mathbf{x} &= \mathbf{a}\\ \begin{pmatrix}3\\-1\end{pmatrix} + \mathbf{x} &= \begin{pmatrix}4\\2\end{pmatrix}\\ \mathbf{x} &= \begin{pmatrix}4\\2\end{pmatrix} - \begin{pmatrix}3\\-1\end{pmatrix}\\ &= \begin{pmatrix}4-3\\2-(-1)\end{pmatrix}\\ &= \begin{pmatrix}1\\3\end{pmatrix}\end{aligned}$$

Practice

1. If **a**= $\begin{pmatrix}3\\5\end{pmatrix}$, **b**= $\begin{pmatrix}4\\-6\end{pmatrix}$, and **c**= $\begin{pmatrix}0\\7\end{pmatrix}$, find:
   a. $\mathbf{a} + \mathbf{b}$ b. $\mathbf{b} + \mathbf{c}$ c. $\mathbf{a} - \mathbf{c}$
   d. $\mathbf{a} - \mathbf{b}$ e. $\mathbf{a} + \mathbf{b} - \mathbf{c}$
2. If **a**= $\begin{pmatrix}3\\2\end{pmatrix}$, **b**= $\begin{pmatrix}5\\-1\end{pmatrix}$, and **c**= $\begin{pmatrix}-4\\-6\end{pmatrix}$, find:
   a. $\mathbf{a} + \mathbf{b}$ b. $\mathbf{a} + \mathbf{c}$ c. $\mathbf{a} + \mathbf{c} + \mathbf{b}$ d. $\mathbf{a} + \mathbf{b} - \mathbf{c}$

3. If **p**= $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, **q**= $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$, and **r**= $\begin{pmatrix} 9 \\ 7 \end{pmatrix}$, find:

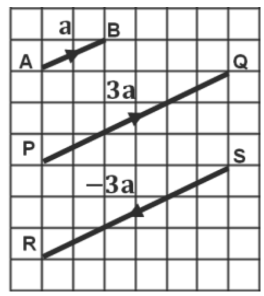
   a. $\mathbf{p} + \mathbf{r}$ b. $\mathbf{r} - \mathbf{p}$ c. $\mathbf{r} - \mathbf{p} + \mathbf{q}$ d. $\mathbf{p} - \mathbf{r} - \mathbf{q}$

4. If **a**= $\begin{pmatrix} 6 \\ 4 \end{pmatrix}$, **b**= $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$, and **c**= $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, solve the equations below to find the column vector **x**.

   a. $\mathbf{a} + \mathbf{x} = \mathbf{b}$ b. $\mathbf{x} - \mathbf{c} = \mathbf{a}$ c. $\mathbf{x} + \mathbf{b} = \mathbf{c}$ d. $\mathbf{b} + \mathbf{x} = \mathbf{a}$

Lesson Title: Multiplication of vectors by scalars	**Theme:** Vectors and Transformations
Lesson Number: PHM3-L076	**Class:** SSS 3
Learning Outcome By the end of the lesson, you will be able to multiply a vector by a scalar to find the scalar multiple.	

Overview



Consider the vector $\overrightarrow{AB} = \mathbf{a} = \binom{2}{1}$ shown in the diagram at right, It can be seen from the diagram that:

$\overrightarrow{PQ} = \binom{6}{3}$

$= 3\binom{2}{1}$

$= 3\mathbf{a}$ 3 times vector **a** in the same direction

$\overrightarrow{RS} = \binom{-6}{-3}$

$= -3\binom{2}{1}$

$= -3\mathbf{a}$ 3 times vector **a** in the opposite direction

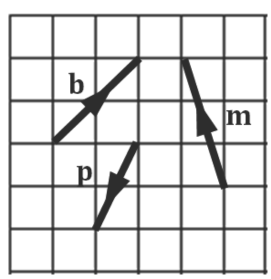
In general, if $\mathbf{a} = \binom{x}{y}$ then

$k\mathbf{a} = \binom{kx}{ky}$ where k is a scalar or number which can be a positive or negative whole number or fraction

Multiplication of a vector by a scalar is called scalar multiplication. Each component of the vector is multiplied by the scalar amount. It has the effect of "scaling" the vector up or down by the factor of the scalar quantity.

- If the scalar is positive, the resulting vector is in the same direction as the original vector.
- If the scalar is negative, the resulting vector is in the opposite direction as the original vector.

Solved Examples

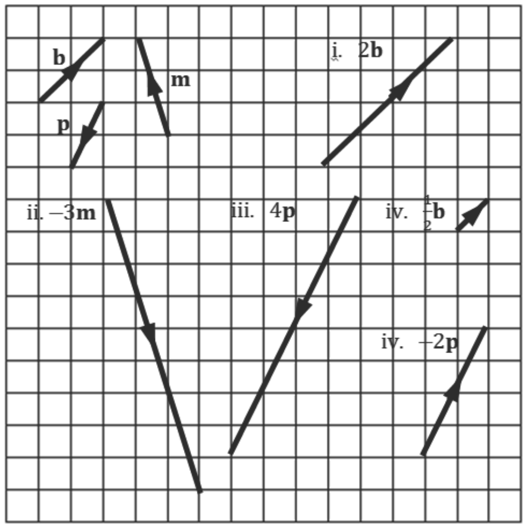


1. Using the vectors **b**, **p** and **m** from the grid shown, draw the following vectors on a grid. Label each vector and show its direction with an arrow.
   a. 2**b** b. –3**m** c. 4**p**
   d. $\frac{1}{2}$**b** e. –2**p**

   Solution:

   Step 1. Assess and extract the given information from the problem.
   Given: vectors **b**, **p** and **m** (shown below)

Step 2. Draw and label each vector on the grid showing its direction with an arrow.



2. If **a**= $\binom{2}{1}$, **b**= $\binom{4}{-1}$ and **c**= $\binom{-2}{-4}$ find:

   a. **a** +2**b**   b. 4**a** + 3**c**   c. 6**a**– 3**b**   d. 5**a** +2**b**−4**c**

Solution:

Given: **a**= $\binom{2}{1}$, **b**= $\binom{4}{-1}$ and **c**= $\binom{-2}{-4}$

a. $3\mathbf{a} + 2\mathbf{b} = 3\binom{2}{1} + 2\binom{4}{-1} = \binom{6}{3} + \binom{8}{-2} = \binom{6+8}{3+(-2)} = \binom{14}{1}$

b. $4\mathbf{a} + 3\mathbf{c} = 4\binom{2}{1} + 3\binom{-2}{-4} = \binom{8}{4} + \binom{-6}{-12} = \binom{8+(-6)}{4+(-12)} = \binom{2}{-8}$

c. $6\mathbf{a} - 3\mathbf{b} = 6\binom{2}{1} - 3\binom{4}{-1} = \binom{12}{6} - \binom{12}{-3} = \binom{12-12}{6-(-3)} = \binom{0}{9}$

d. $5\mathbf{a} + 2\mathbf{b} - 4\mathbf{c} = 5\binom{2}{1} + 2\binom{4}{-1} - 4\binom{-2}{-4} = \binom{10}{5} + \binom{8}{-2} - \binom{-8}{-16} = \binom{10+8+8}{5+(-2)+16} = \binom{26}{19}$

3. If **a**= $\binom{4}{2}$, **b**= $\binom{3}{-1}$, $\mathbf{c} = \binom{-2}{1}$ solve for $\mathbf{x}$ in the equations below.

   a. 3**a**+2**x** =4**b**

   b. **4a**– **x**= **c**

   c. 2**x** + 3**b** = **c**

Solutions:

Given: **a**= $\binom{4}{2}$,**b**= $\binom{3}{-1}$, $\mathbf{c} = \binom{-2}{1}$

a. $3\mathbf{a} + 2\mathbf{x} = 4\mathbf{b}$

$3\binom{4}{2} + 2\mathbf{x} = 4\binom{3}{-1}$

b. $\mathbf{a} - 2\mathbf{x} = 4\mathbf{c}$

$\binom{4}{2} - 2\mathbf{x} = 4\binom{-2}{1}$

$$\begin{pmatrix}12\\6\end{pmatrix} + 2\mathbf{x} = \begin{pmatrix}12\\-4\end{pmatrix}$$
$$2\mathbf{x} = \begin{pmatrix}12\\-4\end{pmatrix} - \begin{pmatrix}12\\6\end{pmatrix}$$
$$= \begin{pmatrix}12-12\\-4-6\end{pmatrix}$$
$$= \begin{pmatrix}0\\-10\end{pmatrix}$$
$$\mathbf{x} = \begin{pmatrix}0\\-10\end{pmatrix} \div 2$$
$$\mathbf{x} = \begin{pmatrix}0\\-5\end{pmatrix}$$

$$\begin{pmatrix}4\\2\end{pmatrix} - 2\mathbf{x} = \begin{pmatrix}-8\\4\end{pmatrix}$$
$$\begin{pmatrix}4\\2\end{pmatrix} - \begin{pmatrix}-8\\4\end{pmatrix} = 2\mathbf{x}$$
$$\begin{pmatrix}4-(-8)\\2-4\end{pmatrix} = 2\mathbf{x}$$
$$2\mathbf{x} = \begin{pmatrix}12\\-2\end{pmatrix}$$
$$\mathbf{x} = \begin{pmatrix}12\\-2\end{pmatrix} \div 2$$
$$= \begin{pmatrix}6\\-1\end{pmatrix}$$

c. $2\mathbf{x} + 3\mathbf{b} = \mathbf{c}$
$$2\mathbf{x} + 3\begin{pmatrix}3\\-1\end{pmatrix} = \begin{pmatrix}-2\\1\end{pmatrix}$$
$$2\mathbf{x} + \begin{pmatrix}9\\-3\end{pmatrix} = \begin{pmatrix}-2\\1\end{pmatrix}$$
$$2\mathbf{x} = \begin{pmatrix}-2\\1\end{pmatrix} - \begin{pmatrix}9\\-3\end{pmatrix}$$
$$= \begin{pmatrix}-2-9\\1-(-3)\end{pmatrix}$$
$$= \begin{pmatrix}-11\\4\end{pmatrix}$$
$$\mathbf{x} = \begin{pmatrix}-11\\4\end{pmatrix} \div 2$$
$$\mathbf{x} = \begin{pmatrix}-5.5\\2\end{pmatrix}$$

Practice

1. If **a**= $\begin{pmatrix}1\\2\end{pmatrix}$, **b**= $\begin{pmatrix}2\\-1\end{pmatrix}$ and **c**= $\begin{pmatrix}0\\-2\end{pmatrix}$ evaluate:
   a. **a** + **b** b. **b – c** c. **a** + **b** + **c** d. **a - 2b**
   e. 3**a** - **b** - 2**c** f. **a** - 2**b** - 3**c** g. 2**a** + 3**b** + 5**c**

2. If **a**= $\begin{pmatrix}2\\1\end{pmatrix}$, **b**= $\begin{pmatrix}6\\-2\end{pmatrix}$ and **c**= $\begin{pmatrix}-3\\-5\end{pmatrix}$ find:
   a. **a** +2**b** b. 4**a** + 3**c** c. 6**a**– 3**b** d. 5**a** +2**b**–4**c**

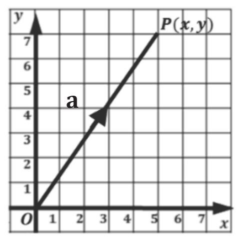
3. If **a**= $\begin{pmatrix}6\\2\end{pmatrix}$, **b**= $\begin{pmatrix}5\\-1\end{pmatrix}$, $\mathbf{c} = \begin{pmatrix}-2\\1\end{pmatrix}$ solve for $\mathbf{x}$ in the equations below.
   a. 3**a**+2**x** =4**b** b. 4**a**– $5\begin{pmatrix}2\\1\end{pmatrix} + 2\begin{pmatrix}4\\-1\end{pmatrix} + 4\begin{pmatrix}-2\\-4\end{pmatrix}$ + **x** = **c** c. $2\mathbf{x} + 6\mathbf{b} = 2\mathbf{c}$

Lesson Title: Position vectors	**Theme:** Vectors and Transformations
Lesson Number: PHM3-L077	**Class:** SSS 3

Learning Outcomes
By the end of the lesson, you will be able to:
1. Define the position vector of a point.
2. Express two given points as a vector.

Overview

This lesson introduces position vectors and some of its uses in describing vectors. The diagram shows the point $P(x, y)$ on the Cartesian plane with origin O.



Vector **a** is the displacement of P from O**.** Remember that displacement is the shortest distance travelled in a given direction. Since displacement gives the position of P relative to the origin O, **a** is called the position vector of P.
From the diagram $\overrightarrow{OP} = \mathbf{a} = \binom{5}{7}$
We already know the co-ordinates of $P = (5,7)$.
Therefore, if a point has co-ordinates (x, y), its position vector is $\binom{x}{y}$.

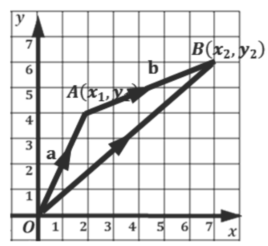


Figure A

Position vectors can be used to express 2 given points as a vector. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any 2 points on a Cartesian plane as shown in Figure A.

From the diagram, we can see that:

$$\begin{aligned} \overrightarrow{OA} + \overrightarrow{AB} &= \overrightarrow{OB} \\ \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} && \text{position vector of } B \text{–position vector of } A \\ &= \mathbf{b} - \mathbf{a} && \text{equivalent representation of position vectors} \\ &= \binom{x_2}{y_2} - \binom{x_1}{y_1} \\ &= \binom{x_2 - x_1}{y_2 - y_1} \end{aligned}$$

The vector joining A and B is given by: $\overrightarrow{AB} = \binom{x_2 - x_1}{y_2 - y_1}$.

Solved Examples

1. State the position vectors relative to the origin of the points:
   a. A b. B c. C d. D

   Solutions:

   a. **Step 1.** Assess and extract the given information from the problem.
   Given: graph showing point A

   Step 2. Write the co-ordinates and position vector of the given point.

co-ordinates of point A = $(3,7)$

position vector of point A = $\overrightarrow{OA}$

= $\binom{3}{7}$

Step 3. Write the answer.

The position vector relative to the origin of point $A(3,7)$: $\overrightarrow{OA} = \binom{3}{7}$.

b. The position vector relative to the origin of $B(5,-3)$: $\overrightarrow{OB} = \binom{5}{-3}$.

c. The position vector relative to the origin of $C(-7,-4)$: $\overrightarrow{OC} = \binom{-7}{-4}$.

d. The position vector relative to the origin of $D(-7,3)$: $\overrightarrow{OD} = \binom{-7}{3}$.

2. $A(2,4)$ and $B(7,6))$ are points on a Cartesian plane (see Figure A). Find:
   a. The position vector of points A and B relative to the origin O.
   b. The vector $\overrightarrow{AB}$.
   c. The vector $\overrightarrow{BA}$.

Solution:

Given: $A(2,4)$ and $B(7,6)$

a. position vector of point A = $\overrightarrow{OA}$ relative to the origin O

= $\binom{2}{4} = \binom{x_1}{y_1}$

position vector of point B = $\overrightarrow{OB}$ relative to the origin O

= $\binom{7}{6} = \binom{x_2}{y_2}$

b. $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \binom{x_2-x_1}{y_2-y_1} = \binom{7-2}{6-4}$

$= \binom{5}{2}$

This is clearly the case as can be verified from Figure A. Above

c. $\overrightarrow{BA} = -\overrightarrow{AB}$

$\overrightarrow{BA} = -\binom{5}{2}$

$= \binom{-5}{-2}$

3. $R(1,6)$ and $S(x,y)$ are points on a Cartesian plane such that $\overrightarrow{RS} = \binom{3}{-5}$

Find the co-ordinates of S.

Solutions:

Given: $R(1,6)$, $S(x,y)$, $\overrightarrow{RS} = \binom{3}{-5}$

a. position vector of point R = $\overrightarrow{OR}$ relative to the origin O

= $\binom{1}{6}$

position vector of point S = $\overrightarrow{OS}$ relative to the origin O

= $\binom{x}{y}$

$$\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR}$$
$$\overrightarrow{RS} = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$
$$\begin{pmatrix} 3 \\ -5 \end{pmatrix} = \begin{pmatrix} x-1 \\ y-6 \end{pmatrix}$$

Equating the components of the vectors gives:

$$3 = x - 1$$
$$\Rightarrow x = 4$$
$$-5 = y - 6$$
$$\Rightarrow y = 1$$

The co-ordinates of S are $(4,1)$

Practice

1. $X(7,-5)$ and $Y(-2,3))$ are points on a Cartesian plane. Find:
    a. The position vector of points X and Y relative to the origin O.
    b. The vector $\overrightarrow{XY}$.
    c. The vector $\overrightarrow{YX}$.

2. Find $\overrightarrow{AB}$ and $\overrightarrow{BA}$ given the following:
    a. $A(6,3)$ and $B(2,-4)$
    b. $A(5,-6)$ and $B(-4-8)$

3. $P(4,7)$ and $T(x,y)$ are points on a Cartesian plane such that $\overrightarrow{PT} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$. Find the co-ordinates of T.

4. A is the point $(4,3)$ such that $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$. Find the position vector of B.

5. $E(-2,4)$ and $F(x,y)$ are points on the $O(x,y)$ plane such that $\overrightarrow{EF} = \begin{pmatrix} -5 \\ -8 \end{pmatrix}$. Find the co-ordinates of F.

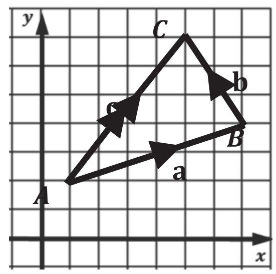
Lesson Title: Triangle law of vector addition	**Theme:** Vectors and Transformations
Lesson Number: PHM3-L078	**Class:** SSS 3
Learning Outcome By the end of the lesson, you will be able to add two vectors using the triangle law of vector addition.	

Overview

To find the sum of the given vectors, we used what we learned from a previous lesson, namely that:

If $\mathbf{a} = \binom{x_1}{y_1}$ and $\mathbf{b} = \binom{x_2}{y_2}$, then

$$\mathbf{a} + \mathbf{b} = \binom{x_1+x_2}{y_1+y_2} \quad (1)$$



Vector addition can also be shown on a diagram.

A point moving from A to B, then from B to C performs the same journey as a point moving from A to C.

We can write this using vector notation as:

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ $\overrightarrow{AC}$ is called the **resultant** of the 2 vectors. Note the end point (B) of the first vector must be the starting point (B) of the second vector.

$$\mathbf{a} + \mathbf{b} = \mathbf{c} \quad (2)$$

$$\Longrightarrow \quad \mathbf{a} + \mathbf{b} - \mathbf{c} = \mathbf{0} \quad (3)$$ This is the zero vector, $\mathbf{0} = \binom{0}{0}$,

Equation (3) is the triangle law of vector addition which states that:

If three vectors are represented by the sides of a triangle **taken in order**, then their vector sum must be equal to the zero vector.

Equation (2) is the usual form of the triangle law to use in solving problems.

The vector found by drawing is the same magnitude and direction as the vector found by calculating.

Using the triangle law of vector addition gives the same result as adding 2 vectors together.

$$\mathbf{c} = \binom{x_1+x_2}{y_1+y_2} \quad (4)$$

We will now do examples to show we get the same answer by drawing using the triangle law of vector addition as by calculating.

Solved Examples

1. If $\mathbf{a} = \binom{4}{2}$ and $\mathbf{b} = \binom{1}{3}$ find $\mathbf{a} + \mathbf{b}$**:**

a. By using the triangle law of vector addition $\mathbf{a} + \mathbf{b} = \mathbf{c}$. (draw a diagram)

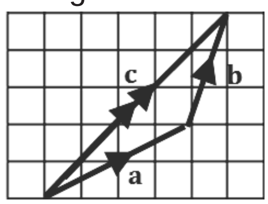
b. By calculation using $\mathbf{a} + \mathbf{b} = \binom{x_1+x_2}{y_1+y_2}$

Solutions:

a. **Step 1.** Assess and extract the given information from the problem.

Given: $\mathbf{a} = \binom{4}{2}$, $\mathbf{b} = \binom{1}{3}$

Step 2. Add by using the triangle law of vector addition $\mathbf{a} + \mathbf{b} = \mathbf{c}$.



From the diagram: $\mathbf{a} + \mathbf{b} = \binom{5}{5}$

b. **Step 3.** Add by calculation using $\mathbf{a} + \mathbf{b} = \binom{x_1+x_2}{y_1+y_2}$

$$\mathbf{a} + \mathbf{b} = \binom{4}{2} + \binom{1}{3} = \binom{4+1}{2+3}$$

$$= \binom{5}{5}$$

Step 4. Write the answer.

The resultant vector $\mathbf{c} = \binom{5}{5}$.

2. Find the sum of the given vectors.

- By using the triangle law of vector addition. (Hint: Draw a diagram)
- By calculation.

a. If $\overrightarrow{AB} = \binom{5}{-6}$ and $\overrightarrow{BC} = \binom{-3}{4}$, find $\overrightarrow{AC}$

b. If $\overrightarrow{AB} = \binom{4}{-5}$ and $\overrightarrow{CB} = \binom{-2}{1}$, find $\overrightarrow{AC}$

Solutions:

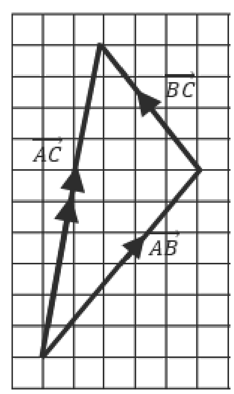
a. Given: $\overrightarrow{AB} = \binom{5}{6}$, $\overrightarrow{BC} = \binom{-3}{4}$

$\overrightarrow{AB} + \overrightarrow{BC} = \binom{2}{10}$ from the diagram

$\overrightarrow{AB} + \overrightarrow{BC} = \binom{5}{6} + \binom{-3}{4} = \binom{5+(-3)}{6+4}$

$= \binom{2}{10}$

The sum $\overrightarrow{AC} = \binom{2}{10}$



b. Given: $\overrightarrow{AB} = \binom{4}{-5}$, $\overrightarrow{CB} = \binom{-2}{1}$

$\overrightarrow{AB} + \overrightarrow{BC} = \binom{6}{-6}$ from the diagram

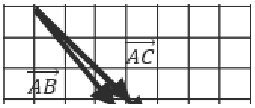
$\overrightarrow{BC} = -\overrightarrow{CB}$

$= -\binom{-2}{1} = \binom{2}{-1}$

$\overrightarrow{AB} + \overrightarrow{BC} = \binom{4}{-5} + \binom{2}{-1} = \binom{4+2}{-5+(-1)}$

$= \begin{pmatrix} 6 \\ -6 \end{pmatrix}$

The sum $\overrightarrow{AC} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$



Practice

1. Find the sum of the given vectors:
   a. By using the triangle law of vector addition. (draw a diagram)
   b. By calculation.

   a. If $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -8 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$, find $\overrightarrow{AC}$ b. If $\overrightarrow{AB} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ and $\overrightarrow{CB} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$, find $\overrightarrow{AC}$

2. If $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $\overrightarrow{PR} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, find $\overrightarrow{QR}$

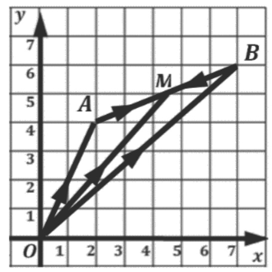
3. If $\overrightarrow{OP} = \begin{pmatrix} 8 \\ -9 \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, find $\overrightarrow{PQ}$

4. If $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ and $\overrightarrow{CB} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, find $\overrightarrow{AC}$ and $\overrightarrow{CA}$

Lesson Title: Mid-point of a line segment	**Theme:** Vectors and Transformations	
Lesson Number: PHM3-L079	**Class:** SSS 3	**Time:** 40 minutes
Learning Outcome By the end of the lesson, you will be able to calculate the mid-point of a line segment.		

Overview

Consider the diagram shown at right. The points $A(x_1, y_1)$ and $B(x_2, y_2)$ lie in a Cartesian plane. M is the mid-point of the line segment AB.



From the diagram, we can see that there are two routes from O to M from O via A to M and from O via B to M.

We can write the position vector $\overrightarrow{OM}$ in terms of the position vectors of A and B and the vector $\overrightarrow{AB}$.

$$\overrightarrow{AM} = \overrightarrow{MB} = \frac{1}{2}\overrightarrow{AB} \qquad \text{since } M \text{ is the mid-point of } AB$$

$$\begin{aligned}
\overrightarrow{OM} &= \overrightarrow{OA} + \overrightarrow{AM} \\
&= \overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{AB} \qquad (1) \\
\overrightarrow{OM} &= \overrightarrow{OB} + \overrightarrow{BM} \\
&= \overrightarrow{OB} - \overrightarrow{MB} \\
&= \overrightarrow{OB} - \tfrac{1}{2}\overrightarrow{AB} \qquad (2) \\
2\overrightarrow{OM} &= \overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{AB} + \overrightarrow{OB} - \tfrac{1}{2}\overrightarrow{AB} \qquad \text{add equations (1) and (2)} \\
& \quad \overrightarrow{OA} + \overrightarrow{OB} \\
\overrightarrow{OM} &= \frac{\overrightarrow{OA} + \overrightarrow{OB}}{2} \\
&= \tfrac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right)
\end{aligned}$$

The position vector of the mid-point of the line segment is an average of the position vectors of the 2 end points.

The co-ordinates of the mid-point can similarly be found by finding the average of the x coordinates and y coordinates respectively.

The mid-point M will have co-ordinates $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. This is called the Mid-point Theorem.

Solved Examples

1. State the position vectors relative to the origin of $A(2,4)$ and $B(7,6)$.
   Find:
   a. The position vector of the mid-point M.
   b. The mid-point of the line segment AB.

Solutions:

Step 1. Assess and extract the given information from the problem.

Given: points $A(2,4)$ and $B(7,6)$.

$A(2,4)$: $\overrightarrow{OA} = \binom{2}{4}$

$B(7,6)$: $\overrightarrow{OB} = \binom{7}{6})$

a. **Step 2.** Find the position vector and co-ordinates of the mid-point.

$$\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$$

$$= \frac{1}{2}\left(\binom{2}{4} + \binom{7}{6}\right)$$

$$= \frac{1}{2}\binom{2+7}{4+6} = \frac{1}{2}\binom{9}{10}$$

$$\overrightarrow{OM} = \binom{4.5}{5}$$

b. co-ordinates of M $= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$= (4.5, 5)$ write directly using $\overrightarrow{OM}$

Step 3. Write the answer.

The position vector $\overrightarrow{OM} = \binom{4.5}{5}$.

The co-ordinates of $M = (4.5, 5)$.

2. $P(p,2)$ and $Q(-1,q)$ are points on a Cartesian plane. Find the values of p and q so that $\binom{-2}{3}$ gives the position vector of the mid-point of PQ.

Solution:

Given: $P(p,2)$ and $Q(-1,q)\overrightarrow{OM} = \binom{-2}{3}$

Let the position vectors $\overrightarrow{OP} = \binom{p}{2}$ and $\overrightarrow{OQ} = \binom{-1}{q}$

$$\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OP} + \overrightarrow{OQ})$$

$$\binom{-2}{3} = \frac{1}{2}\left(\binom{p}{2} + \binom{-1}{q}\right)$$

Multiply both sides by two

$$\binom{-4}{6} = \binom{p}{2} + \binom{-1}{q}$$

$$= \binom{p-1}{2+q}$$

$$-4 = p - 1$$

$$-4 + 1 = p$$

$$p = -3$$

$$\begin{aligned} 6 &= 2+q \\ 6-2 &= q \\ q &= 4 \end{aligned}$$

$$p=-3, q=4$$

3. $A(4,-7), B(-2,3)$ and $Y(5,1)$ are three points on a Cartesian plane. If X is the mid-point of AB, find $\overrightarrow{XY}$.

Solution:

Given: $A(4,-7), B(-2,3)$ and $Y(5,1)$

Let the position vectors $\overrightarrow{OA}=\begin{pmatrix}4\\-7\end{pmatrix}, \overrightarrow{OB}=\begin{pmatrix}-2\\3\end{pmatrix}$ and $\overrightarrow{OY}=\begin{pmatrix}5\\1\end{pmatrix}$

$$\begin{aligned} \overrightarrow{OX} &= \frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OB}\right) \\ &= \frac{1}{2}\left(\begin{pmatrix}4\\-7\end{pmatrix}+\begin{pmatrix}-2\\3\end{pmatrix}\right) \\ &= \frac{1}{2}\begin{pmatrix}4+(-2)\\-7+3\end{pmatrix}=\frac{1}{2}\begin{pmatrix}4-2\\-7+3\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix}2\\-4\end{pmatrix} \\ \overrightarrow{OX} &= \begin{pmatrix}1\\-2\end{pmatrix} \\ \overrightarrow{XY} &= \overrightarrow{OY}-\overrightarrow{OX} \\ \overrightarrow{XY} &= \begin{pmatrix}5\\1\end{pmatrix}-\begin{pmatrix}1\\-2\end{pmatrix} \\ &= \begin{pmatrix}5-1\\1-(-2)\end{pmatrix} \\ \overrightarrow{XY} &= \begin{pmatrix}4\\3\end{pmatrix} \end{aligned}$$

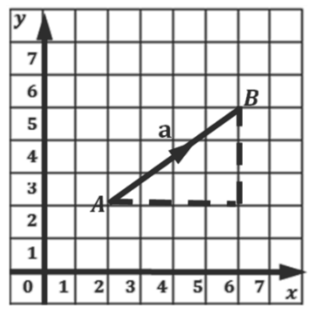
The vector $\overrightarrow{XY}=\begin{pmatrix}4\\3\end{pmatrix}$

Practice

1. $P(1,3)$ and $Q(2,6)$ are points in a Cartesian plane. If M is the mid-point of PQ, find the position vector of M.
2. The position vector of the mid-point of the line segment XY is given by $\begin{pmatrix}6\\-4\end{pmatrix}$. If point X has co-ordinates $(5,4)$, find the co-ordinates of point Y.
3. $P(11,-4), Q(-5,8)$ and $T(7,3)$ are three points in a Cartesian plane. If K is the mid-point of PQ, find $\overrightarrow{KT}$.
4. $R(-6,3)$ and $S(-4,5)$ are points in a Cartesian plane. If M is the mid-point of RS, find the position vector of M.
5. The position vector of the mid-point of the line segment DE is given by $\begin{pmatrix}2\\9\end{pmatrix}$. If point D has co-ordinates $(-5,7)$, find the co-ordinates of point E.
6. $A(-5,-6), B(2,-3)$ and $C(3,-4)$ are three points in a Cartesian plane. If P is the mid-point of AB, find $\overrightarrow{PC}$.

Lesson Title: Magnitude of a vector	**Theme:** Vectors and Transformations
Lesson Number: PHM3-L080	**Class:** SSS 3
Learning Outcome By the end of the lesson, you will be able to calculate the magnitude of a vector.	

Overview



Consider the diagram showing the 2 points $A(2,2)$ and $B(6,5)$.
From the diagram we can write the column vector for AB as:

$$\overrightarrow{AB} = \begin{pmatrix} 6-2 \\ 5-2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

We can use Pythagoras' Theorem to find the magnitude or length of the vector $\overrightarrow{AB}$.
The magnitude of vector $\overrightarrow{AB}$ can be written with the modulus or absolute value notation.

Examples: $|\overrightarrow{AB}|$, $|AB||\mathbf{a}|$, $|\underline{a}|$.

Now, we already know that for any 2 points:

$A(x_1, y_1)$ and $B(x_2, y_2)$, vector $\overrightarrow{AB} = \begin{pmatrix} x_2-x_1 \\ y_2-y_1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$.

Since $\overrightarrow{AB} = \mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ then, using Pythagoras' theorem, this formula can be derived:

$$|\overrightarrow{AB}|^2 = |\mathbf{a}|^2 = x^2 + y^2$$

$$|\overrightarrow{AB}| = |\mathbf{a}| = \sqrt{x^2 + y^2} \qquad (1)$$

Alternatively, we can find the magnitude of $\overrightarrow{AB}$ by substituting directly in equation (1) using the co-ordinates of the given points:

$$|\overrightarrow{AB}| = |\mathbf{a}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad (2)$$

Solved Examples

1. Find the magnitude of $\overrightarrow{AB}$ from the column vector above.

   Solution:

   Step 1. Assess and extract the given information from the problem.

   Given: vector $\overrightarrow{AB} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ from the column vector above.

   Step 2. Substitute into equation (1).

   $$\begin{aligned} |\overrightarrow{AB}| &= \sqrt{x^2 + y^2} \quad \text{Pythagoras' Theorem} \\ &= \sqrt{4^2 + 3^2} \\ &= \sqrt{16 + 9} \\ |\overrightarrow{AB}| &= \sqrt{25} \\ &= 5 \end{aligned}$$

   Step 3. Write the answer.

The magnitude of $\overrightarrow{AB}$: $|\overrightarrow{AB}| = 5$ units

2. Find the magnitude of the vectors below.

   a. $\overrightarrow{AB} = \binom{3}{4}$ b. $\overrightarrow{BC} = \binom{5}{0}$ c. $\overrightarrow{CD} = \binom{-3}{4}$ d. $\overrightarrow{DA} = \binom{-5}{0}$

   e. What shape is $ABCD$?

Solutions:

a. Given: $\overrightarrow{AB} = \binom{3}{4}$

$$\begin{aligned} |\overrightarrow{AB}| &= \sqrt{x^2 + y^2} \\ &= \sqrt{3^2 + 4^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ |\overrightarrow{AB}| &= 5 \end{aligned}$$

The magnitude of $\overrightarrow{AB} = 5$ units

b. Given: $\overrightarrow{BC} = \binom{5}{0}$

$$\begin{aligned} |\overrightarrow{BC}| &= \sqrt{x^2 + y^2} \\ &= \sqrt{5^2 + 0^2} \\ &= \sqrt{25 + 0} \\ &= \sqrt{25} \\ |\overrightarrow{BC}| &= 5 \end{aligned}$$

The magnitude of $\overrightarrow{BC} = 5$ units

c. Given: $\overrightarrow{CD} = \binom{-3}{4}$

$$\begin{aligned} |\overrightarrow{CD}| &= \sqrt{x^2 + y^2} \\ &= \sqrt{-3^2 + 4^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ |\overrightarrow{CD}| &= 5 \end{aligned}$$

The magnitude of $\overrightarrow{CD} = 5$ units

d. Given: $\overrightarrow{DA} = \binom{-5}{0}$

$$\begin{aligned} |\overrightarrow{CD}| &= \sqrt{x^2 + y^2} \\ &= \sqrt{-5^2 + 0^2} \\ &= \sqrt{25 + 0} \\ &= \sqrt{25} \\ |\overrightarrow{CD}| &= 5 \end{aligned}$$

The magnitude of $\overrightarrow{CD} = 5$ units

e. The shape is a square.

4. A column vector $\binom{x}{6}$ has a magnitude of 10. Find x:

Solution:

Given: magnitude of $\binom{x}{6} = 10$

$$\sqrt{x^2 + y^2} = 10$$
$$\sqrt{x^2 + 6^2} = 10$$

Square both sides:

$$x^2 + 36 = 100$$
$$x^2 = 100 - 36$$
$$x^2 = 64$$
$$x = \sqrt{64}$$
$$= 8$$

Practice

1. Find the magnitude of the given vectors to 1 decimal place:
   a. $\overrightarrow{AB} = \binom{-5}{7}$ b. $\overrightarrow{RT} = \binom{2}{4}$

2. Find the magnitude of the vector $\mathbf{r} = \binom{-4}{12}$.
   Give your answer: a. In surd form b. To 2 decimal places

3. XYZ is a triangle with vertices X(1,-3),Y(7,5) and Z(-3,5).
   a. If O is the origin, express $\overrightarrow{XY}$, $\overrightarrow{YZ}$ and $\overrightarrow{ZX}$ as column vectors.
   b. Show that triangle is XYZ isosceles.

4. A column vector $\binom{7}{y}$ has a magnitude of 25. Find y.

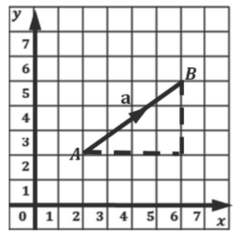
5. A column vector $\binom{x}{12}$ has a magnitude of 13. Find x.

Lesson Title: Direction of a vector	Theme: Vectors and Transformations
Lesson Number: PHM3-L081	**Class:** SSS 3

Learning Outcome
By the end of the lesson, you will be able to will be able to calculate the direction of a vector.

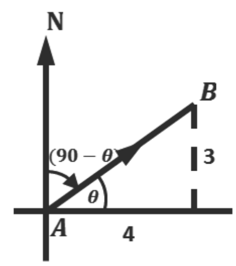
Overview

During the last lesson, we looked at finding the magnitude of vectors such as the one shown on the right. This lesson focuses on finding the direction of a vector.



We start by writing the vector joining the points A and B.
This is given as: $A(2,2), B(6,5)$; $\overrightarrow{AB} = \binom{6-2}{5-2} = \binom{4}{3}$
Before the direction of the vector can be found, we need a diagram of the problem.

It is not always the case that we are given a diagram in the question, so we start by drawing a sketch of the problem, as shown. This will assist us in finding the direction of the vector. The direction of the vector is given by the angle it makes when measured from the north in a clockwise direction.
We find this angle by first finding the acute angle θ, the vector makes with the x-axis.



This angle is given by $\tan\theta = \frac{y}{x}$, where x, y are the components of the resultant vector. From our sketch, we can then deduce the angle the vector makes when measured from the north in a clockwise direction.

In our example, this angle is given by $(90 - \theta)$. This is the same as finding the bearing of B from A. We will now work through how to find the direction of the vector above.

Solved Examples

1. Find the direction of the column vector $\overrightarrow{AB} = \binom{4}{3}$ to the nearest whole number.
   Solution:
   Step 1. Assess and extract the given information from the problem.
   Given: vector $\overrightarrow{AB} = \binom{4}{3}$
   Step 2. Draw a sketch of the vector. (shown above)
   Step 3. Find the direction (bearing) of the vector.
   From the diagram the direction of $\overrightarrow{AB}$ is at an angle $(90 - \theta)$ when measured clockwise from the north.

$\tan\theta \;=\; \frac{3}{4} \;=\; 0.75$ from diagram, use tan ratio

$\theta \;=\; \tan^{-1}(0.75)$

$\;=\; 36.87°$

The direction of $\overrightarrow{AB}$ measured from the north:

$= 90 - 36.87$

$= 53.13°$

Step 4. Write the answer.

The direction of $\overrightarrow{AB}$ measured from the north $=$ 53° to the nearest degree.

2. If $\overrightarrow{XY} = \binom{2}{1}$ and $\overrightarrow{ZY} = \binom{3}{-5}$, find:
   a. $\overrightarrow{ZX}$
   b. The bearing of Z from X correct to the nearest degree

Given: $\overrightarrow{XY} = \binom{2}{1}$, $\overrightarrow{ZY} = \binom{3}{-5}$,

a. $\overrightarrow{YZ} = -\binom{3}{-5}$

$= \binom{-3}{5}$

$\overrightarrow{XZ} = \overrightarrow{XY} + \overrightarrow{YZ}$

$= \binom{2}{1} + \binom{-3}{5}$

$= \binom{2+(-3)}{1+5}$

$\overrightarrow{XZ} = \binom{-1}{6}$

$\overrightarrow{ZX} = -\binom{-1}{6}$

$= \binom{1}{-6}$

b. Find the acute angleθ, the vector makes with the x-axis:

$\tan\theta = \frac{6}{1}$

$\theta = \tan^{-1}(6)$

$= 80.54°$

The direction of $\overrightarrow{ZX}$ measured from the north:

$= 90 + 80.54$

$= 170.54°$

The direction of $\overrightarrow{XZ}$ measured from the north $= 171°$ to the nearest degree.

3. The points $A(0,-1), B(4,-1), C(7,-2)$ and $D(3,-2)$ are the vertices of a parallelogram. Find:
   a. $\overrightarrow{AC}$
   b. the bearing of A from C correct to the nearest degree

Given: $A(0,-1)$ and $C(7,-2)$,

Let position vectors $\overrightarrow{OA} = \binom{0}{-1}$ and $\overrightarrow{OC} = \binom{7}{-2}$:

a. $\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$\overrightarrow{AC} = \binom{7}{-2} - \binom{0}{-1}$

$= \binom{7-0}{-2-(-1)}$

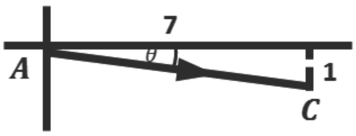
$\overrightarrow{AC} = \binom{7}{-1}$

b. Find the acute angle θ, the vector makes with the x-axis:

$\tan\theta = \frac{1}{7}$

$\theta = \tan^{-1}(\frac{1}{7})$

$= 8.13°$



The direction of $\overrightarrow{AC}$ measured from the north:

$= 90 + 8.13$

$= 98.13°$

The direction of $\overrightarrow{AC}$ measured from the north $= 98°$ to the nearest degree.

Practice

1. Find the direction of the given vectors to the nearest whole number:

   a. $\overrightarrow{XY} = \binom{9}{-3}$ b. $\overrightarrow{PR} = \binom{-4}{-8}$

2. $P(-1,2)$ *and* $Q(x,y)$ are points on the Oxy plane such that $\overrightarrow{PQ} = \binom{3}{-4}$. Find:
   a. The coordinates of Q.
   b. The bearing of P from Q to the nearest degree.

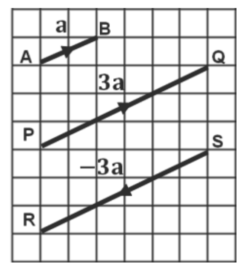
3. $A(1,2), B(4,6), C(2,7)$ and $D(x,y)$ are the vertices of the parallelogram ABCD. Find:
   a. x and y
   b. $\overrightarrow{AC}$
   c. The bearing of A from C, correct to the nearest degree.

4. If $\overrightarrow{AB} = \binom{3}{-5}$ and $\overrightarrow{CB} = \binom{2}{1}$, find:
   a. $\overrightarrow{CA}$
   b. The bearing of C from A correct to the nearest degree.

Lesson Title: Parallel and perpendicular vectors	Theme: Vectors and Transformations
Lesson Number: PHM3-L082	**Class:** SSS 3
Learning Outcome By the end of the lesson, you will be able to solve problems with parallel and perpendicular vectors.	

Overview

Parallel Vectors



We know from a previous lesson that:

If $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ then

$k\mathbf{a} = \begin{pmatrix} kx \\ ky \end{pmatrix}$ where k is a scalar or number which can be a positive or negative whole number or fraction

From this, we know that the vectors $\mathbf{a}, \mathbf{3a}$ and $-\mathbf{3a}$ are all scalar multiples of each other. From the diagram we can see that they are also all **parallel** to each other.
They all make the same acute angle with the x-axis.

$$\tan\theta = \frac{2}{1} = \frac{6}{3} = 2$$
$$\theta = \tan^{-1}(2) = 63.42$$
$$= 63° \text{ to the nearest degree}$$

By definition,

If $\mathbf{b} = k\mathbf{a}$ Then $\mathbf{a}$ and $\mathbf{b}$ are parallel

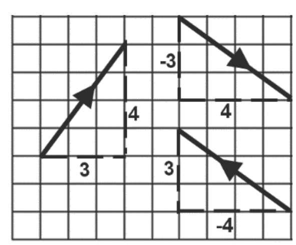
- If $k > 0$, then $\mathbf{a}$ and $\mathbf{b}$ have the same direction.
- If $k < 0$, then $\mathbf{a}$ and $\mathbf{b}$ have opposite directions.
- If $k = 1$, then $\mathbf{a}$ and $\mathbf{b}$ are equal (that is, they have the same magnitude and direction).

It follows from the above that the corresponding components of 2 parallel vectors are in the same ratio with each other.

If $\binom{a}{b} = k\binom{c}{d}$ that is, they are parallel vectors

then $a : c = b : d$ corresponding ratios in the same order are equal

Perpendicular Vectors



There are times when we are asked to find vectors which are **perpendicular** to each other.
Consider the diagram shown on the right.
We can see that the vectors $\binom{4}{-3}$ and $\binom{-4}{3}$ are perpendicular to vector $\binom{3}{4}$.

Their scalar multiples $\binom{4k}{-3k}$ and $\binom{-4k}{3k}$ where k is a **positive** number are also perpendicular to $\binom{3}{4}$.

In general, the vectors perpendicular to $\binom{x}{y}$ are:

$$\binom{y}{-x}\binom{-y}{x}$$

and scalar multiples: $\binom{ky}{-kx}\binom{-ky}{kx}$

Solved Examples

1. Which of the following vectors are parallel to $\binom{10}{15}$?

a. $\binom{6}{9}$ b. $\binom{2}{3}$ c. $\binom{8}{-12}$ d. $\binom{-4}{-6}$ e. $\binom{15}{10}$

Solutions:

Step 1. Assess and extract the given information from the problem.

Given: vector $\binom{10}{15}$, 5 other vectors

Step 2. Compare each vector with $\binom{10}{15}$.

If the vectors are parallel, then:

$a : 10 = b : 15$ where a and b are components of the vector to be compared

$$\frac{a}{10} = \frac{b}{15}$$

a. $\binom{6}{9}$: $\frac{6}{10} = \frac{9}{15} = \frac{3}{5}$ parallel

b. $\binom{2}{3}$: $\frac{2}{10} = \frac{3}{15} = \frac{1}{5}$ parallel

c. $\binom{8}{-12}$: $\frac{8}{10} \neq \frac{-12}{15}$ not parallel

d. $\binom{-4}{-6}$: $\frac{-4}{10} = \frac{-6}{15} = -\frac{2}{5}$ parallel

e. $\binom{15}{10}$: $\frac{15}{10} \neq \frac{10}{15}$ not parallel

Step 3. Write the answer.

The vectors $\binom{6}{9}$, $\binom{2}{3}$ and $\binom{-4}{-6}$ are parallel to $\binom{10}{15}$.

2. Which of the following vectors are perpendicular to $\binom{7}{5}$?

a. $\binom{-5}{-7}$ b. $\binom{10}{-14}$ c. $\binom{-5}{7}$ d. $\binom{-14}{-10}$ e. $\binom{5}{-7}$

Solutions:

Given: vector $\binom{7}{5}$, 5 other vectors

The vectors perpendicular to $\binom{7}{5}$ are $\binom{5}{-7}$, $\binom{-5}{7}$ and their scalar multiples.

Compare each vector with the known perpendicular vectors.

a. $\binom{-5}{-7}$: not perpendicular

b. $\binom{10}{-14}$: perpendicular – scalar multiple of $\binom{5}{-7}$

c. $\begin{pmatrix}-5\\7\end{pmatrix}$: perpendicular

d. $\begin{pmatrix}-14\\-10\end{pmatrix}$: not perpendicular

e. $\begin{pmatrix}5\\-7\end{pmatrix}$: perpendicular

The vectors $\begin{pmatrix}10\\-14\end{pmatrix}$, $\begin{pmatrix}-5\\-7\end{pmatrix}$ and $\begin{pmatrix}5\\-7\end{pmatrix}$ are perpendicular to $\begin{pmatrix}7\\5\end{pmatrix}$.

3. Three of the vectors below are parallel. Which are the parallel vectors? Give reasons for your answer.

a. $\overrightarrow{AB} = 2\mathbf{m} + 4\mathbf{n}$ b. $\overrightarrow{CD} = 6\mathbf{m} - 12\mathbf{n}$ c. $\overrightarrow{EF} = 4\mathbf{m} + 8\mathbf{n}$

d. $\overrightarrow{GH} = -\mathbf{m} - 2\mathbf{n}$ e. $\overrightarrow{IJ} = 6\mathbf{m} + 16\mathbf{n}$

Given: 5 vectors

a.	$\overrightarrow{AB}$	=	$2\mathbf{m} + 4\mathbf{n}$	=	$2(\mathbf{m} + 2\mathbf{n})$
b.	$\overrightarrow{CD}$	=	$6\mathbf{m} - 12\mathbf{n}$	=	$6(\mathbf{m} - 2\mathbf{n})$
c.	$\overrightarrow{EF}$	=	$4\mathbf{m} + 8\mathbf{n}$	=	$4(\mathbf{m} + 2\mathbf{n})$
d.	$\overrightarrow{GH}$	=	$-\mathbf{m} - 2\mathbf{n}$	=	$-(\mathbf{m} + 2\mathbf{n})$
e.	$\overrightarrow{IJ}$	=	$6\mathbf{m} + 16\mathbf{n}$	=	$2(\mathbf{3m} + 8\mathbf{n})$

From the above it is clear that $\overrightarrow{AB}$, $\overrightarrow{EF}$ and $\overrightarrow{GH}$ are parallel, as they are scalar multiples of $(\mathbf{m} + 2\mathbf{n})$.

Practice

1. Which of the following vectors are parallel to $\begin{pmatrix}16\\20\end{pmatrix}$?

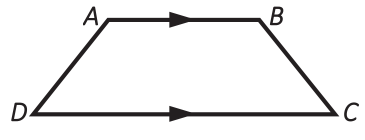
a. $\begin{pmatrix}20\\16\end{pmatrix}$ b. $\begin{pmatrix}-12\\-15\end{pmatrix}$ c. $\begin{pmatrix}8\\10\end{pmatrix}$ d. $\begin{pmatrix}-8\\10\end{pmatrix}$ e. $\begin{pmatrix}4\\5\end{pmatrix}$

2. The vector $\begin{pmatrix}k\\15\end{pmatrix}$ is parallel to the vector $\begin{pmatrix}4\\6\end{pmatrix}$, find k.

3. Which of the following vectors are perpendicular to $\begin{pmatrix}3\\8\end{pmatrix}$?

a. $\begin{pmatrix}-8\\-3\end{pmatrix}$ b. $\begin{pmatrix}24\\-9\end{pmatrix}$ c. $\begin{pmatrix}-8\\3\end{pmatrix}$ d. $\begin{pmatrix}-3\\-8\end{pmatrix}$ e. $\begin{pmatrix}8\\-3\end{pmatrix}$ f. $\begin{pmatrix}-3\\8\end{pmatrix}$ g. $\begin{pmatrix}3\\-8\end{pmatrix}$

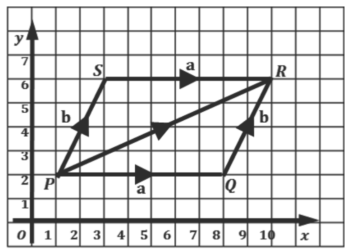
4. A regular trapezium $ABCD$ is shown in the diagram. $\overrightarrow{DC}$ is parallel to $\overrightarrow{AB}$ and $\overrightarrow{DC} : \overrightarrow{AB} = 2 : 1$. If $\overrightarrow{AB} = \mathbf{m}$, express $\overrightarrow{CD}$ in terms of $\mathbf{m}$.



Lesson Title: Parallelogram law of vector addition	**Theme:** Vectors and Transformations
Lesson Number: PHM3-L083	**Class:** SSS 3
Learning Outcome By the end of the lesson, you will be able to add two vectors using the parallelogram law of vector addition.	

Overview

We are often required to find the resultant vector of 2 vectors starting from the same origin.
Consider the parallelogram $PQRS$:
The opposite sides are equal in length and are parallel.



Since $\overrightarrow{PQ}$ and $\overrightarrow{SR}$ are in the same direction,

$$\overrightarrow{PQ} = \overrightarrow{SR} \quad (1)$$
$$= \mathbf{a}$$

Similarly,

$$\overrightarrow{PS} = \overrightarrow{QR} \quad (2)$$
$$= \mathbf{b}$$

From the triangle law of vector addition,

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$
$$= \mathbf{a} + \mathbf{b}$$
$$\Longrightarrow \quad \overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{PS} \quad \text{from equations (1) and (2)}$$

Let $\overrightarrow{PR} = \mathbf{c}$

then $\mathbf{a} + \mathbf{b} = \mathbf{c}$ (3) where $\mathbf{c}$ is the **resultant vector** of the two vectors $\mathbf{a}$ and $\mathbf{b}$

The parallelogram law of vector addition states that when two vectors are represented by two adjacent sides of a parallelogram by magnitude and direction, then the resultant of these vectors is represented in magnitude and direction by the diagonal of the parallelogram starting from the same point.

If $\mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$,

then $\mathbf{a} + \mathbf{b} = \mathbf{c} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$ as before

Solved Examples

1. Use the diagram above to write the column vectors for $\mathbf{a}$ and $\mathbf{b}$.
   Find the resultant vector$\mathbf{c}$ using the parallelogram law of vector addition.
   Solution:
   Step 1. Assess and extract the given information from the problem.

Given: from above.$\overrightarrow{PQ} = \overrightarrow{SR} = \mathbf{a}$, $\overrightarrow{PS} = \overrightarrow{QR} = \mathbf{b}$

Step 2. Write down the column vectors for a and b

From the diagram: $\mathbf{a} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$

$\mathbf{b} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

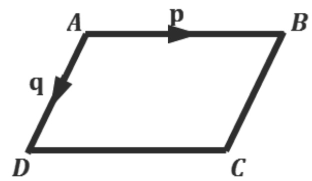
Step 3. Add by using the parallelogram law of vector addition $\mathbf{a} + \mathbf{b} = \mathbf{c}$,

where $\mathbf{c} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$.

$$\mathbf{a} + \mathbf{b} = \mathbf{c}$$
$$= \begin{pmatrix} 7 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 7+2 \\ 0+4 \end{pmatrix}$$
$$= \begin{pmatrix} 9 \\ 4 \end{pmatrix}$$

Step 4. Write the answer.

The resultant vector $\mathbf{c} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$



2. $ABCD$ is a parallelogram.
   $\overrightarrow{AB} = \mathbf{p}$ and is parallel to $\overrightarrow{DC}$. $\overrightarrow{AD} = \mathbf{q}$ and is parallel to $\overrightarrow{BC}$.
   Express in terms of $\mathbf{p}$ and $\mathbf{q}$:

   a. $\overrightarrow{AC}$ b. $\overrightarrow{BD}$

   c. AC and BD intersect at T. Express $\overrightarrow{AT}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

Solutions:

Given: $\overrightarrow{AB} = \mathbf{p}, \overrightarrow{AD} = \mathbf{q}$

a. $\overrightarrow{DC} = \mathbf{p}$ parallel

$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}$

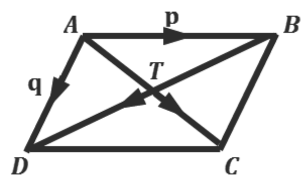
$\overrightarrow{AC} = \mathbf{p} + \mathbf{q}$

b. $\overrightarrow{BC} = \mathbf{q}$ parallel

$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$

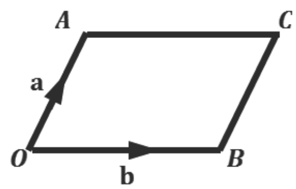
$\overrightarrow{AC} = \mathbf{q} + (-\mathbf{p})$

$= \mathbf{q} - \mathbf{p}$



c. $\overrightarrow{AT} = \frac{1}{2}\overrightarrow{AC}$ T is the mid-point of $\overrightarrow{AC}$

$= \frac{1}{2}(\mathbf{p} + \mathbf{q})$



3. $OACB$ is a parallelogram with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.
   P is the mid-point of AB. Q is the mid-point of OC.
   Express in terms of $\mathbf{a}$ and $\mathbf{b}$:

   a. $\overrightarrow{OP}$ b. $\overrightarrow{QO}$

c. What do your answers show about the points P and Q?

d. What property of a parallelogram has been proved by this question?

Solutions:

Given: $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$

a.
$$\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$$
$$\overrightarrow{AB} = -\mathbf{a} + \mathbf{b}$$
$$\overrightarrow{\boldsymbol{OP}} = \mathbf{p} = \frac{1}{2}(-\mathbf{a} + \mathbf{b})$$

b.
$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$$
$$\overrightarrow{OC} = \mathbf{a} + \mathbf{b}$$
$$\overrightarrow{\boldsymbol{OQ}} = \mathbf{q} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$$

c. Points P and Q are the same point.

d. The diagonals of a parallelogram bisect each other.

Practice

1. In a parallelogram ABCD, $\overrightarrow{AB} = \mathbf{a} = m\binom{-1}{2}$ and $\overrightarrow{AD} = \mathbf{b} = n\binom{4}{3}$. Find m and n such that the resultant vector$\overrightarrow{AC}$= $\mathbf{c} = \binom{0}{22}$.

2. The points $A(-2,1), B(2,1)$, $C(5,3)$ and $D(p, q)$ are the vertices of a parallelogram ABCD. Write down $\overrightarrow{AB}$, $\overrightarrow{DC}$ and $\overrightarrow{BD}$ as column vectors and deduce the values of p and q.

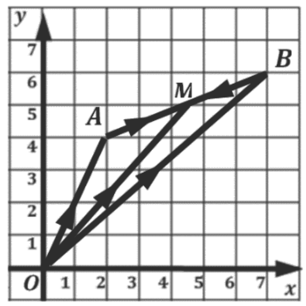
3. The points $A(-2,3), B(2,-1)$, $C(5,0)$ and $D(x, y)$ are the vertices of a parallelogram ABCD.
   a. Find $\overrightarrow{AB}$ and $\overrightarrow{DC}$.
   b. Find the coordinates of D.

4. The points $P(3,4), Q(1,1)$, $R(6,2)$ and $S(x, y)$.are the vertices of a parallelogram PQRS.
   a. Write down $\overrightarrow{PQ}$ $\overrightarrow{SR}$ and $\overrightarrow{QS}$ as column vectors.
   b. Deduce the values of x and y.

Lesson Title: Application of vectors – Part 1	**Theme:** Vectors and Transformations
Lesson Number: PHM3-L084	**Class:** SSS 3

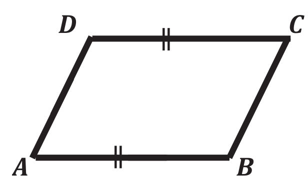
Learning Outcome
By the end of the lesson, you will be able to apply vectors to solve simple geometric problems.

Overview



Vectors can be used to solve simple problems in geometry.
We can describe any point in the Cartesian plane relative to the origin using position vectors.
We know that given any point $A(x, y)$, the position vector relative to O is $\overrightarrow{OA} = \binom{x}{y}$.
We also know that line segments can be described using position vectors.
For example, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$, where AB is a line segment and $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are position vectors of the end-points of AB.
We can also find the position vector $\overrightarrow{OM}$ of the mid-point of any line segment using the formula $\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$.

Solved Examples



1. A parallelogram $ABCD$ has vertices $A(-6, -3), B(1, -3)$, and $C(3,1)$. Find the co-ordinates of the vertex D.

Solution:

Step 1. Assess and extract the given information from the problem.
Given: co-ordinates of vertices $A(-6, -3), B(1, -3)$, and $C(3,1)$

Step 2. Write down the position vectors of each point
(remember position vectors are relative to the origin)

Let $\overrightarrow{OA} = \mathbf{a} = \binom{-6}{-3}$ $\quad \overrightarrow{OB} = \mathbf{b} = \binom{1}{-3}$

$\overrightarrow{OC} = \mathbf{c} = \binom{3}{1}$ $\quad \overrightarrow{OD} = \mathbf{b} = \binom{x}{y}$

Since $ABCD$ is a parallelogram.

$\overrightarrow{AB} = \overrightarrow{DC}$ (1)

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \mathbf{b} - \mathbf{a}$

$= \binom{1}{-3} - \binom{-6}{-3} \quad = \binom{7}{0}$

$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD}$

$= \mathbf{c} - \mathbf{d}$

$= \binom{3}{1} - \binom{x}{y} \quad = \binom{3-x}{1-y}$

$$\begin{aligned} \begin{pmatrix}7\\0\end{pmatrix} &= \begin{pmatrix}3-x\\1-y\end{pmatrix} && \text{from equation (1)} \\ 7 &= 3-x \\ \Longrightarrow \quad x &= -4 \\ 0 &= 1-y \\ \Longrightarrow \quad y &= 1 \end{aligned}$$

Step 3. Write the answer.

The co-ordinates of $D = (-4,1)$.

2. A triangle ABC has vertex $A(3,7)$. $\overrightarrow{BA} = \begin{pmatrix}-1\\6\end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix}4\\-2\end{pmatrix}$.

   Find the co-ordinates of: a. B b. C

   Solutions:

   Given: triangle ABC with $A(3,7)$, $\overrightarrow{BA} = \begin{pmatrix}-1\\6\end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix}4\\-2\end{pmatrix}$

   Let position vectors for A: $\overrightarrow{OA} = \begin{pmatrix}3\\7\end{pmatrix}$, B: $\overrightarrow{OB}$ and C: $\overrightarrow{OC}$

   a.
$$\begin{aligned} \overrightarrow{BA} &= \overrightarrow{OA} - \overrightarrow{OB} \\ \begin{pmatrix}-1\\6\end{pmatrix} &= \begin{pmatrix}3\\7\end{pmatrix} - \overrightarrow{OB} \\ \overrightarrow{OB} &= \begin{pmatrix}3\\7\end{pmatrix} - \begin{pmatrix}-1\\6\end{pmatrix} \\ &= \begin{pmatrix}3--1\\7-6\end{pmatrix} \\ &= \begin{pmatrix}4\\1\end{pmatrix} \end{aligned}$$

   The co-ordinates of $B = (4,1)$.

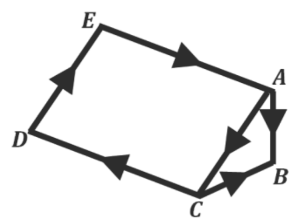
   b.
$$\begin{aligned} \overrightarrow{BC} &= \overrightarrow{OC} - \overrightarrow{OB} \\ \begin{pmatrix}4\\-2\end{pmatrix} &= \overrightarrow{OC} - \begin{pmatrix}4\\1\end{pmatrix} \\ \overrightarrow{OC} &= \begin{pmatrix}4\\-2\end{pmatrix} + \begin{pmatrix}4\\1\end{pmatrix} \\ &= \begin{pmatrix}4+4\\-2+1\end{pmatrix} \\ &= \begin{pmatrix}8\\-1\end{pmatrix} \end{aligned}$$

   The co-ordinates of $C = (8,-1)$.

3. The points A, B, C, D and E are the vertices of a pentagon. O is the origin.

   $\overrightarrow{AB} = \begin{pmatrix}0\\-2\end{pmatrix}$, $\overrightarrow{CB} = \begin{pmatrix}2\\1\end{pmatrix}$, $\overrightarrow{CD} = \begin{pmatrix}-5\\2\end{pmatrix}$ and $\overrightarrow{DE} = \begin{pmatrix}2\\3\end{pmatrix}$.

   Show that $ACDE$ is a parallelogram.



not to scale

   Solution:

   Given: pentagon $ABCDE$, origin O,

   $\overrightarrow{AB} = \begin{pmatrix}0\\-2\end{pmatrix}$, $\overrightarrow{CB} = \begin{pmatrix}2\\1\end{pmatrix}$, $\overrightarrow{CD} = \begin{pmatrix}-5\\2\end{pmatrix}$ and $\overrightarrow{DE} = \begin{pmatrix}2\\3\end{pmatrix}$.

   If $ACDE$ is a parallelogram, then $\overrightarrow{AC}$ should be parallel to $\overrightarrow{DE}$ and $\overrightarrow{EA}$ should be parallel to $\overrightarrow{CD}$.

$$\begin{aligned} \overrightarrow{AC} &= \overrightarrow{AB} + \overrightarrow{BC} && \text{triangle law} \\ \overrightarrow{BC} &= \begin{pmatrix}-2\\-1\end{pmatrix} \\ \overrightarrow{AC} &= \begin{pmatrix}0\\-2\end{pmatrix} + \begin{pmatrix}-2\\-1\end{pmatrix} \\ &= \begin{pmatrix}0+(-2)\\-2+(-1)\end{pmatrix} = \begin{pmatrix}-2\\-3\end{pmatrix} \end{aligned}$$

   $\overrightarrow{DE} = \begin{pmatrix}2\\3\end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix}-2\\-3\end{pmatrix}$ therefore they are parallel

$$\begin{aligned} \overrightarrow{EC} &= \overrightarrow{EA} + \overrightarrow{AC} && \text{triangle law} \\ \overrightarrow{EA} &= \overrightarrow{EC} - \overrightarrow{AC} \\ \overrightarrow{CE} &= \overrightarrow{CD} + \overrightarrow{DE} && \text{find } \overrightarrow{EC} = -\overrightarrow{CE} \end{aligned}$$

$$= \begin{pmatrix}-5\\2\end{pmatrix} + \begin{pmatrix}2\\3\end{pmatrix}$$

$$= \begin{pmatrix}-5+(2)\\2+3\end{pmatrix} = \begin{pmatrix}-3\\5\end{pmatrix}$$

$$\overrightarrow{EC} = \begin{pmatrix}3\\-5\end{pmatrix}$$

$$\overrightarrow{EA} = \begin{pmatrix}3\\-5\end{pmatrix} - \begin{pmatrix}-2\\-3\end{pmatrix}$$

$$= \begin{pmatrix}3-(-2)\\-5-(-3)\end{pmatrix} = \begin{pmatrix}5\\-2\end{pmatrix}$$

$\overrightarrow{CD} = \begin{pmatrix}-5\\2\end{pmatrix}$ and $\overrightarrow{EA} = \begin{pmatrix}5\\-2\end{pmatrix}$ therefore they are parallel

Hence $ACDE$ is a parallelogram.

Practice

1. PQRS is a quadrilateral with P (2, 2), S (4, 4) and R (6, 4). If $\overrightarrow{PQ} = 4\overrightarrow{SR}$, find the co-ordinates of Q.

2. ABCD is a quadrilateral. A, B and D have co-ordinates (4,-7), (2, 4) and (10, 5) respectively. If $\overrightarrow{AD} = 3\overrightarrow{BC}$, Find the co-ordinates of C.

3. *O* (0, 0), A (3, 4) and *B* (5, 2) are the co-ordinates of a quadrilateral OABC. If $\overrightarrow{BC} = \begin{pmatrix}4\\1\end{pmatrix}$, find: a. The co-ordinates of C; b. $\overrightarrow{AC}$.

4. A (1,-2) is a vertex of the quadrilateral ABCD. $\overrightarrow{AB} = \begin{pmatrix}2\\1\end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix}1\\-1\end{pmatrix}$, and $\overrightarrow{CD} = \begin{pmatrix}-1\\-2\end{pmatrix}$.
   a. Find the co-ordinates of B, C and D.
   b. If M is the mid-point of $\overrightarrow{AB}$, find $\overrightarrow{MC}$.

5. A is the point (5,- 3) and B is the point (0,9).
   a. Express $\overrightarrow{AB}$ in the form $\begin{pmatrix}x\\y\end{pmatrix}$.
   b. Calculate the length AB.

Lesson Title: Application of vectors – Part 2	Theme: Vectors and Transformations
Lesson Number: PHM3-L085	**Class:** SSS 3
Learning Outcome By the end of the lesson, you will be able to apply vectors to solve simple real-world problems.	

Overview

Vectors can be used to solve simple problems in the real world.
Problems usually require us to find the magnitude and direction of the resultant vector of forces, velocities or displacement.

For a vector $\binom{x}{y}$:

- The magnitude of the vector is given by $\sqrt{x^2 + y^2}$ (Pythagoras' Theorem).
- To find the direction or bearing of the vector:
  - Draw a sketch of the problem; find the acute angle θ, the vector makes with the x-axis. This is given by $\tan\theta = \frac{y}{x}$.
  - Use the sketch to find the direction of the vector.

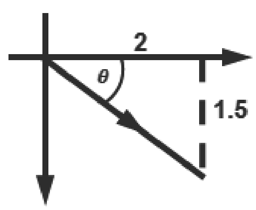
Solved Examples

1. A man paddles his canoe east at 2 m/s across a river. If the river flows south with a current of 1.5 m/s.
   a. Express the resultant velocity of the canoe as a column vector.
   b. Find the resultant velocity of the canoe giving the direction as a bearing.
   c. If the river is 20 m across, how long does it take the man to cross the river?

Solutions:

a. **Step 1.** Assess and extract the given information from the problem.
Given: speed of canoe east = 2 m/s,
speed of river current south = 1.5 m/s

Step 2. Write the position vectors of canoe and current.
Sketch the diagram (not to scale)
Since movement east is positive on the x-axis and movement south is negative on the y-axis



position vector of canoe $= \binom{2}{0}$

position vector of current $= \binom{0}{-1.5}$

$\therefore$ resultant velocity of canoe $= \binom{2}{0} + \binom{0}{-1.5}$

$= \binom{2}{-1.5}$

Step 3. Find the magnitude of the velocity.

magnitude of velocity $= \sqrt{(2)^2 + (-1.5)^2} = \sqrt{4 + 2.25}$

$= \sqrt{6.25}$

$= 2.5$ m/s

Step 4. Write the answer to a.

The velocity has magnitude 2.5 m/s.

b. **Step 5.** Find the acute angle θ, the vector makes with the x-axis.

$\tan\theta = \frac{1.5}{2} = 0.75$ from diagram

$\theta = \tan^{-1}(0.75)$

$= 36.87°$

Step 6. Find the direction of the velocity.

direction of velocity $= 90 + 36.87$ from diagram

$= 126.87°$

The velocity has direction $127°$.

c. **Step 7.** Find the time it takes the man to cross river.

width of river $= 20$ m given

time taken to cross $t = \frac{20}{2.5}$ from distance formula $d = st$

$= 8$ s

The time taken to cross the river is 8 s.

2. The location of two towns are shown on a map as $X(3,5)$ and $Y(\mathrm{a,b})$ relative to a point O on the map. The displacement between the towns is given as $\overrightarrow{XY} = \binom{4}{-2}$.

Find:

a. The position of Y on the map.

b. The distance on the map between the towns.

c. The bearing of X from Y correct to the nearest degree.

Solutions:

Given: $X(3,5)$, $Y(\mathrm{a,b})$ and $\overrightarrow{XY} = \binom{4}{-2}$

Let position vectors for X: $\overrightarrow{OX} = \binom{3}{5}$, Y: $\overrightarrow{OY} = \binom{a}{b}$

a.

$\overrightarrow{XY} = \overrightarrow{OY} - \overrightarrow{OX}$

$\binom{4}{-2} = \binom{a}{b} - \binom{3}{5}$

$\binom{a}{b} = \binom{4}{-2} + \binom{3}{5}$

$= \binom{4+3}{-2+5}$

$= \binom{7}{3}$

Position of Y on map = $(7,3)$

b. distance between the towns, $\overrightarrow{XY} = \sqrt{(4)^2 + (-2)^2}$

$= \sqrt{16+4}$

$= \sqrt{20}$

$= 4.472$

$= 4.5$ units

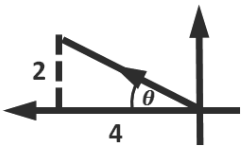
distance on the map between the towns = 4.5 units to 1 d.p.

c. $\tan\theta = \frac{2}{4} = 0.5$

$\theta = \tan^{-1}(0.5) = 26.565$

bearing $= 270 + 26.565$

$= 296.565$



The bearing of X from $Y = 297°$ to the nearest degree.

Practice

1. An aeroplane flew from town A, travels 200 km on a bearing of $135^{\circ}$to town B. It then travels 250 km from town B on a bearing of $045^{\circ}$to town C. Find using the vector approach:
    a. The distance of A from C.to the nearest kilometre.
    b. The bearing of A from C to nearest degree.

2. Village P is 10 km from a lorry station, Q, on a bearing 065°. Another village R, is 8 km from Q, on a bearing 155°. Calculate:
    a. The distance of R from P, to the nearest kilometre.
    b. The bearing of R from P, to the nearest degree.

3. An aircraft flies from a point A to a point B, x km away on a bearing of 135°. From B, the aircraft flies to a point C, 2x km away on a bearing of 225°. If the distance from A to C is 350 km, find, correct to the nearest whole number:
    a. The distance from A to B.
    b. The bearing of C from A.

4. A boy running in a northern direction later changes his direction and starts running toward in an eastern direction. If the distance he covered in the northern direction is 4 km and in the eastern direct eastern direction is 3 km, determine:
    a. The magnitude of his resultant.
    b. His direction. Give your answer to the nearest whole number.

5. A village X is 15 km from a train station, Y, on a bearing 078°. Another village Z is 11 km from Y, on a bearing 168°. Calculate:

Lesson Title: Application of vectors – Part 3	**Theme:** Vectors and Transformations
Lesson Number: PHM3-L086	**Class:** SSS 3
Learning Outcome By the end of the lesson, you will be able to apply vectors to solve more real-world problems.	

Overview

This lesson will look at more problems using vectors to solve real-world problems.

The solution to Question 1 provides a step-by-step method of solving a problem when we are given both the magnitude and direction (bearing) of a vector.

Follow the solution carefully. You will notice that in general we can write the components of any vector given as a magnitude and direction or bearing as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\theta \\ a\sin\theta \end{pmatrix} \qquad (1)$$

where a is the magnitude of the vector
θ is the acute angle the bearing makes with the x-axis

It is useful to always draw a diagram to then help us with writing the direction of any resultant vector measured from the North.
Question 2 provides a Solved Example on how to use equation (1) to find the components of the resultant vector.

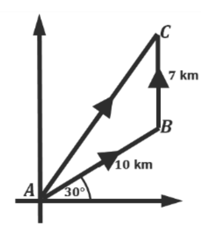
Solved Examples

1. A cyclist starts from A and travels a distance of 10 km in the direction $060°$ to B. She then travels 7 km north to C.
   a. Draw a sketch of the journey from A to C.
   b. Write $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ as column vectors.
   c. Find the cyclist's distance and bearing from A.

   Solutions:

   a. **Step 1.** Assess and extract the given information from the problem.
   Given: 2 stage journey 1st stage: from A to B in direction $060°$ for 10 km
   2nd stage: from B to C north for 7 km

   Step 2. Sketch the diagram – shown at right (not to scale) i.



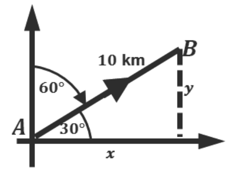
   b. Write vector $\overrightarrow{AB}$ in the form $\begin{pmatrix} x \\ y \end{pmatrix}$

   From the diagram on the next page, for the 1st stage, a $060°$ bearing makes an acute angle of $30°$ with the x-axis

   $\therefore \quad \cos 30° = \frac{x}{10}$ use the cosine ratio

$$x = 10\cos 30° = 10 \times 0.8660$$

ii.



$$= 8.660$$
$$= 8.7 \text{ km}$$

Similarly $\sin 30° = \frac{y}{10}$

$$y = 10\sin 30° = 10 \times 0.5000$$
$$= 5 \text{ km}$$

$\therefore \quad \overrightarrow{AB} = \begin{pmatrix} 8.7 \\ 5 \end{pmatrix}$

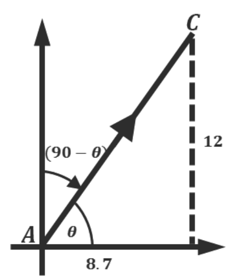
For the 2nd stage of the journey

$\overrightarrow{BC} = \begin{pmatrix} 0 \\ 7 \end{pmatrix}$ since $\overrightarrow{BC}$ is 7 km North

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$
$$= \begin{pmatrix} 8.7 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ 7 \end{pmatrix}$$
$$= \begin{pmatrix} 8.7+0 \\ 5+7 \end{pmatrix}$$
$$\overrightarrow{AC} = \begin{pmatrix} 8.7 \\ 12 \end{pmatrix}$$

c. Magnitude of $\overrightarrow{AC} = \sqrt{(8.7)^2 + (12)^2}$ use Pythagoras' Theorem

$$= \sqrt{75.69 + 144}$$
$$= \sqrt{219.69}$$
$$= 14.862$$
$$= 14.9 \text{ km}$$



From the diagram

$$\tan\theta = \frac{12}{8.7}$$
$$\theta = \tan^{-1}\left(\frac{12}{8.7}\right)$$
$$= 54.058$$

Direction of $\overrightarrow{AC} = 90 - 54.058 = 35.942$

The cyclist is at a distance of 14.9 km from A at direction $036°$ to the nearest degree.

2. A plane flies from P to Q traveling a distance of 300 km at a bearing of $150°$ to Q. It then travels 350 km at a bearing of $60°$ to R.
   a. Draw a sketch of the journey from P to R.
   b. Write $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{PR}$ as column vectors.
   c. Find the plane's distance and bearing from P to the nearest whole number.

Solution:

Given: 2 stage journey 1st stage: 300 km from P to Q at a bearing of $150°$ to Q;
2nd stage: 350 km from Q to R at a bearing of $60°$ to R.

a. Sketch of journey shown below (not to scale)

b. From the sketch:

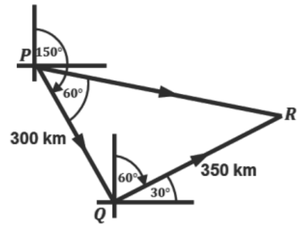
$$\overrightarrow{PQ} = \begin{pmatrix} 300\cos 60° \\ -300\sin 60° \end{pmatrix} = \begin{pmatrix} 150 \\ -260 \end{pmatrix}$$

$$\overrightarrow{QR} = \begin{pmatrix} 350\cos 30° \\ 350\sin 30° \end{pmatrix} = \begin{pmatrix} 303 \\ 175 \end{pmatrix}$$

$$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} = \begin{pmatrix} 150 \\ -260 \end{pmatrix} + \begin{pmatrix} 303 \\ 175 \end{pmatrix} = \begin{pmatrix} 150+303 \\ -260+175 \end{pmatrix}$$

$$\overrightarrow{PR} = \begin{pmatrix} 453 \\ -85 \end{pmatrix}$$

i.



Magnitude of $\overrightarrow{PR}$

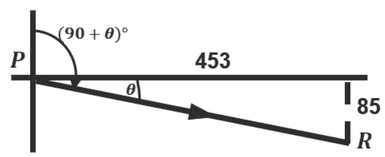
$$|\overrightarrow{PR}| = \sqrt{(453)^2 + (-85)^2} = \sqrt{205209 + 7225} = \sqrt{212434} = 460.906 = 461 \text{ km}$$

Direction of $\overrightarrow{PR}$

$$\tan\theta = \frac{85}{453} = 0.1876$$

$$\theta = \tan^{-1}(0.1876) = 10.625°$$

$$\text{bearing} = 90 + 10.625° = 100.625°$$



The plane is 461 km from P at a bearing of $101°$.

Practice

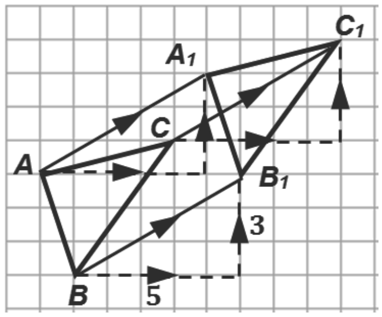
1. A boat starts from P and travels a distance of 8 km in the direction $040°$ to Q.
   a. Draw a sketch of the journey from P to Q.
   b. Write the resultant displacement $\overrightarrow{PQ}$ as a column vector.
2. A boat starts from A and travels a distance of 9 km in the direction $070°$ to B.
   a. Draw a sketch of the journey from A to B.
   b. Write the resultant displacement $\overrightarrow{AB}$ as a column vector.
3. A cyclist starts from X and travels a distance of 12 km in the direction $075°$ to Y. She then travels 8 km north to Z.
   a. Draw a sketch of the journey from X to Z.
   b. Write$\overrightarrow{XY}$, $\overrightarrow{YZ}$ and $\overrightarrow{XZ}$ as column vectors.
   c. Find the cyclist's distance and bearing from $X to\ Z$.
4. A ship travels at a speed of 3 m/s due east. A current moves the water due south at a speed of 1.2 m/s.
   a. Write the velocity of the ship and water as a column vector.
   b. Find the actual velocity of the ship giving the direction as a bearing.

Lesson Title: Translation – Part 1	**Theme:** Vectors and Transformations
Lesson Number: PHM3-L087	**Class:** SSS 3

Learning Outcomes
By the end of the lesson, you will be able to:
1. Identify that translation moves an object without changing its size or shape.
2. Use vectors to translate given points and images.

Overview

Consider the triangle *ABC* shown at right. It was translated from its original position 5 units right and 3 units up.
A translation moves all the points of an object in the same direction and the same distance without changing its shape or size.



The column vectors which show the movements of *A*, *B* and *C* to their new positions A_1, B_1 and C_1 are given by:

$$\overrightarrow{AA_1} = \binom{5}{3}, \overrightarrow{BB_1} = \binom{5}{3} \text{ and } \overrightarrow{CC_1} = \binom{5}{3}$$

Since all 3 points move according to the same column vector, we can conclude that all the points on triangle ABC moved by the same column vector $\binom{5}{3}$ to $A_1B_1C_1$.
The vector $\binom{5}{3}$ is called a **translation** vector, $\mathbf{v}$.
In general, a translation vector $\binom{a}{b}$ moves a point $A(x, y)$ along the x- and y-axes by the amount of the components of the vector.

The new point $A_1(x_1, y_1)$, called the **image point,** will have co-ordinates $x_1 = x + a$, $y_1 = y + b$.

We can write a mapping for the translation as:

$$\binom{x}{y} \longrightarrow \binom{x}{y} + \binom{a}{b} = \binom{x+a}{y+b}$$

Similarly, $(x, y) \longrightarrow (x + a, y + b)$

Solved Examples

1. Find the coordinates of the image of the point P $(4,2)$ when it is translated by the vector $\binom{-4}{3}$.

   Solution:

   Step 1. Assess and extract the given information from the problem.
   Given: point P $(4,2)$, translation vector $\binom{-4}{3}$

   Step 2. Write the mapping for the translation.

   $$\binom{x}{y} \longrightarrow \binom{x}{y} + \binom{a}{b} = \binom{x+a}{y+b}$$

$$\begin{pmatrix}4\\2\end{pmatrix} \longrightarrow \begin{pmatrix}4\\2\end{pmatrix}+\begin{pmatrix}-4\\3\end{pmatrix} = \begin{pmatrix}4+(-4)\\2+3\end{pmatrix}=\begin{pmatrix}0\\5\end{pmatrix}$$
$$\begin{pmatrix}4\\2\end{pmatrix} \longrightarrow \begin{pmatrix}0\\5\end{pmatrix}$$

Step 3. Write the answer.

The co-ordinates of the image $= (0,5)$.

2. $A'(-2,4)$ is the image of a point A under the translation by the vector $\begin{pmatrix}2\\-1\end{pmatrix}$. Find the co-ordinates of point A.

Solution:

Given: image $A'(-2,4)$ of point A under translation vector $\begin{pmatrix}2\\-1\end{pmatrix}$

$$\begin{pmatrix}x\\y\end{pmatrix} \longrightarrow \begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}x+a\\y+b\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix} \longrightarrow \begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}2\\-1\end{pmatrix}$$ using the translation vector

$$\begin{pmatrix}x\\y\end{pmatrix} \longrightarrow \begin{pmatrix}-2\\4\end{pmatrix}$$ using the image point

$$\Longrightarrow \begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}-2\\4\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-2\\4\end{pmatrix}-\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}-2-2\\4-(-1)\end{pmatrix}$$

$$= \begin{pmatrix}-4\\5\end{pmatrix}$$

$$\begin{pmatrix}-4\\5\end{pmatrix} \longrightarrow \begin{pmatrix}-2\\4\end{pmatrix}$$ mapping for the translation

The co-ordinates of point A under translation vector $\begin{pmatrix}2\\-1\end{pmatrix}$ to image point $A'(-2,4) = (-4,5)$.

3. $Q'(1,-3)$ is the image of the point $Q(5,2)$ under translation by a vector. Find the translation vector**.**

Solution:

Given: point $Q(5,2)$, image $Q'(1,-3)$ under a translation vector

$$\begin{pmatrix}x\\y\end{pmatrix} \longrightarrow \begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}x+a\\y+b\end{pmatrix}$$

$$\begin{pmatrix}5\\2\end{pmatrix} \longrightarrow \begin{pmatrix}5\\2\end{pmatrix}+\begin{pmatrix}a\\b\end{pmatrix}$$ using the translation vector

$$\begin{pmatrix}5\\2\end{pmatrix} \longrightarrow \begin{pmatrix}1\\-3\end{pmatrix}$$ using the image point

$$\Longrightarrow \begin{pmatrix}5\\2\end{pmatrix}+\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}1\\-3\end{pmatrix}$$

$$\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}1\\-3\end{pmatrix}-\begin{pmatrix}5\\2\end{pmatrix} = \begin{pmatrix}1-5\\-3-2\end{pmatrix}$$

$$= \begin{pmatrix}-4\\-5\end{pmatrix}$$ translation vector

The translation vector which maps $Q(5,2)$ to $Q'(1,-3) = \begin{pmatrix}-4\\-5\end{pmatrix}$.

4. $P'(5\,,2)$ is the image of the point $P(2\,,-5)$ by the translation vector $\mathbf{v}$. Find
   a. The vector $\mathbf{v}$**.**
   b. The co-ordinates of point Q which maps onto point $Q'(-5,-2)$ under $\mathbf{v}$**.**

Solutions:

Given: point $P(5\,,2)$, image $P'(2\,,-5)$, translation vector $\mathbf{v}$

$$\begin{pmatrix}x\\y\end{pmatrix} \longrightarrow \begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}x+a\\y+b\end{pmatrix}$$

a. vector from $P(5\,,2)$ to $P'(2\,,-5)$

$$\begin{pmatrix}5\\2\end{pmatrix} \longrightarrow \begin{pmatrix}5\\2\end{pmatrix} + \mathbf{v} \qquad \mathbf{v} = \begin{pmatrix}a\\b\end{pmatrix}$$

$$\begin{pmatrix}5\\2\end{pmatrix} \longrightarrow \begin{pmatrix}2\\-5\end{pmatrix}$$

$$\begin{pmatrix}5\\2\end{pmatrix} + \mathbf{v} = \begin{pmatrix}2\\-5\end{pmatrix}$$

$$\mathbf{v} = \begin{pmatrix}2\\-5\end{pmatrix} - \begin{pmatrix}5\\2\end{pmatrix}$$

$$= \begin{pmatrix}2-5\\-5-2\end{pmatrix}$$

$$= \begin{pmatrix}-3\\-7\end{pmatrix}$$

The translation vector $= \begin{pmatrix}-3\\-7\end{pmatrix}$.

b. Q given $Q'(-5,-2)$, $\mathrm{v} = \begin{pmatrix}-3\\-7\end{pmatrix}$

$$\begin{pmatrix}x\\y\end{pmatrix} \longrightarrow \begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}-3\\-7\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix} \longrightarrow \begin{pmatrix}-5\\-2\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}-3\\-7\end{pmatrix} = \begin{pmatrix}-5\\-2\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-5\\-2\end{pmatrix} - \begin{pmatrix}-3\\-7\end{pmatrix}$$

$$= \begin{pmatrix}-5-(-3)\\-2-(-7)\end{pmatrix}$$

$$= \begin{pmatrix}-2\\5\end{pmatrix}$$

Co-ordinates of $Q = (-2{,}5)$.

Practice

1. Find the co-ordinates of the images of the following points when they are translated by the vector $\begin{pmatrix}2\\3\end{pmatrix}$:
   a. A(3,2) b. B(-1,5) c. C(-1,-2) d. D(3,-3)

2. Find co-ordinates of the images of the following points when they are translated by the vector $\begin{pmatrix}-1\\-4\end{pmatrix}$:
   a. A(5, 6) b. B(-2,3) c. C(1,-4) d. D(-3,-3)

3. A'(2, 6) is the image of point A under the translation by the vector $\begin{pmatrix}-1\\2\end{pmatrix}$.
   Find the point A.

4. P'(-3, 4) is the image of point P(2,1) under a translation by a vector. Find the translation.

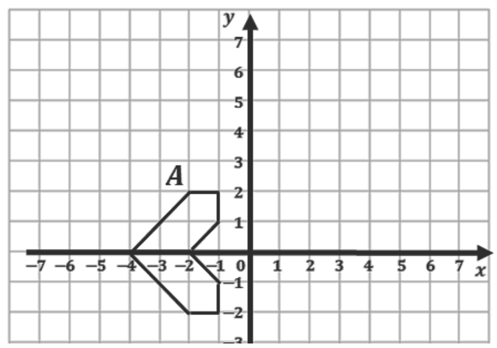
5. S'(-2, -6) is the image of a point S under the translation by the vector $\begin{pmatrix}2\\2\end{pmatrix}$. Find the point S.

Lesson Title: Translation – Part 2	**Theme:** Vectors and Transformations
Lesson Number: PHM3-L088	**Class:** SSS 3
Learning Outcome By the end of the lesson, you will be able to translate plane figures.	

Overview

This lesson focuses on using the information from the last lesson to translate plane figures.

Consider the figure shown at right. To translate the shape by any given translation vector, we can first translate the point A. We can then use the image A' obtained as a reference point to translate the whole shape by the given vector.



Solved Examples

1. On the same axes, draw the image of the shape above under the translation by the given vectors such that A maps onto A'.

   For each translation find the co-ordinates of A' and use it as the reference point to translate the object.

   a. $\binom{5}{1}$ b. $\binom{-3}{3}$ c. $\binom{7}{2}$

   Solutions:

   a. **Step 1.** Assess and extract the given information from the problem.

   Given: a shape with point A marked, translation vector $\binom{5}{1}$

   Step 2. Find the image point A' under the translation vector.

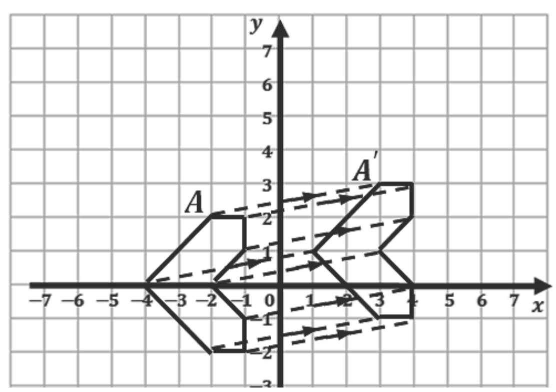
   co-ordinates of $A = (-2,2)$;

   $\binom{-2}{2} \longrightarrow \binom{-2}{2} + \binom{5}{1}$

   $= \binom{-2+5}{2+1}$

   $= \binom{3}{3}$

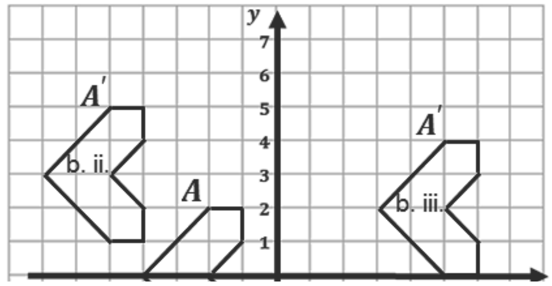
   co-ordinates of $A' = (3,3)$.



   Step 3. Draw the image using A' as the reference image point.

   b. Given: Use translation vector $\binom{-3}{3}$ to translate given object.

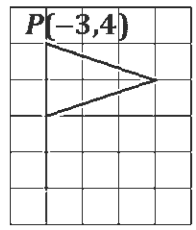
co-ordinates of $A = (-2,2)$;
$\binom{-2}{2} \longrightarrow \binom{-2}{2} + \binom{-3}{3} = \binom{-2+(-3)}{2+3} = \binom{-5}{5}$
co-ordinates of $A' = (-5,5)$
translation shown right.



c. Given: Use translation vector $\binom{7}{2}$ to translate given object.

co-ordinates of $A = (-2,2)$; $\binom{-2}{2} \longrightarrow \binom{-2}{2} + \binom{7}{2} = \binom{-2+7}{2+2} = \binom{5}{4}$
co-ordinates of $A' = (5,4)$

2. Using the axes as before:
   a. Draw the shape given.
   b. Translate the shape using the vector $\binom{4}{2}$ and $P(-3,4)$ as the reference point to give P_1.
   c. Translate the image using the vector $\binom{1}{-4}$ and P_1 the reference image point to give P_2.
   d. Which vector would be needed to translate the final image back to the position of the original shape?

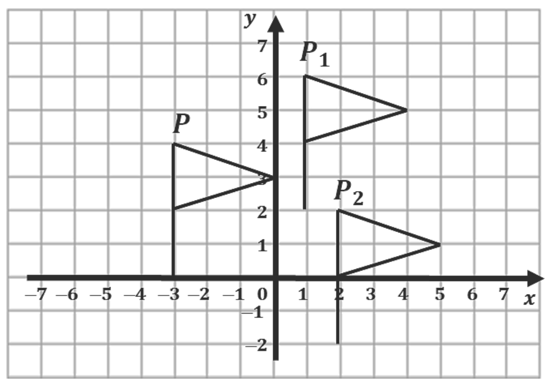


Solutions:

Given: shape to be translated

a. Shape and all images shown below.

b. Use translation vector $\binom{4}{2}$ to translate given object.
co-ordinates of $P = (-3,4)$;
$\binom{-3}{4} \longrightarrow \binom{-3}{4} + \binom{4}{2} = \binom{-3+4}{4+2} = \binom{1}{6}$
co-ordinates of $P_1 = (1,6)$



c. Use translation vector $\binom{1}{-4}$ to translate given object.
co-ordinates of $P_1 = (1,6)$
$\binom{1}{6} \longrightarrow \binom{1}{6} + \binom{1}{-4} = \binom{1+1}{6+(-4)} = \binom{2}{2}$
co-ordinates of $P_2 = (2,2)$

d. Since the object was translated using the vector sum $\binom{4}{2} + \binom{1}{-4}$, the inverse vector will take the final image back to the position of the original shape.

$$\text{translation vector } \mathbf{v} = -\left(\binom{4}{2} + \binom{1}{-4}\right) = -\binom{4+1}{2+(-4)}$$
$$= -\binom{5}{-2} = \binom{-5}{2}$$

Translation vector $\mathbf{v} = \binom{-5}{2}$ takes the final image back to the position of the original shape.

Practice

1. Using a scale of 2 cm to 2 units on each axis, draw on a graph sheet two perpendicular axes Ox and Oy for the intervals $-12 \le x \le 12$ and $-12 \le y \le 12$.
    a. Draw ∆EFG with vertices E(-2,7) F(-6,7) and G(-2,2).
    b. Draw the image $E_1F_1\,G_1$ of ∆EFG where $E \to E_1$, $F \to F_1$, $G \to G_1$, when it is translated by the vector $\binom{2}{4}$.
    c. Draw the image $E_2F_2\,G_2$ of ∆EFG where $E \to E_2$, $F \to F_2$, $G \to G_2$, when it is translated by the vector $\binom{-3}{-8}$.
    d. Draw the image $E_3F_3G_3$ of ∆EFG where $E \to E_3$, $F \to F_3$, $G \to G_3$, when it is translated by the vector $\binom{7}{-2}$.

2. Using a scale of 2 cm to 2 units on each axis, draw on a graph sheet two perpendicular axes Ox and Oy for the intervals $-2 \le x \le 12$ and $-12 \le y \le 12$.
    a. Draw ∆ABC with vertices A(4,-4) B(8,-8) and C(4,-8).
    b. Draw the image $A_1B_1\,C_1$ of ∆ABC where $A \to A_1$, $B \to B_1$, $C \to C_1$, when it is translated by the vector $\binom{3}{5}$.
    c. Draw the image $A_2B_2C_2$ of ∆ABC where $A \to A_2$, $B \to B_2$, $C \to C_2$, when it is translated by the vector $\binom{-3}{-3}$.
    d. Draw the image $A_3B_3\,C_3$ of ∆ABC where $A \to A_3$, $B \to B_3$, $C \to C_3$, when it is translated by the vector $\binom{3}{9}$.

3. Using a scale of 2 cm to 2 units on each axis, draw on a graph sheet two perpendicular axes Ox and Oy for the intervals $-12 \le x \le 12$ and $-12 \le y \le 12$.
    a. Draw ABCD with vertices A (2, 8) B (2, 4) C (8, 4) and D (8, 8).
    b. Draw the image $A_1B_1\,C_1D_1$ of ∆ABCD where $A \to A_1$, $B \to B_1$, $C \to C_1$, $D \to D_1$ when it is translated by the vector $\binom{-3}{1}$.
    c. Draw the image $A_2B_2C_2D_2$ of ∆ABCD where $A \to A_2$, $B \to B_2$, $C \to C_2$, $D \to D_2$ when it is translated by the vector $\binom{2}{-9}$.
    d. Draw the image $A_3B_3C_3D_3$ of ∆ABCD where $A \to A_3$, $B \to B_3$, $C \to C_3$, $D \to D_3$ when it is translated by the vector $\binom{-10}{-10}$.

Lesson Title: Reflection – Part 1	**Theme:** Vectors and Transformations
Lesson Number: PHM3-L089	**Class:** SSS 3

Learning Outcomes
By the end of the lesson, you will be able to:
1. Identify and explain the reflection of an object in the line $y = k$.
2. Identify and explain the reflection of an object in the line $x = k$.

Overview

Reflection is the image we see when we look at an object in a mirror.
Every point on the reflected image is the same distance away from the **line of reflection** or **mirror line** as the object. Distances from both object and image are always measured at right angles to the mirror line. The image has the same angles, lengths and area as the object, but its figure is reversed.

We can write the mappings for various reflections as shown below:

Reflection in the x-axis (i.e. $y = 0$) is given by:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ -y \end{pmatrix} \quad \text{giving} \quad (x, y) \longrightarrow (x, -y)$$

Reflection in the y-axis (i.e. $x = 0$) is given by:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} -x \\ y \end{pmatrix} \quad \text{giving} \quad (x, y) \longrightarrow (-x, y)$$

Reflection in the line $y = k$ or $y - k = 0$ is given by:

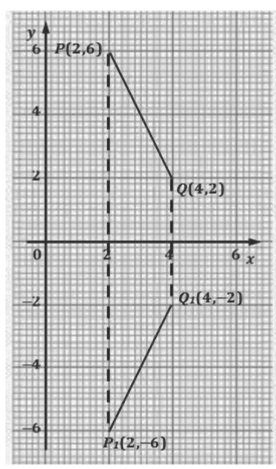
$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ 2k-y \end{pmatrix} \quad \text{giving} \quad (x, y) \longrightarrow (x, 2k - y)$$

Reflection in the line $x = k$ or $x - k = 0$ is given by:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 2k-x \\ y \end{pmatrix} \quad \text{giving} \quad (x, y) \longrightarrow (2k - x, y)$$

Solved Examples

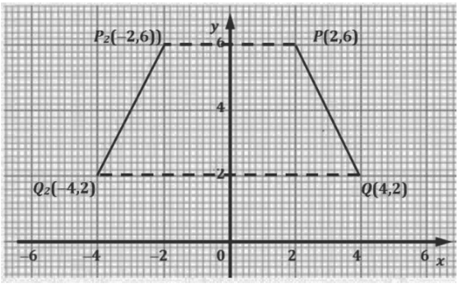
1. Points $P(2,6)$, $Q(4,2)$ are two points on the given axes. Find the co-ordinates, P_1 and Q_1, of the image of the line joining the points under reflection in the x-axis.



Solution:
Step 1. Assess and extract the given information from the problem. Given points $P(2,6)$, $Q(4,2)$
Step 2. Draw the x- and y- axes (if not already drawn). Locate the points P and Q on the graph. Draw the line joining the points.
Step 3. Draw a line at right angles from P to the mirror line (the x-axis). Measure this distance.

Step 4. Measure the same distance on the opposite side of the mirror line (the x-axis) to locate the point P_1 on the graph.
Step 5. Write the co-ordinates of $P_1 = (2, -6)$. Notice that the x-co-ordinates are the same as for P, the y-co-ordinates have opposite signs.
Step 6. Follow the same procedure for the point Q to give $Q_1(4, -2)$; see graph.

2. For the given points in question a., and using the same axes, find the co-ordinates, P_2 and Q_2, of the image under reflection in the y-axis ($x = 0$).
Write the mapping for each point. What do you notice?
Solution:
Given: points $P(2,6)$, $Q(4,2)$ as in question 1.
From the graph, the co-ordinates are $P_2(-2, 6)$ and $Q_2(-4,2)$



mapping for P $\binom{2}{6} \longrightarrow \binom{-2}{6}$ giving $P_2(-2,6)$
mapping for Q $\binom{4}{2} \longrightarrow \binom{-4}{2}$ giving $Q_2(-4,2)$
The x-co-ordinates have opposite signs, the y-co-ordinates are the same.

3. Using a scale of 2 cm to 2 units on both axis, draw on a graph sheet two perpendicular axes Ox and Oy for the intervals $-10 \leq x \leq 10$ and $-6 \leq y \leq 6$.
   a. Plot the points with co-ordinates: $A(1,3)$, $B(4,4)$, $C(4,2)$, $D(2,2)$.
   Join the points in that order to form shape $ABCD$.
   b. Reflect the object in the x- axis. Write down the co-ordinates of the corners of the image $A_1B_1C_1D_1$.
   c. Reflect $A_1B_1C_1D_1$, in the y- axis. Write down the co-ordinates of the image $A_2B_2C_2D_2$.
   d. Reflect $A_2B_2C_2D_2$ in the line $x = -5$. Write down the co-ordinates of $A_3B_3C_3D_3$.
   e. Reflect $A_3B_3C_3D_3$ in the line $y - 1 = 0$. Write down the co-ordinates of $A_4B_4C_4D_4$.

Solutions:
Given: $A(1,3), B(4,4), C(4,2), D(2,2)$
   a. See diagram at the end of the question.
   b. Reflect in $y = 0$ (x-axis): $\binom{x}{y} \longrightarrow \binom{x}{-y}$

$\binom{1}{3} \longrightarrow \binom{1}{-3}$ $\binom{4}{2} \longrightarrow \binom{4}{-2}$
$\binom{4}{4} \longrightarrow \binom{4}{-4}$ $\binom{2}{2} \longrightarrow \binom{2}{-2}$

Co-ordinates:: $A_1(1,-3)$, $B_1(4,-4)$, $C_1(4,-2), D_1(2,-2)$
   c. Reflect $A_1(1,-3)$, $B_1(4,-4)$, $C_1(4,-2), D_1(2,-2)$ in $x = 0$ (y-axis)

$\binom{x}{y} \longrightarrow \binom{-x}{y}$
$\binom{1}{-3} \longrightarrow \binom{-1}{-3}$ $\binom{4}{-2} \longrightarrow \binom{-4}{-2}$
$\binom{4}{-4} \longrightarrow \binom{-4}{-4}$ $\binom{2}{-2} \longrightarrow \binom{-2}{-2}$

Co-ordinates: $A_2(-1,-3), B_2(-4,-4), C_2(-4,-2), D_2(-2,-2)$

d. Reflect $A_2(-1,-3), B_2(-4,-4), C_2(-4,-2), D_2(-2,-2)$ in $x=-5 \Longrightarrow\ k=-5$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 2k-x \\ y \end{pmatrix} = \begin{pmatrix} 2(-5)-x \\ y \end{pmatrix} = \begin{pmatrix} -10-x \\ y \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ -3 \end{pmatrix} \longrightarrow \begin{pmatrix} -10-(-1) \\ -3 \end{pmatrix} = \begin{pmatrix} -9 \\ -3 \end{pmatrix} \qquad \begin{pmatrix} -4 \\ -2 \end{pmatrix} \longrightarrow \begin{pmatrix} -10-(-4) \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} -4 \\ -4 \end{pmatrix} \longrightarrow \begin{pmatrix} -10-(-4) \\ -4 \end{pmatrix} = \begin{pmatrix} -6 \\ -4 \end{pmatrix} \qquad \begin{pmatrix} -2 \\ -2 \end{pmatrix} \longrightarrow \begin{pmatrix} -10-(-2) \\ -2 \end{pmatrix} = \begin{pmatrix} -8 \\ -2 \end{pmatrix}$$

Co-ordinates: $A_3(-9,-3), B_3(-6,-4), C_3(-6,-2), D_3(-8,-2)$

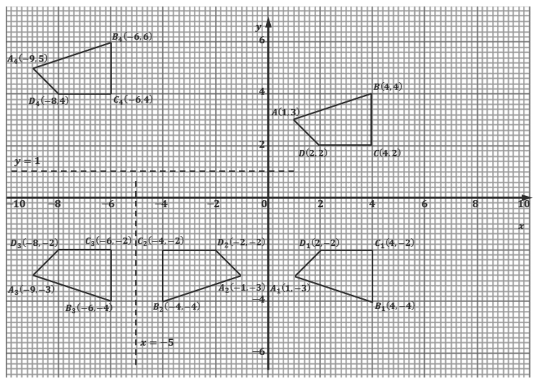
e. Reflect $A_3(-9,-3), B_3(-6,-4), C_3(-6,-2), D_3(-8,-2)$ in $y-1=0 \Longrightarrow\ k=1$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ 2k-y \end{pmatrix} = \begin{pmatrix} x \\ 2(1)-y \end{pmatrix} = \begin{pmatrix} x \\ 2-y \end{pmatrix}$$

$$\begin{pmatrix} -9 \\ -3 \end{pmatrix} \longrightarrow \begin{pmatrix} -9 \\ 2-(-3) \end{pmatrix} = \begin{pmatrix} -9 \\ 5 \end{pmatrix} \qquad \begin{pmatrix} -6 \\ -2 \end{pmatrix} \longrightarrow \begin{pmatrix} -6 \\ 2-(-2) \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} -6 \\ -4 \end{pmatrix} \longrightarrow \begin{pmatrix} -6 \\ 2-(-4) \end{pmatrix} = \begin{pmatrix} -6 \\ 6 \end{pmatrix} \qquad \begin{pmatrix} -8 \\ -2 \end{pmatrix} \longrightarrow \begin{pmatrix} -8 \\ 2-(-2) \end{pmatrix} = \begin{pmatrix} -8 \\ 4 \end{pmatrix}$$

Co-ordinates: $A_4(-9,5), B_4(-6,6), C_4(-6,4), D_4(-8,4)$



Practice

1. Points $A(1,4)$, $B(2,3), C(3,5)$ and $D(4,6)$ are points on the given axes. Find the co-ordinates, A_1, B_1, C_1 and D_1, of the image of the points under a reflection in the x-axis (y=0).
2. Points $P(1,4)$, $Q(2,3), R(3,5)$ and $S(4,6)$ are points on the given axes. Find the co-ordinates, P_1, Q_1, R_1 and S_1 of the image of the points under a reflection in the y-axis (x=0).
3. Find the image of the point $A(1,-6)$ when reflected in the lines:
   a. $y=2$ b. $x=5$ c. $x=-4$ d. $x+1=0$ e. $y+1=0$
4. Using a scale of 2 cm to 2 units on each axis, draw on a graph sheet two perpendicular axes Ox and Oy for the intervals $-10 \le x \le 10$ and $-10 \le y \le 10$.
   a. Draw ΔEFG with vertices E(2,9) F(5,9) and G(2,4).
   b. Draw the image $\Delta E_1F_1\,G_1$ of ΔEFG under a reflection in the y-axis where $E \to E_1, F \to F_1, G \to G_1$.
   c. Draw the image $\Delta E_2F_2\,G_2$ of ΔEFG under a reflection in the x-axis where $E \to E_2, F \to F_2, G \to G_2$.

Lesson Title: Reflection – Part 2	Theme: Vectors and Transformations
Lesson Number: PHM3-L090	**Class:** SSS 3

Learning Outcomes

By the end of the lesson, you will be able to:

1. Identify and explain the reflection of an object in the line $y = x$
2. Identify and explain the reflection of an object in the line $y = -x$.

Overview

In the last lesson, we looked at reflection in horizontal and vertical lines, including the x- and y-axes. Reflection of points or shapes can be done on other lines apart from horizontal and vertical lines.

We can write the following mappings:

- **Reflection in the line** $y = x$ as:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} y \\ x \end{pmatrix} \quad \text{giving} \quad (x, y) \longrightarrow (y, x)$$

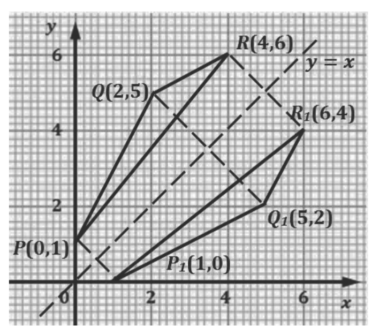
- **Reflection in in the line** $y = -x$ as:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} -y \\ -x \end{pmatrix} \quad \text{giving} \quad (x, y) \longrightarrow (-y, -x)$$

Solved Examples

1. Points $P(2,4), Q(2,6)$ and $R(3,6)$ are points on the given axes. Find the co-ordinates, P_1, Q_1 and R_1 of the image of the triangle formed under reflection in the line $y = x$.

Solution:



Step 1. Assess and extract the given information from the problem. Given: points $P(0,1), Q(2,5)$ and $R(4,6)$, line $y = x$

Step 2. Draw the x- and y- axes (if not already drawn). Locate the points P, Q and R on the graph. Draw the lines joining the points.

Step 3. Draw the line $y = x$.

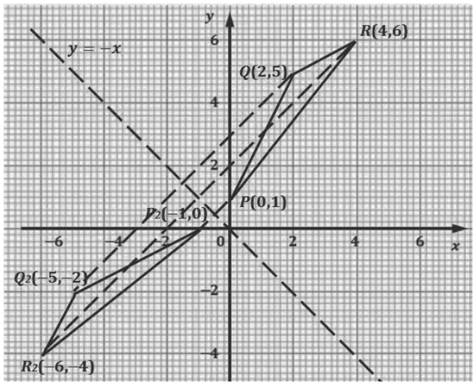
Step 4. Draw a line at right angles from P to the mirror line ($y = x$). Measure this distance.

Step 5. Measure the same distance on the opposite side of the mirror line ($y = x$) to locate the point P_1 on the graph.

Step 6. Write the co-ordinates of $P_1 = (1,0)$.
Notice that the x- and y-co-ordinates have interchanged

Step 7. Follow the same procedure for points Q and R giving $Q_1(5,2)$ and $R_1(6,4)$ (see graph).

2. For the given points in question b., and using the same axes, find the co-ordinates, P_2, Q_2, and R_2 of the image under reflection in the line $y = -x$.
   Write the mapping for each point. What do you notice?
   Solution:



Given: points $P(0,1), Q(2,5)$ and $R(4,6)$,
line $y = -x$

mapping for P $\begin{pmatrix}0\\1\end{pmatrix} \longrightarrow \begin{pmatrix}-1\\0\end{pmatrix}$

mapping for Q $\begin{pmatrix}2\\5\end{pmatrix} \longrightarrow \begin{pmatrix}-5\\-2\end{pmatrix}$

mapping for R $\begin{pmatrix}4\\6\end{pmatrix} \longrightarrow \begin{pmatrix}-6\\-4\end{pmatrix}$

The co-ordinates are $P_2(-1,0), Q_2(-5,-2)$ and $R_2(-6,-4)$
The x and y-co-ordinates have interchanged and have opposite signs.

3. Using a scale of 2 cm to 2 units on both axes, draw and label on a sheet of graph paper two perpendicular axes Ox and Oy for $-8 \le x \le 8$ and $-8 \le y \le 8$.
   Draw on the same axes, showing clearly the co-ordinates of all vertices:
   a. The quadrilateral with vertices $A(2,8), B(4,8), C(4,6)$ and $D(3,5)$.
   b. The image $A_1B_1C_1D_1$ of $ABCD$ under a reflection in the line $y = x$ where $A \longrightarrow A_1$, $B \longrightarrow B_1$, $C \longrightarrow C_1$ and $D \longrightarrow D_1$.
   c. The image $A_2B_2C_2D_2$ under a reflection in the line $y = -x$ where $A \longrightarrow A_2$, $B \longrightarrow B_2$, $C \longrightarrow C_2$ and $D \longrightarrow D_2$.
   d. The image $A_3B_3C_3D_3$ under a reflection in the line $y = -x$ where $A_1 \longrightarrow A_3$, $B_1 \longrightarrow B_3$, $C_1 \longrightarrow C_3$ and $D_1 \longrightarrow D_3$.

Solution:
Given: quadrilateral with vertices $A(2,8), B(4,8), C(4,6)$ and $D(3,5)$

a. All diagrams for this question can be found at the end of the question.

b. Reflect in line $y = x$, mapping is given by $\begin{pmatrix}x\\y\end{pmatrix} \longrightarrow \begin{pmatrix}y\\x\end{pmatrix}$

$\begin{pmatrix}2\\8\end{pmatrix} \longrightarrow \begin{pmatrix}8\\2\end{pmatrix}$ giving $A_1(8,2)$ $\quad$ $\begin{pmatrix}4\\8\end{pmatrix} \longrightarrow \begin{pmatrix}8\\4\end{pmatrix}$ giving $B_1(8,4)$

$\begin{pmatrix}4\\6\end{pmatrix} \longrightarrow \begin{pmatrix}6\\4\end{pmatrix}$ giving $C_1(6,4)$ $\quad$ $\begin{pmatrix}3\\5\end{pmatrix} \longrightarrow \begin{pmatrix}5\\3\end{pmatrix}$ giving $D_1(5,3)$

c. Reflect in line $y = -x$, mapping is given by $\begin{pmatrix}x\\y\end{pmatrix} \longrightarrow \begin{pmatrix}-y\\-x\end{pmatrix}$

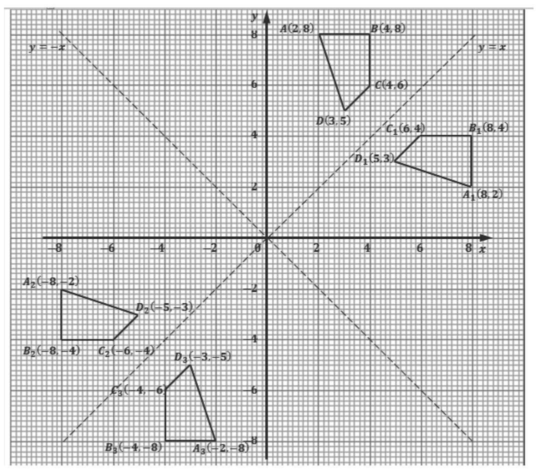
$\begin{pmatrix}2\\8\end{pmatrix} \longrightarrow \begin{pmatrix}-8\\-2\end{pmatrix}$ giving $A_2(-8,-2)$ $\quad$ $\begin{pmatrix}4\\8\end{pmatrix} \longrightarrow \begin{pmatrix}-8\\-4\end{pmatrix}$ giving $B_2(-8,-4)$

$\begin{pmatrix}4\\6\end{pmatrix} \longrightarrow \begin{pmatrix}-6\\-4\end{pmatrix}$ giving $C_2(-6,-4)$ $\quad$ $\begin{pmatrix}3\\5\end{pmatrix} \longrightarrow \begin{pmatrix}-5\\-3\end{pmatrix}$ giving $D_2(-5,-3)$

Reflect $A_1(8,2), B_1(8,4), C_1(6,4)$ and $D_1(5,3)$ in line $y = -x$

$\begin{pmatrix}8\\2\end{pmatrix} \longrightarrow \begin{pmatrix}-2\\-8\end{pmatrix}$ giving $A_3(-2,-8)$ $\quad$ $\begin{pmatrix}8\\4\end{pmatrix} \longrightarrow \begin{pmatrix}-4\\-8\end{pmatrix}$giving $B_3(-4,-8)$

$\begin{pmatrix}6\\4\end{pmatrix} \longrightarrow \begin{pmatrix}-4\\-6\end{pmatrix}$ giving $C_3(-4,-6)$ $\quad$ $\begin{pmatrix}5\\3\end{pmatrix} \longrightarrow \begin{pmatrix}-3\\-5\end{pmatrix}$ giving $D_3(-3,-5)$



Practice

1. Find the image of the point $K(2,-4)$ when reflected in the line $x + 3 = 0$.
2. Points $A(2,4), B(1,3)$ and $C(5,4)$ are points on the Cartesian plane. Find the co-ordinates, A_1, B_1 and C_1 of the image of the triangle formed under a reflection in the line $y = -x$.
3. Points $X(1,3), Y(4,5)$ and $Z(3,7)$ are points on the Cartesian plane. Find the co-ordinates, X_1, $Y_1$and Z_1 of the image of the triangle formed under a reflection in the line $y = x$.
4. Using a scale of 2 cm to 2 units on each axis, draw on a graph sheet two perpendicular axes Ox and Oy for the intervals $-12 \leq x \leq 8$ and $-10 \leq y \leq 10$.
   a. Draw ΔPQR with vertices P (2,6) Q (2,2) and R (6,2).
   b. Draw the image $P_1Q_1\,R_1$ of ΔPQR under a reflection in the line y= 3 where $P \to P_1, Q \to Q_1, R \to R_1$.

c. Draw the image $P_2Q_2R_2$ of ∆PQR under a reflection in the line y= -x where $P \to P_2, Q \to Q_2, R \to R_2$.
d. Draw the image $P_3Q_3\,R_3$ of ∆PQR under a reflection in the line x= -3 where $P \to P_3, Q \to Q_3, R \to R_3$.

5. Using a scale of 2 cm to 2 units on each axis, draw on a graph sheet two perpendicular axes Ox and Oy for the intervals $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$.
   a. Draw ABCD with vertices A (1, 2) B (4, 4) C (3, 7) and D (1, 5).
   b. Draw the image $A_1B_1\,C_1D_1$ of ABCD under a reflection in the line y=x where $A \to A_1, B \to B_1, C \to C_1, D \to D_1$.
   c. Draw the image $A_2B_2C_2D_2$ of ABCD under a reflection in the line y= -x where $A \to A_2, B \to B_2, C \to C_2, D \to D_2$.
   d. Draw the image $A_3B_3C_3D_3$ of ABCD under a reflection in the line x+1=0 where $A \to A_3, B \to B_3, C \to C_3, D \to D_3$.

Lesson Title: Rotation – Part 1	**Theme:** Vectors and Transformations
Lesson Number: PHM3-L091	**Class:** SSS 3

Learning Outcomes

By the end of the lesson, you will be able to:

1. Identify that rotation is a movement around a fixed point.
2. Find the image of an object under rotation about the origin.

Overview

Rotation is a movement around a fixed point, called the **centre of rotation**. Rotation is always done in a specified angle and direction about a specified point. Anti- clockwise rotation is the standard rotation used, and can be assumed if no direction is given.

The mappings for rotation are given below:

- **Rotation through $90°$ anti-clockwise or $270°$ clockwise about the origin O:**

  $\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} -y \\ x \end{pmatrix}$ giving $(x, y) \longrightarrow (-y, x)$
- **Rotation through $270°$ anti-clockwise or $90°$ clockwise about the origin O:**

  $\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} y \\ -x \end{pmatrix}$ giving $(x, y) \longrightarrow (y, -x)$
- **Rotation through $180°$ (half-turn) anti-clockwise about the origin O:**

  $\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} -x \\ -y \end{pmatrix}$ giving $(x, y) \longrightarrow (-x, -y)$

  Since $180°$ is a half-turn, both anti-clockwise and clockwise rotation result in the same image.

Solved Examples

1. Points $P(3,2), Q(4,6)$ and $R(2,4)$ are points on the given axes. Find the co-ordinates, P_1, Q_1 and R_1 of the image of the triangle formed under an anti-clockwise rotation of $90°$ about the origin, O.

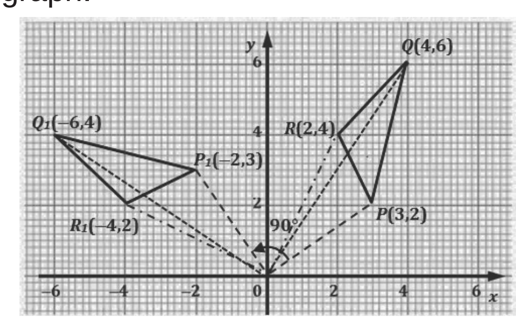
   Solution:

   Step 1. Draw the x- and y- axes (if not already drawn).
   Locate the points P, Q and R on the graph.
   Draw the lines joining the points.

   Step 2. Mark the centre of rotation $O(0,0)$.
   Draw a straight line from O to P
   Measure the distance OP.

   Step 3. Measure an angle of $90°$ in an anti-clockwise direction.

   Step 4. Measure the distance OP along this line to locate the point P_1.



Step 5. Write the co-ordinates of $P_1 = (-2,3)$.
Notice that the x- and y-co-ordinates have interchanged, and the y-co-ordinate has also changed signs.
Step 6. Follow the same procedure for points Q and R giving $Q_1(-6,4)$ and $R_1(-4,2)$ (see graph).

2. A rotation about the origin maps $A(-4,-7)$ to $A_1(-7,4)$ and $B(3,-5)$ to B_1. Find the co-ordinates of B_1. Describe the rotation for the mapping.

Solution:

Given: $A(-4,-7)$, $A_1(-7,4)$ and $B(3,-5)$

$$\begin{pmatrix}-4\\-7\end{pmatrix} \longrightarrow \begin{pmatrix}-7\\4\end{pmatrix}$$

this is the same as $\begin{pmatrix}x\\y\end{pmatrix} \longrightarrow \begin{pmatrix}y\\-x\end{pmatrix}$

$$\therefore \begin{pmatrix}3\\-5\end{pmatrix} \longrightarrow \begin{pmatrix}-5\\-3\end{pmatrix}$$

The point B was rotated $270°$ anti-clockwise about the origin *O.*

3. Using a scale of 2 cm to 2 units on both axes, draw on a sheet of graph paper two perpendicular axes Ox and Oy for $-8 \leq x \leq 8$ and $-8 \leq y \leq 8$.
Draw on the same axes showing clearly the co-ordinates of all vertices:
   a. The triangle PQR with $P(5,3), Q(8,4)$ and $R(2,6)$.
   b. The image $\Delta P_1Q_1R_1$ of ΔPQR under an anti-clockwise rotation of $90°$ about the origin where $P \longrightarrow P_1,\ Q \longrightarrow Q_1\ R \longrightarrow R_1$.
   c. The image $\Delta P_2Q_2R_2$ under an anti-clockwise rotation of $270°$ about the origin where $P \longrightarrow P_2,\ Q \longrightarrow Q_2\ R \longrightarrow R_2$.
   d. The image $\Delta P_3Q_3R_3$ under a rotation of $180°$ about the origin where $P \longrightarrow P_3,\ Q \longrightarrow Q_3\ R \longrightarrow R_3$.

Solutions:

Given: triangle PQR with $P(5,3), Q(8,4)$ and $R(2,6)$

a. All diagrams for this question can be found at the end of the question.

b. Rotate $90°$ about the origin, mapping is given by $\begin{pmatrix}x\\y\end{pmatrix} \longrightarrow \begin{pmatrix}-y\\x\end{pmatrix}$

$\begin{pmatrix}5\\3\end{pmatrix} \longrightarrow \begin{pmatrix}-3\\5\end{pmatrix}$ giving $P_1(-3,5)$ $\qquad$ $\begin{pmatrix}8\\4\end{pmatrix} \longrightarrow \begin{pmatrix}-4\\8\end{pmatrix}$ giving $Q_1(-4,8)$

$\begin{pmatrix}2\\6\end{pmatrix} \longrightarrow \begin{pmatrix}-6\\2\end{pmatrix}$ giving $R_1(-6,2)$

c. Rotate $270°$ about the origin, mapping is given by $\begin{pmatrix}x\\y\end{pmatrix} \longrightarrow \begin{pmatrix}y\\-x\end{pmatrix}$

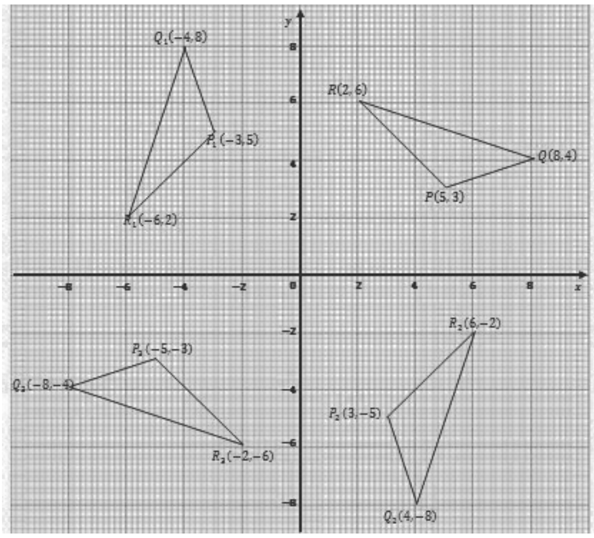
$\begin{pmatrix}5\\3\end{pmatrix} \longrightarrow \begin{pmatrix}3\\-5\end{pmatrix}$ giving $P_2(3,-5)$ $\qquad$ $\begin{pmatrix}8\\4\end{pmatrix} \longrightarrow \begin{pmatrix}4\\-8\end{pmatrix}$ giving $Q_2(4,-8)$

$\begin{pmatrix}2\\6\end{pmatrix} \longrightarrow \begin{pmatrix}6\\-2\end{pmatrix}$ giving $R_2(6,-2)$

d. Rotate $180°$ about the origin, mapping is given by $\begin{pmatrix}x\\y\end{pmatrix} \longrightarrow \begin{pmatrix}-x\\-y\end{pmatrix}$

$\begin{pmatrix}5\\3\end{pmatrix} \longrightarrow \begin{pmatrix}-5\\-3\end{pmatrix}$ giving $P_2(-5,-3)$ $\qquad$ $\begin{pmatrix}8\\4\end{pmatrix} \longrightarrow \begin{pmatrix}-8\\-4\end{pmatrix}$ giving $Q_2(-8,-4)$

$\binom{2}{6} \longrightarrow \binom{-2}{-6}$ giving $R_2(-2,-6)$



Practice

1. Find the image of the point $Q(3,6)$ formed under a clockwise rotation of 90° about the origin.
2. Points $A(3,4), B(2,3)$ and $C(2,4)$ are points on the Cartesian plane. Find the co-ordinates, A_1, B_1 and C_1 of the image of the triangle formed under an anti-clockwise rotation of 90° about the origin.
3. Using a scale of 2 cm to 2 units on each axis, draw on a graph sheet two perpendicular axes Ox and Oy for the intervals $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$.
   a. Draw ΔEFG with vertices E(-2,7) F(-6,7) and G(-2,2).
   b. Draw the image $E_1F_1G_1$ of ΔEFG under an anti-clockwise rotation of 90° about the origin where $E \to E_1$, $F \to F_1$, $G \to G_1$.
   c. Draw the image $E_2F_2G_2$ of ΔEFG under a clockwise rotation of 90° about the origin where $E \to E_2$, $F \to F_2$, $G \to G_2$.
4. Using a scale of 2 cm to 2 units on each axis, draw on a graph sheet two perpendicular axes Ox and Oy for the intervals $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$.
   a. Draw ΔPQR with vertices P (3,1) Q(6,2) and R(2,4).
   b. Draw the image $P_1Q_1R_1$ of ΔPQR under an anti-clockwise rotation of 270° about the origin where $P \to P_1$, $Q \to Q_1$, $R \to R_1$.
   c. Draw the image $P_2Q_2R_2$ of ΔPQR under a clockwise rotation of 270° about the origin where $P \to P_2$, $Q \to Q_2$, $R \to R_2$.

   d. Draw the image $P_3Q_3R_3$ of ΔPQR under an anti-clockwise or clockwise rotation of 180° about the origin where $P \to P_3$, $Q \to Q_3$, $R \to R_3$.

Lesson Title: Rotation – Part 2	**Theme:** Vectors and Transformations
Lesson Number: PHM3-L092	**Class:** SSS 3

Learning Outcome
By the end of the lesson, you will be able to find the image of an object under rotation about any point (a, b).

Overview

We can rotate a given point or shape about any point (a, b) by following the procedure from the last lesson. We can also use a formula to calculate the new co-ordinates obtained after a specified rotation.

We follow the steps given below to find the formula for **rotation through $90°$ anti-clockwise or $270°$ clockwise about the point (a, b)**.

Step 1.	Subtract the co-ordinates of the centre of rotation (a, b) from (x, y)	$\begin{pmatrix} x-a \\ y-b \end{pmatrix}$
Step 2.	Apply the appropriate rotation formula	$\begin{pmatrix} x-a \\ y-b \end{pmatrix} \longrightarrow \begin{pmatrix} -(y-b) \\ x-a \end{pmatrix}$
Step 3.	Add the result in Step 2 to the centre of rotation to get the image point	$\begin{pmatrix} -(y-b)+a \\ (x-a)+b \end{pmatrix}$
Step 4.	Write the co-ordinates of the image point	$(-(y-b)+a,\ (x-a)+b)$

Using the steps above, we also get:

- **Rotation through $270°$ anti-clockwise or $90°$ clockwise about (a, b)**

$\begin{pmatrix} x-a \\ y-b \end{pmatrix}$	subtract components of the centre of rotation from the given point
$\begin{pmatrix} x-a \\ y-b \end{pmatrix} \longrightarrow \begin{pmatrix} y-b \\ -(x-a) \end{pmatrix}$	apply the appropriate formula
$\begin{pmatrix} (y-b)+a \\ -(x-a)+b \end{pmatrix}$	add the result to (a, b)
$((y-b)+a, -(x-a)+b)$	co-ordinates of the image point

- **Rotation through $180°$ (half turn) anti-clockwise about (a, b)`**

$\begin{pmatrix} x-a \\ y-b \end{pmatrix}$	subtract components of the centre of rotation from the given point
$\begin{pmatrix} x-a \\ y-b \end{pmatrix} \longrightarrow \begin{pmatrix} -(x-a) \\ -(y-b) \end{pmatrix}$	apply the appropriate formula
$\begin{pmatrix} -(x-a)+a \\ -(y-b)+b \end{pmatrix} = \begin{pmatrix} -(x-2a) \\ -(y-2b) \end{pmatrix}$	add the result to (a, b)
$(-(x-2a), -(y-2b))$	co-ordinates of the image point

Solved Examples

1. Use the appropriate formula to find the co-ordinates of the image point when point $Y(-2,-5)$ is rotated $90°$ clockwise about the point $(1,-4)$.

Solution:

Given: point $Y(-2,-5)$, rotated $90°$ clockwise about the point $(1,-4)$

$\begin{pmatrix} x-a \\ y-b \end{pmatrix} = \begin{pmatrix} -2-1 \\ -5-(-4) \end{pmatrix} = \begin{pmatrix} -2-1 \\ -5+4 \end{pmatrix}$ subtract components of the centre of rotation from the given point

$= \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

$\begin{pmatrix} x-a \\ y-b \end{pmatrix} \longrightarrow \begin{pmatrix} y-b \\ -(x-a) \end{pmatrix}$ apply the appropriate formula

$\begin{pmatrix} -3 \\ -1 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 \\ -(-3) \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$\begin{pmatrix} -1 \\ 3 \end{pmatrix} \longrightarrow \begin{pmatrix} -1+1 \\ 3+(-4) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ add the result to $(1,-4)$

$Y(-2,-5)$ rotated $90°$ clockwise about the point $(1,-4)$ gives $(0,-1)$

2. Using a scale of 2 cm to 2 units on both axes, draw on a sheet of graph paper two perpendicular axes Ox and Oy for $-8 \leq x \leq 8$ and $-6 \leq y \leq 10$.

Draw on the same axes showing clearly the co-ordinates of all vertices:

a. The triangle PQR with $P(2,4), Q(4,8)$ and $R(8,7)$.

b. The image $\Delta P_1Q_1R_1$ of triangle PQR under an anti-clockwise rotation of $90°$ about the point $(1,2)$ where $P \longrightarrow P_1,\ Q \longrightarrow Q_1\ R \longrightarrow R_1$.

c. The image $\Delta P_2Q_2R_2$ under an anti-clockwise rotation of $270°$ about the point $(1,2)$ where $P \longrightarrow P_2,\ Q \longrightarrow Q_2\ R \longrightarrow R_2$.

d. The image $\Delta P_3Q_3R_3$ under a rotation of $180°$ about the point $(1,2)$ where $P \longrightarrow P_3,\ Q \longrightarrow Q_3\ R \longrightarrow R_3$.

Solutions:

Given: triangle PQR with $P(2,4), Q(4,8)$ and $R(8,7)$

a. All diagrams for this question can be found at the end of the question.

b. Mapping under an anti-clockwise rotation of $90°$ about the point (a,b) is given by: $\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} -(y-b)+a \\ (x-a)+b \end{pmatrix}$

for rotation about centre of rotation, $C(1,2)$ $\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} -(y-2)+1 \\ (x-1)+2 \end{pmatrix} = \begin{pmatrix} -y+3 \\ x+1 \end{pmatrix}$

$\begin{pmatrix} 2 \\ 4 \end{pmatrix} \longrightarrow \begin{pmatrix} -4+3 \\ 2+1 \end{pmatrix}$

$\longrightarrow \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ giving $P_1(-1,3)$

$\begin{pmatrix} 4 \\ 8 \end{pmatrix} \longrightarrow \begin{pmatrix} -8+3 \\ 4+1 \end{pmatrix}$

$\longrightarrow \begin{pmatrix} -5 \\ 5 \end{pmatrix}$ giving $Q_1(-5,5)$

$\begin{pmatrix} 8 \\ 7 \end{pmatrix} \longrightarrow \begin{pmatrix} -7+3 \\ 8+1 \end{pmatrix}$

$\longrightarrow \begin{pmatrix} -4 \\ 9 \end{pmatrix}$ giving $R_1(-4,9)$

c. Mapping is given by $\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} (y-b)+a \\ -(x-a)+b \end{pmatrix} = \begin{pmatrix} (y-2)+1 \\ -(x-1)+2 \end{pmatrix} = \begin{pmatrix} y-1 \\ -x+3 \end{pmatrix}$

$\begin{pmatrix} 2 \\ 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 4-1 \\ -2+3 \end{pmatrix}$

$\begin{pmatrix} 4 \\ 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 8-1 \\ -4+3 \end{pmatrix}$

$\longrightarrow \binom{3}{1}$ giving $P_2(3,1)$ $\qquad\longrightarrow \binom{7}{-1}$ giving $Q_2(7,-1)$

$\binom{8}{7} \longrightarrow \left(\binom{7-1}{-8+3}\right)$

$\longrightarrow \binom{6}{-5}$ giving $R_2(6,-5)$

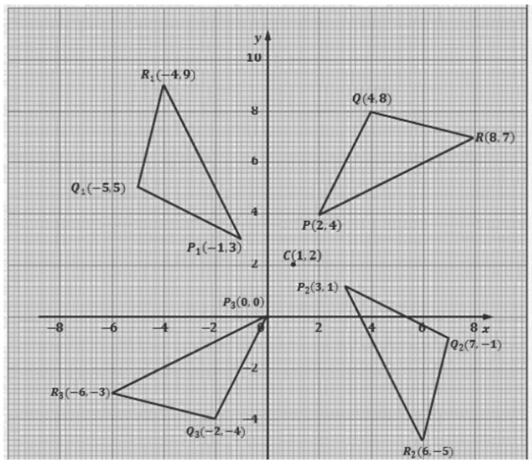
d. Mapping is given by $\binom{x}{y} \longrightarrow \binom{-(x-2a)}{-(y-2b)} = \binom{-(x-2(1))}{-(y-2(2))} = \binom{-x+2}{-y+4}$

$\binom{2}{4} \longrightarrow \binom{-2+2}{-4+4}$ $\qquad \binom{4}{8} \longrightarrow \binom{-4+2}{-8+4}$

$\longrightarrow \binom{0}{0}$ giving $P_3(0,0)$ $\qquad \longrightarrow \binom{-2}{-4}$ giving $Q_3(-2,-4)$

$\binom{8}{7} \longrightarrow \binom{-8+2}{-7+4}$

$\longrightarrow \binom{-6}{-3}$ giving $R_3(-6,-3)$



Practice

1. a. Determine the formula for the co-ordinates of the image point when (x ,y) is rotated clockwise about the point (a, b).
   b. Find the co-ordinates of the image point when $R(4,3)$ is rotated 90° clockwise about the point(-5,2).
2. a. Determine the formula for the co-ordinates of the image point when (x ,y) is rotated 90°anti-clockwise about the point (a, b).\
   b. Find the co-ordinates of the image point when $P(-5,-2)$ is rotated 90°anti-clockwise about the point(4,7).
3. a. Determine the formula for the co-ordinates of the image point when (x ,y) is rotated 180°clockwise or anti-clockwise about the point (a, b).
   b. Find the co-ordinates of the image point when $Q(6,-3)$ is rotated 180 clockwise or anti-clockwise about the point(-2,-3).
4. Points $A(3,4), B(2,3)$ and $C(2,4)$ are points on the Cartesian plane. Find the co-ordinates, A_1, $B_1$and $C_1$of the image of the triangle formed under:
   a. An anti-clockwise rotation of 270°about the point(-3.-2).
   b. A rotation of 180°about the point(4,-3).
5. Using a scale of 2 cm to 2 units on each axis, draw on a graph sheet two perpendicular axes Ox and Oy for the intervals $-8 \leq x \leq 12$ and $-12 \leq y \leq 12$.
   a. Draw ΔABC with vertices A(4,-4) B(8,-3) and C(4,-3).

b. Draw the image $A_1B_1\,C_1$ of ΔABC under an anti-clockwise rotation of 90°about the point (2, 4) where$A \to A_1$, $B \to B_1$, $C \to C_1$.
c. Draw the image $A_2B_2C_2$ of ΔABC under a rotation of 180°about the point (2,4) where$A \to A_2$, $B \to B_2$, $C \to C_2$.
d. Draw the image $A_3B_3\,C_3$ of ΔABC under an clockwise rotation of 90°about the point (2,4) where$A \to A_3$, $B \to B_3$, $C \to C_3$.

Lesson Title: Enlargement – Part 1	Theme: Vectors and Transformations
Lesson Number: PHM3-L093	**Class:** SSS 3
Learning Outcome By the end of the lesson, you will be able to use scalar multiplication to enlarge given shapes.	

Overview

An enlargement is a transformation which enlarges or reduces the size of an image.
It is described by a centre of enlargement and a scale factor, k.
Two different formulas are given for enlargement:

- The formula **for enlargement from the origin O by a scale factor k** is given by:

  $\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow k\begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} kx \\ kx \end{pmatrix}$ where k is positive or negative whole number or fraction

  $(x, y) \longrightarrow (kx, ky)$

- The formula for **enlargement from any point (a, b) other than the origin O by a scale factor k** can be found by following the steps given below.

Step 1.	Subtract the co-ordinates of the centre of rotation (a, b) from (x, y)	$\begin{pmatrix} x-a \\ y-b \end{pmatrix}$
Step 2.	Enlarge using the given scale factor	$\begin{pmatrix} x-a \\ y-b \end{pmatrix} \longrightarrow \begin{pmatrix} k(x-a) \\ k(y-b) \end{pmatrix}$
Step 3.	Add the result in Step 2 to the centre of rotation to get the image point	$\begin{pmatrix} k(x-a)+a \\ k(y-b)+b \end{pmatrix}$
Step 4.	Write the co-ordinates of the image point	$(k(x-a)+a,\ k(y-b)+b)$

A negative scale factors give images at the opposite side of the centre of enlargement. The image is turned upside down (inverted). An object under enlargement with a scale factor which is a fraction, results in a smaller image than the object. It is a **reduction**.

Solved Examples

1. Find the image of $(-1, -6)$ under the enlargement with scale factor of 4 from:
   a. The origin b. The point $(2,4)$

   Solutions:

   Given: $(-1, -6)$, enlarge with scale factor 4

   a. $\begin{pmatrix} -1 \\ -6 \end{pmatrix} \longrightarrow 4\begin{pmatrix} -1 \\ -6 \end{pmatrix} = \begin{pmatrix} 4\times(-1) \\ 4\times(-6) \end{pmatrix} = \begin{pmatrix} -4 \\ -24 \end{pmatrix}$

   b. Given: $(-1, -6)$ enlarge about the point $(2,4)$ with scale factor 4

   $\begin{pmatrix} x-a \\ y-b \end{pmatrix} \longrightarrow \begin{pmatrix} -1-2 \\ -6-4 \end{pmatrix} = \begin{pmatrix} -3 \\ -10 \end{pmatrix}$ subtract components of the centre of rotation from the given point

   $\begin{pmatrix} -3 \\ -10 \end{pmatrix} \longrightarrow 4\begin{pmatrix} -3 \\ -10 \end{pmatrix} = \begin{pmatrix} -12 \\ -40 \end{pmatrix}$ enlarge using the given scale factor

   $\begin{pmatrix} -12 \\ -40 \end{pmatrix} \longrightarrow \begin{pmatrix} -12+2 \\ -40+4 \end{pmatrix} = \begin{pmatrix} -10 \\ -36 \end{pmatrix}$ add back components of the centre of rotation

2. Draw on the given axes showing clearly the co-ordinates of all vertices.

a. The triangle PQR with $P(1,1), Q\ (3,1)$ and $R(1,4)$.

b. The image triangle $P_1Q_1R_1$ of triangle PQR under an enlargement from the origin with scale factor 2 where $P \longrightarrow P_1,\ Q \longrightarrow Q_1\ R \longrightarrow R_1$.

c. The image triangle $P_2Q_2R_2$ of triangle PQR under an enlargement from the origin with scale factor -2 where $P \longrightarrow P_2,\ Q \longrightarrow Q_2\ R \longrightarrow R_2$.

d. What do you notice about the enlargement $P_2Q_2R_2$?

Solutions:

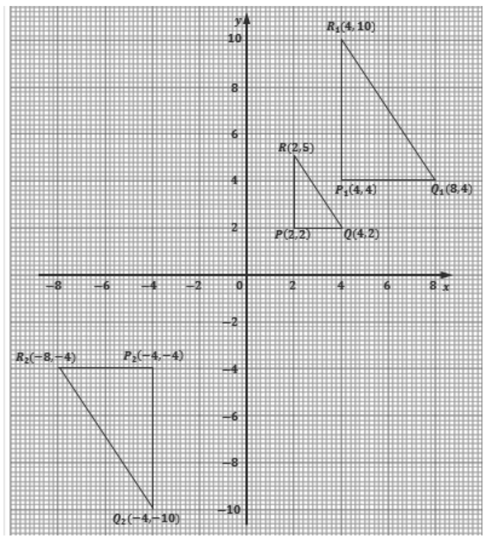
Given: triangle PQR with points $P(1,1),\ Q(3,1)$ and $R(1,4)$

a. All diagrams for this question can be found at the end of the question.

b. Mapping under an enlargement from the origin with scale factor 2 $\quad \binom{x}{y} \longrightarrow \binom{kx}{kx} = \binom{2x}{2x}$

$$\binom{1}{1} \longrightarrow 2\binom{1}{1} = \binom{2}{2}$$
$$\binom{3}{1} \longrightarrow 2\binom{3}{1} = \binom{6}{2}$$
$$\binom{1}{4} \longrightarrow 2\binom{1}{4} = \binom{2}{8}$$

$P_1(2,2), Q_1(6,2)$ and $R_1(2,8)$



c. Mapping under an enlargement from the origin with scale factor 2

$$\binom{x}{y} \longrightarrow \binom{kx}{kx} = \binom{2x}{2x}$$
$$\binom{1}{1} \longrightarrow -2\binom{1}{1} = \binom{-2}{-2}$$
$$\binom{3}{1} \longrightarrow -2\binom{3}{1} = \binom{-6}{-2}$$
$$\binom{1}{4} \longrightarrow -2\binom{1}{4} = \binom{-2}{-8}$$

$P_2(-2,-2), Q_2(-6,-2)$ and $R_2(-2,-8)$

d. The image $P_2Q_2R_2$ is upside down.

3. A square has vertices $A(8,4),\ B(8,-8),\ C(-4,-8)$ and $D(-4,4)$.

   a. Find the co-ordinates of the vertices of the image square $A_1B_1C_1D_1$ of $ABCD$ under an enlargement from the origin with scale factor $\frac{1}{2}$ where $A \longrightarrow A_1,\ B \longrightarrow B_1,\ C \longrightarrow C_1$ and $D \longrightarrow D_1$.

   b. What do you notice about the co-ordinates of $A_1B_1C_1D_1$?

Using a scale of 2 cm to 2 units on both axes, draw on a sheet of graph paper two perpendicular axes Ox and Oy for $-4 \le x \le 8$ and $-8 \le y \le 4$.

Draw on the same axes, showing clearly the co-ordinates of all vertices:

c. The square $ABCD$ above.

d. The image $A_1B_1C_1D_1$ of $ABCD$.

Solutions:

Given: $A(8,4),\ B(8,-8),\ C(-4,-8)$ and $D(-4,4)$

a. $\binom{8}{4} \longrightarrow \frac{1}{2}\binom{8}{4} = \binom{4}{2}$

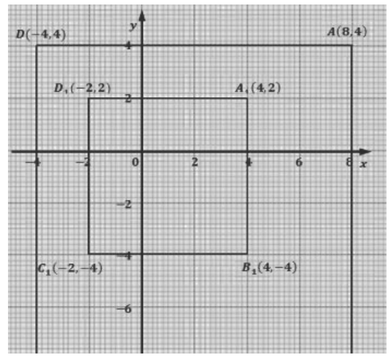
c.

$$\begin{pmatrix}8\\-8\end{pmatrix} \longrightarrow \frac{1}{2}\begin{pmatrix}8\\-8\end{pmatrix} = \begin{pmatrix}4\\-4\end{pmatrix}$$
$$\begin{pmatrix}-4\\-8\end{pmatrix} \longrightarrow \frac{1}{2}\begin{pmatrix}-4\\-8\end{pmatrix} = \begin{pmatrix}-2\\-4\end{pmatrix}$$
$$\begin{pmatrix}-4\\4\end{pmatrix} \longrightarrow \frac{1}{2}\begin{pmatrix}-4\\4\end{pmatrix} = \begin{pmatrix}-2\\2\end{pmatrix}$$

$A_1(4,2), B_1(4,-4), C_1(-2,-4), D_1(-2,2)$

b. The co-ordinates of $A_1B_1C_1D_1$ are half that of the original $ABCD$.

d.



Practice

For all enlargements using a scale of 2 cm to 2 units on each axis, draw on a graph sheet two perpendicular axes Ox and Oy for the intervals $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$.

1. Find the image of D $(-4,2)$ under the enlargement with scale factor of 3 from:
   a. The origin b. The point (5,6)
2. a. Draw ∆EFG: with vertices E(2,6) F(6,4) and G(2,4).
   b. Draw the image $E_1F_1G_1$ of ∆EFG under the enlargement from the origin with a scale factor of $\frac{1}{2}$ where $E \rightarrow E_1, F \rightarrow F_1, G \rightarrow G_1$.
   c. Draw the image $E_2F_2G_2$ of ∆EFG under the enlargement from the origin with a scale factor of 2 where $E \rightarrow E_2, F \rightarrow F_2, G \rightarrow G_2$.
   d. Draw the image $E_3F_3G_3$ of ∆EFG under the enlargement from the origin with a scale factor of -1 where $E \rightarrow E_3, F \rightarrow F_3, G \rightarrow G_3$,
3. a. Draw ∆ABC with vertices A(1,2), B(2,1) and C(3,1).
   b. Draw the image $A_1B_1C_1$ of ∆ABC under the enlargement from the origin with a scale factor of 2 where $A \rightarrow A_1, B \rightarrow B_1, C \rightarrow C_1$.
   c. Draw the image $A_2B_2C_2$ of ∆ABC under the enlargement from the origin with a scale factor of 3 where $A \rightarrow A_2, B \rightarrow B_2, C \rightarrow C_2$.
   d. Draw the image $A_3B_3C_3$ of ∆ABC under the enlargement with a scale factor of -1 from the point (2, 3) where $A \rightarrow A_3, B \rightarrow B_3, C \rightarrow C_3$.
4. a. Draw ABCD with vertices A (2, 8), B (2, 4), C (8, 4) and D (8, 8).
   b. Draw the image $A_1B_1C_1D_1$ of ABCD under the enlargement from the origin with a scale factor of $\frac{1}{2}$ where $A \rightarrow A_1, B \rightarrow B_1, C \rightarrow C_1, D \rightarrow D_1$.
   c. Draw the image $A_2B_2C_2D_2$ of ABCD under the enlargement from the origin with a scale factor of $\frac{1}{4}$ where $A \rightarrow A_2, B \rightarrow B_2, C \rightarrow C_2, D \rightarrow D_2$.
   d. Draw the image $A_3B_3C_3D_3$ of ABCD under the enlargement with a scale factor of -1 from the point (-1, -2) where $A \rightarrow A_3, B \rightarrow B_3, C \rightarrow C_3, D \rightarrow D_3$.

Lesson Title: Enlargement – Part 2	Theme: Vectors and Transformations
Lesson Number: PHM3-L094	**Class:** SSS 3

Learning Outcome
By the end of the lesson, you will be able to find the relationship between scale factor, length, area and volume of enlarged shapes.

Overview

When an object is enlarged or reduced by a scale factor, k, then the ratio of any length on the object to the corresponding length on the image is given by:

$$k = \frac{\text{length of image}}{\text{length of corresponding side of object}} \quad (1)$$

Ratios can also be found for the area and volume such that:

$$k^2 = \frac{\text{area of enlarged image}}{\text{area of original object}} \quad (2)$$

and,

$$k^3 = \frac{\text{volume of enlarged image}}{\text{volume of original object}} \quad (3)$$

Note that enlargement always produces images which are similar figures to the objects where they are in proportion to the scale factor of the enlargement.
For triangles, if either one of the following is true, they are **similar triangles:**

- The ratio of corresponding sides are equal.
- The angles of one triangle are equal to corresponding angles in the other triangle.

Solved Examples

1. A triangle has vertices with co-ordinates $A(2,1), B(5,1)$ and $C(5,5)$. It is enlarged from the origin to give an image having vertices $A_1(6,3)$, $B_1(15,3)$ and $C_1(15,15)$.
   a. What is the scale factor of the enlargement?
   b. Draw on the given axes showing clearly the co-ordinates of all vertices triangles ABC and $A_1B_1C_1$.
   c. Measure corresponding lengths of the triangle and image on the graph. Copy and complete the table given. What do you notice?

Length of triangle		Length of image	
AB		A_1B_1	
BC		B_1C_1	
AC		A_1C_1	

Solutions:

a. **Step 1.** Assess and extract the given information from the problem.
Given: triangle with vertices $A(2,1), B(5,1)$ and $C(5,5)$; enlarged from the origin to give image having vertices $A_1(6,3)$, $B_1(15,3)$ and $C_1(15,15)$

Step 2. Find the formula for the mapping

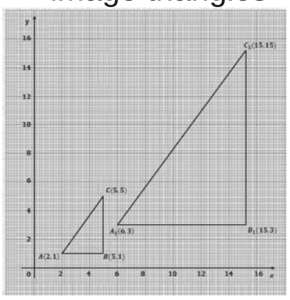
mapping for A $\begin{pmatrix}2\\1\end{pmatrix} \longrightarrow \begin{pmatrix}6\\3\end{pmatrix}$ for A_1

mapping for B $\begin{pmatrix}5\\1\end{pmatrix} \longrightarrow \begin{pmatrix}15\\3\end{pmatrix}$ for B_1

mapping for C $\binom{5}{5} \longrightarrow \binom{15}{3}$ for C_1

from the images it is clear the mapping is $\binom{x}{y} \longrightarrow 3\binom{x}{y}$, $\therefore$ the scale factor is 3.

b. **Step 3.** Draw the object and image triangles



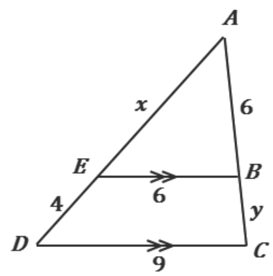
c. **Step 4.** Measure corresponding lengths and complete the table

Length of triangle		Length of image	
AB	3	A_1B_1	9
BC	4	B_1C_1	12
AC	5	A_1C_1	15

d. **Step 5.** Write the relationship connecting scale factor to length.

length of image $= 3 \times$ length of corresponding side of object
where $3 =$ scale factor, k, of the enlargement

2. For the figure shown on the right:
   a. Explain why triangles ABE and ADE are similar.
   b. Find the lengths of x and y. All measurements are in cm.



Solutions:

Given: ΔABE with $AB = 6$ cm, $AE = x$, $EB = 6$ cm
and ΔADE with $BC = y$, $ED = 4$ cm, $DC = 9$ cm

a. $\angle ABE = \angle ACD$
$\angle AEB = \angle ADC$ — since the lines BE and CD are parallel
$\angle EAB = \angle DAC$ — common vertex A

Since the 3 angles are the same in both triangles, they are similar triangles.

b. $k = \frac{\text{length of image}}{\text{length of corresponding side of object}}$

For $EB:DC$ $\quad k = \frac{6}{9} = 1.5$

For $AC:AB$
$$1.5 = \frac{AC}{6}$$
$$AC = 6 \times 1.5$$
$$= 9 \text{ cm}$$
$$\therefore BC = y = 3 \text{ cm}$$

For $AD:AE$
$$1.5 = \frac{4+x}{x}$$
$$1.5x = 4 + x$$
$$0.5x = 4$$
$$\therefore AE = x = 8 \text{ cm}$$

3. The length of a side of a regular pentagon A is 4 times the length of a side of another regular pentagon B.
   a. Find the ratio of the area of A to B.
   b. If the volume of B is 4 cm^3, what is the volume of A?

Solutions:

Given: $\frac{\text{length of one side of pentagon A}}{\text{length of corresponding side of pentagon B}} = \text{scale factor} = 4$

$$\frac{\text{area of pentagon A}}{\text{area of pentagon B}} = 4^2 = 16$$

$$\frac{\text{volume of pentagon A}}{\text{volume of pentagon B}} = 4^3$$

$$\frac{\text{volume of pentagon A}}{4} = 64$$

$\therefore$ Volume of pentagon A $= 4 \times 64 = 256$ cm^3

Practice

1. If ΔABC has vertices with co-ordinates $A(6,4), B(-8,2)$ and $C(10,-4)$, find the image $\Delta A_1B_1C_1$ obtained when ΔABC is enlarged with a scale factor of $-\frac{1}{2}$.
2. A triangle has vertices with co-ordinates $P(3,4), Q(5,4)$ and $R(4,6)$. It is enlarged from the origin to give an image having vertices with co-ordinates $P_1(6,8),\ Q_1(10,8)$ and $R_1(8,12)$.
   a. What is the scale factor of the enlargement?
   b. Use a scale of 2 cm to 2 units on both axes, draw showing clearly the co-ordinates of all vertices triangles PQR and $P_1Q_1R_1$.
   c. Measure corresponding lengths of the triangle and image on the graph.

   Copy and complete the table below.

Length of triangle		Length of image	
PQ		P_1Q_1	
QR		Q_1R_1	
PR		P_1R_1	

   d. What do you notice?
   e. Prove that the ratio of the area of triangles $P_1Q_1R_1$ to PQR is equal to the square of the ratio of the scale factor.
3. A figure is enlarged with a scale factor of 3. If the area of the original figure is 15 cm^2, find the area of the enlarged figure.
4. The length of the side of a cube X is 3 times the length of the side of another cube Y. If the volume of Y is 8 cm^3,
   a. What is the volume of X?
   b. What is the length of cube X?

Lesson Title: Combination of transformations	**Theme:** Vectors and Transformations
Lesson Number: PHM3-L095	**Class:** SSS 3
Learning Outcome By the end of the lesson, pupils will be able to perform a combination of transformations on a plane shape.	

Overview

An object can undergo more than one transformation. This **combination of transformations** can be described by a mapping which gives the single transformation that produces the same end result. We will look at a few examples in this lesson.

Solved Examples

Using a scale of 2 cm to 2 units on both axes, draw two perpendicular axes, Ox and Oy for $-8 \leq x \leq 8$ and $-8 \leq y \leq 8$. All mappings are given as standard (e.g. $A \longrightarrow A_1,\ B \longrightarrow B_1,\ C \longrightarrow C_1$).

1. Draw on the given axes, showing clearly the co-ordinates of all vertices:
   a. Triangle ABC with vertices $A(2,1), B(6,1)$ and $C(2,5)$.
   b. The image $\Delta A_1B_1C_1$ of ΔABC under a reflection in the x-axis $(y = 0)$.
   c. The image $\Delta A_2B_2C_2$ of $\Delta A_1B_1C_1$ under a rotation of $180°$ about the origin.
   d. Describe fully the single transformation that will map ΔABC onto $\Delta A_2B_2C_2$.

Solutions:

a. **Step 1.** Assess and extract the given information from the problem.
Given: $A(2,1), B(6,1)$ and $C(2,5)$

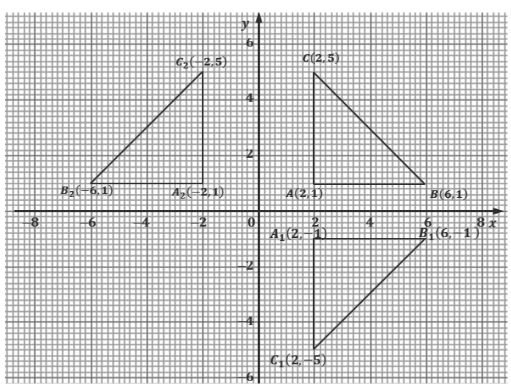
Step 2. Draw the triangle with the given vertices.
ΔABC shown below.

b. **Step 3.** Apply the appropriate mapping formula for reflection in the x-axis.

$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ -y \end{pmatrix}$

$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $\quad$ $\begin{pmatrix} 6 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 6 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 2 \\ 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 \\ -5 \end{pmatrix}$



Step 4. Write the answer and draw the image triangle (shown at right).
$A_1(2,-1), B_1(6,-1)$ and $C_1(2,-5)$

c. Given: $A_1(2,-1), B_1(6,-1)$ and $C_1(2,-5)$

$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} -x \\ -y \end{pmatrix}$ $\quad$ rotation of $180°$ about the origin

$\begin{pmatrix} 2 \\ -1 \end{pmatrix} \longrightarrow \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ $\quad$ $\begin{pmatrix} 6 \\ -1 \end{pmatrix} \longrightarrow \begin{pmatrix} -6 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 2 \\ -5 \end{pmatrix} \longrightarrow \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ We get, $A_2(-2,1), B_2(-6,1)$ and $C_2(-2,5)$

d. From the graph, the reflection on the y-axis $(x = 0)$ maps ΔABC onto $\Delta A_2B_2C_2$

mapping can be written: $\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ -y \end{pmatrix} \longrightarrow \begin{pmatrix} -x \\ -(-y) \end{pmatrix}$

i.e. $\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} -x \\ y \end{pmatrix}$ reflection in the y-axis $(x = 0)$.

2. Draw on the given axes, showing clearly the co-ordinates of all vertices. All mappings are given as standard (e.g. $A \longrightarrow A_1$).
   a. The quadrilateral $ABCD$ with $A(1,5)$, $B(3,5)$, $C(3,8)$ and $D(1,7)$.
   b. The image $A_1B_1C_1D_1$ of $ABCD$ under a clockwise rotation of $90°$ about the origin.
   c. The image $A_2B_2C_2D_2$ of $A_1B_1C_1D_1$ under a translation by the vector $\begin{pmatrix} -4 \\ -2 \end{pmatrix}$.
   d. The image $A_3B_3C_3D_3$ of $A_1B_1C_1D_1$ under a reflection in the line $x = 1$.
   e. Describe the single transformation that will map $ABCD$ onto $A_3B_3C_3D_3$.

Solutions:

Given: $A(1,5)$, $B(3,5)$, $C(3,8)$ and $D(1,7)$

a. All diagrams for this question can be found below.

b. Clockwise rotation of $90°$ about the origin.

$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} y \\ -x \end{pmatrix}$

$\begin{pmatrix} 1 \\ 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 5 \\ -1 \end{pmatrix}$, A_1

$\begin{pmatrix} 3 \\ 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 5 \\ -3 \end{pmatrix}$, B_1

$\begin{pmatrix} 3 \\ 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 8 \\ -3 \end{pmatrix}$, C_1

$\begin{pmatrix} 1 \\ 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 7 \\ -1 \end{pmatrix}$, D_1

c. Given: $A_1(5,-1)$, $B_1(5,-3)$, $C_1(8,-3)$, $D_1(7,-1)$, translation vector $\begin{pmatrix} -4 \\ -2 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x+a \\ y+b \end{pmatrix} = \begin{pmatrix} x+(-4) \\ y+(-2) \end{pmatrix} = \begin{pmatrix} x-4 \\ y-2 \end{pmatrix}$

$\begin{pmatrix} 5 \\ -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 5-4 \\ -1-2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, A_2

$\begin{pmatrix} 5 \\ -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 5-4 \\ -3-2 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$, B_2

$\begin{pmatrix} 8 \\ -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 8-4 \\ -3-2 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$, C_2

$\begin{pmatrix} 7 \\ -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 7-4 \\ -1-2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$, D_2

d. Given: $A_1(5,-1)$, $B_1(5,-3)$, $C_1(8,-3)$ and $D_1(7,-1)$ and line $x = 1$

$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 2k-x \\ y \end{pmatrix} = \begin{pmatrix} 2(1)-x \\ y \end{pmatrix} = \begin{pmatrix} 2-x \\ y \end{pmatrix}$

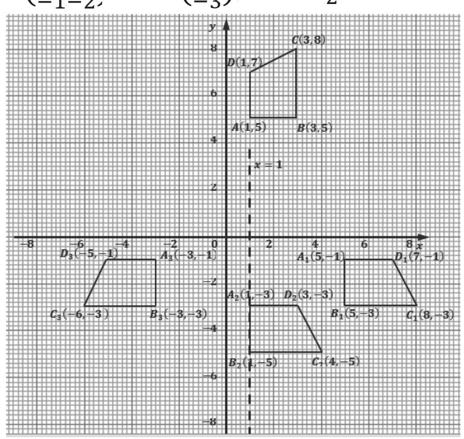
$\begin{pmatrix} 5 \\ -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2-5 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 5 \\ -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 2-5 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix}$

$\begin{pmatrix} 8 \\ -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 2-8 \\ -3 \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \end{pmatrix}$

$\begin{pmatrix} 7 \\ -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2-7 \\ -1 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$

$A_3(-3,-1), B_3(-3,-3), C_3(-6,-3), D_3(-5,-1)$



e. The single transformation is given by the combined mapping:

$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} y \\ -x \end{pmatrix}$ from b.

$\longrightarrow \begin{pmatrix} 2-y \\ -x \end{pmatrix}$ from d. since $x = y$ and $y = -x$

Practice

1. Using a scale of 2 cm to 2 units on each axis, draw on a graph sheet two perpendicular axes Ox and Oy for the intervals $-10 \leq x \leq 10$ and $-12 \leq y \leq 12$.
   a. Draw triangle ABC with co-ordinates A(4,2) B(10,2) and C(6,8).
   b. Draw the image triangle $A_1B_1\,C_1$ of triangle ABC under a reflection in the line x=0 (y-axis) where$A \rightarrow A_1$, $B \rightarrow B_1$, $C \rightarrow C_1$.
   c. Draw the image of triangle $A_2B_2\,C_2$ of triangle$A_1B_1\,C_1$ under a rotation through 180° about the origin, where$A_1 \rightarrow A_2$, $B_1 \rightarrow B_2$, $C_1 \rightarrow C_2$.
   d. Draw the image of triangle $A_3B_3\,C_3$ of triangle $A_2B_2\,C_2$ under an anti-clockwise rotation through 270° about the origin, where$A_2 \rightarrow A_3$, $B_2 \rightarrow B_3$, $C_2 \rightarrow C_3$.

2. Using a scale of 2 cm to 2 units on each axis, draw on a graph sheet two perpendicular axes Ox and Oy for the intervals $-10 \leq x \leq 10$ and $-12 \leq y \leq 12$.
   a. Draw triangle PQR with co-ordinates P(3,3) Q(6,3) and R(3,6).
   b. Draw the image triangle $P_1Q_1\,R_1$ of triangle PQR under a reflection in the line y=x where$P \rightarrow P_1$, $Q \rightarrow Q_1$, $R \rightarrow R_1$.
   c. Draw the image of triangle $P_2Q_2\,R_2$ of triangle P_1QR_1 under a clockwise rotation through 270° about the origin, where $P_1 \rightarrow P_2$, $Q_1 \rightarrow Q_2$, $R_1 \rightarrow R_2$.
   d. Draw the image of triangle $P_3Q_3\,R_3$ of triangle $P_2Q_2\,R_2$ under a reflection in the x-axis (y=0), where $P_2 \rightarrow P_3$, $Q_2 \rightarrow Q_3$, $R_2 \rightarrow R_3$.
   e. Describe the single transformation that maps ΔPQR→ $\Delta P_3Q_3\,R_3$.

3. Using a scale of 2 cm to 2 units on each axis, draw on a graph sheet two perpendicular axes Ox and Oy for the intervals $-12 \leq x \leq 8$ and $-10 \leq y \leq 10$.
   a. Draw triangle ABC with co-ordinates A(-1,5) B(3,6) and C(4,0).
   b. Draw the image triangle $A_1B_1\,C_1$ of triangle ABC under a reflection in the line y-x=0 where$A \rightarrow A_1$, $B \rightarrow B_1$, $C \rightarrow C_1$.
   c. Draw the image of triangle $A_2B_2\,C_2$ of triangle $A_1B_1\,C_1$ under an enlargement from the origin with scale factor -2, where$A_1 \rightarrow A_2$, $B_1 \rightarrow B_2$, $C_1 \rightarrow C_2$.

Lesson Title: Application of transformations	Theme: Vectors and Transformations
Lesson Number: PHM3-L096	**Class:** SSS 3
Learning Outcome By the end of the lesson, pupils will be able to solve problems using transformations.	

Overview

This lesson focuses on application of transformations.

Solved Examples

1. Using a scale of 2 cm to 2 units on both axes, draw two perpendicular axes Ox and Oy for $-8 \leq x \leq 8$ and $0 \leq y \leq 8$.
   Draw on the given axes, showing clearly the co-ordinates of all vertices:
   a. The triangle PQR with $P(3,2)$, $\overrightarrow{PQ} = \binom{4}{1}$ and $\overrightarrow{QR} = \binom{-1}{2}$.
   b. The image triangle $P_1Q_1R_1$ of triangle PQR under an anti-clockwise rotation of $90°$ about the origin where $P \longrightarrow P_1$, $Q \longrightarrow Q_1$ $R \longrightarrow R_1$.
   c. Use the graph to find $\overrightarrow{P_1R_1}$ and hence $|\overrightarrow{P_1R_1}|$ leaving the answer in surd form.

Solutions:

a. **Step 1.** Assess and extract the given information from the problem.
   Given: $P(3,2)$, $\overrightarrow{PQ} = \binom{4}{1}$ and $\overrightarrow{QR} = \binom{-1}{2}$

   Step 2. Find the co-ordinates of Q and R.

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$\binom{4}{1} = \overrightarrow{OQ} - \binom{3}{2}$$
$$\overrightarrow{OQ} = \binom{4}{1} + \binom{3}{2}$$
$$= \binom{7}{3}$$

$$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ}$$
$$\binom{-1}{2} = \overrightarrow{OR} - \binom{7}{3}$$
$$\overrightarrow{OR} = \binom{-1}{2} + \binom{7}{3}$$
$$= \binom{6}{5}$$

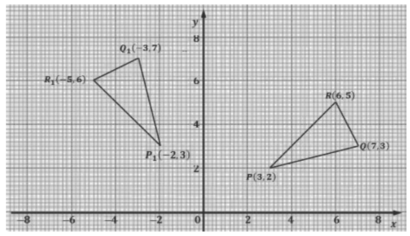
   $P(3,2), Q(7,3)$ and $R(6,5)$.

   Step 3. Draw the triangle with vertices above.
   ΔPQR shown below.

b. **Step 4.** Apply the appropriate mapping formula.

$$\binom{x}{y} \longrightarrow \binom{-y}{x}$$
$$\binom{3}{2} \longrightarrow \binom{-2}{3} \qquad \binom{7}{3} \longrightarrow \binom{-3}{7}$$
$$\binom{6}{5} \longrightarrow \binom{-5}{6}$$

   Step 5. Write the answer and draw the image triangle. (shown below)
   $P_1(-2,3), Q_1(-3,7)$ and $R_1(-5,6)$



c. **Step 6.** Use the graph to find $\overrightarrow{P_1R_1}$ and hence $\left|\overrightarrow{P_1R_1}\right|$.

$$\overrightarrow{P_1R_1} = \begin{pmatrix}-3\\3\end{pmatrix}$$

$$\left|\overrightarrow{P_1R_1}\right| = \sqrt{(-3)^2+3^2}$$

$$= \sqrt{9+9} = \sqrt{18}$$

$$= 3\sqrt{2} \text{ units}$$

2. Using a scale of 2 cm to 2 units on both axes, draw two perpendicular axes Ox and Oy for $-8 \leq x \leq 4$ and $-8 \leq y \leq 4$.
   a. Draw on the same axes, showing clearly the co-ordinates of all vertices:
      ci. The triangle $A(1,0)$, $B(1,3)$ and $C(4,3)$.
      cii. The image $\Delta A_1B_1C_1$ of ΔABC under an enlargement about the origin with scale factor -2 where $A \longrightarrow A_1$, $B \longrightarrow B_1$ and $C \longrightarrow C_1$.
   b. Find the equation of the line BB_1.

Solutions:

Given: $A(1,0)$, $B(1,3)$, $C(4,3)$, scale factor $k=-2$, centre of enlargement

i.

$$\begin{pmatrix}x\\y\end{pmatrix} \longrightarrow k\begin{pmatrix}x\\y\end{pmatrix} = -2\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{pmatrix}1\\0\end{pmatrix} \longrightarrow -2\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-2\\0\end{pmatrix}$$

$$\begin{pmatrix}1\\3\end{pmatrix} \longrightarrow -2\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}-2\\-6\end{pmatrix}$$

$$\begin{pmatrix}4\\3\end{pmatrix} \longrightarrow -2\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}-8\\-6\end{pmatrix}$$

$A_1(-2,0)$, $B_1(-2,-6)$ and $C_1(-8,-6)$

ii. Given: $B(1,3), B_1(-2,-6)$,

gradient $= \frac{y_2-y_1}{x_2-x_1}$ Let $x_1=1,\ y_1=3, x_2=-2,\ y_2=-6$

$= \frac{-6-3}{-2-1}$ substitute the assigned variables

$= \frac{-9}{-3} = 3$

Using standard formula for equation of a straight line and point $B(1,3)$

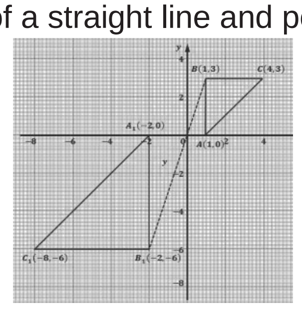
$$\frac{y-3}{x-1} = 3$$

$$y-3 = 3(x-1)$$

$$y-3 = 3x-3$$

$$y = 3x$$

The equation of line BB_1 is $y=3x$



Practice

1. a. Using a scale of 2 cm to 2 units on each axis draw on a graph sheet two perpendicular axes Ox and Oy for the intervals $-10 \leq x \leq 10$ and $-12 \leq y \leq 12$

   b. Draw on the same graph sheet, clearly labelling the vertices and indicating the co-ordinates.

      i. Triangle PQR with P (4,8), $\overrightarrow{QP} = \binom{-2}{2}$ and $\overrightarrow{RP} = \binom{2}{4}$.

      ii. The image triangle $P_1Q_1\,R_1$ of triangle PQR under a reflection in the line y= -2 where $P \to P_1$, $Q \to Q_1$ and $R \to R_1$.

      iii. The image of triangle $P_2Q_2\,R_2$ of triangle PQR under a translation by a vector$\binom{-2}{4}$, where $P \to P_2$, $Q \to Q_2$ and $R \to R_2$.

      iv. The image of triangle $P_3Q_3\,R_3$ of triangle PQR under a rotation through 180° about the origin, where$P \to P_3$, $Q \to Q_3$ and $R \to R_3$

   c. Find $\overrightarrow{Q_2\,Q_3}$.

2. a. Using a scale of 2 cm to 2 units on each axis, draw on a graph sheet two perpendicular axes Ox and Oy for the intervals $-10 \leq x \leq 10$ and $-12 \leq y \leq 12$.

   b. Draw on the same graph sheet indicating the clearly the co-ordinates of all vertices:

      i. The square PQRS with co-ordinates P (2, 2) Q (6, 2) R (6, 6) and S (2, 6).

      ii. The image $P_1Q_1\,R_1S_1$ of the square PQRS under a reflection in y-axis where $p \to p_1$, $Q \to Q_1$, $R \to R_1$, $S \to S_1$.

      iii. The image $P_2Q_2R_2S_2$ of square PQRS under a translation by the vector $\binom{4}{-8}$,where $P \to P_2$, $Q \to Q_2$, $R \to R_2$and $S \to S_2$.

      iv. The image $P_3Q_3R_3S_3$ of square PQRS under a rotation through 180° about the origin where $P \to P_3$, $Q \to Q_3$, $R \to R_3$and $S \to S_3$.

   c. Find the vector$\overrightarrow{P_2\,P_3}$.

3. a. Using a scale of 2 cm to 2 units on each axis, draw on a graph sheet two perpendicular axes Ox and Oy for the intervals $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$.

   b. Given that the point G (4,6) and the vectors $\overrightarrow{GH} = \binom{4}{0}$ and $\overrightarrow{HK} = \binom{-4}{-4}$, draw on the same graph, showing clearly the co-ordinates of their vertices.

      i. Triangle GKH.

      ii. The image $G_1H_1\,K_1$ of triangle GHK under an enlargement from O, the origin, with scale factor $-\frac{1}{2}$ where $G \to G_1$, $H \to H_1$ and $K \to K_1$.

      iii. The image of triangle $G_2H_2\,K_2$ of triangle GHK under an anti-clockwise rotation of 90°about O, the origin, where $G \to G_2$, $H \to H_2$ and $K \to K_2$.

   c. i. Find $\overrightarrow{HH_1}$.

      ii. Describe the single transformation T under which $K_1 \to K_2$.

SSS 3 Maths Term 2 Answer Key

Lesson Title: Expression of ratios
Lesson Number: PHM3-L049

1. a. L=16 cm, w=15 cm b. 5 : 6
2. a. 140 kg b. Le 4,400.00 c. 4.32 km d. 210 mangoes
3. a. 48 m b. 40 min c. 3.5 litres d. 45 pupils

Lesson Title: Comparison of Ratios
Lesson Number: PHM3-L050

1. a. 0.375 : 1 b. 0.75 : 1 c. 1.7 : 1 d. 0.4 : 1
2. a. 1 : 1.4 b. 1 : 1.6 c. 1 : 32 d. 1 : 20,000
3. a. 8 : 17 $>$ 7 : 15 b. 13 : 7 $>$ 11 : 6 c. 7 g : 8 g $>$ 1 1m : 13 m
   d. Le 170.00 : Le 90.00 $>$ Le 300.00 : Le 160.00

Lesson Title: Rate
Lesson Number: PHM3-L051

1. 450 litres 2. 8 labourers 3. Le 486,000.00
4. 37 5. Le 3,800.00 6. 68 km/h

Lesson Title: Proportional division
Lesson Number: PHM3-L052

1. 81 2. 500 3. 240
4. Momodu=Le 1,000.00 Musa= Le 400.00 5. Le 3,712,000.00

Lesson Title: Scales – Part 1
Lesson Number: PHM3-L053

1. a. 1:480,000 b. 72 km
2. a. Width=40 cm length=55 cm b. Width=32 cm length=44 cm
c. Width=16 cm length=22 cm d. Width= 8 cm length=11 cm
3. 1:30
4. a. 59.5 cm b. 47.6 c. 200 cm d. 23.8 cm

Lesson Title: Scales – Part 2
Lesson Number: PHM3-L054

1. a. 96 cm b. 7 cm
2. a. 50 cm b. 7.4 km
3. a. 1:500000 b. 18.8 cm
4. 1:176

Lesson Title: Speed – Part 1
Lesson Number: PHM3- L055

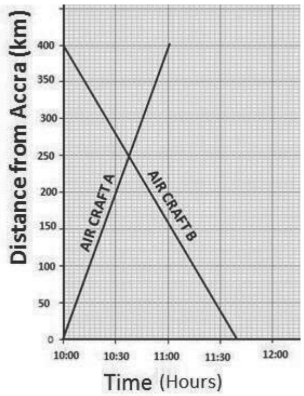
1. 90 km
2. 3.6 km/h
3. 13.3 km/h
4. 4.5 m/s
5. 6 m/s
6. 22.22 m/s

Lesson Title: Speed – Part 2
Lesson Number: PHM3-L056

1. a. y=9 hours b. 10 km/hr
2. 6 km
3. a. k=5 hours b. 20 km/hr
4. a. p=7 hours b. 20 km/hr c. 28 km/hr

Lesson Title: Travel graphs
Lesson Number: PHM3-L057

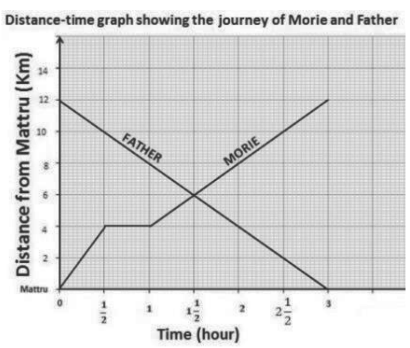
1. a.



b. Aircraft A= 400 km/h
Aircraft B= 300 km/h

c. 1037GMT, 245 km from Accra

2. a.



b. $1\frac{1}{2}$ hours, 6 km from Mattru

Lesson Title: Density
Lesson Number: PHM3-L058

1. 4,800
2. 92
3. 1,850 km^2
4. 2,040,000
5. 1,300
6. 1.03 g/cm^2

Lesson Title: Rates of Pay
Lesson Number: PHM3-L059

1. Le 325,200.00
2. Le 1,875,000.00
3. Le 600,000.00
4. Le 159,500.00
5. Margaret = Le 601,350.00
   Kabba = Le 727,950.00
   Tommy = Le 791,250.00
   Christiana = Le 991,700.00
6. Le 989,583.33

Lesson Title: Commission
Lesson Number: PHM3-L060

1. Le 297,500.00
2. a. Le 450.00 b. Le 1,200.00 c. Le 3,600.00 d. Le 88,500.00
3. Le 860,000.00
4. a. Le 18,690.00 b. Le 62,000.00
5. Le 121,000.00
6. 2.5%

Lesson Title: Income tax
Lesson Number: PHM3-L061

1. a. Le 69,000.00 b. Le 280,000.00
2. Le 1,600,000.00
3. a. Le 650,000.00 b. Le 750,000.00 c. Le 125,000.00
   d. Le 1, 500,000.00 e. Le 1,275,000.00 f. Le 15,300,000.00

Lesson Title: Simple interest
Lesson Number: PHM3-L062

1f 7%
2. Le 40,000.00
3. $3\frac{3}{5} years$
4. Le 80,000.00
5. Le 45,000.00

Lesson Title: Compound interest – Part 1
Lesson Number: PHM3-L063

1. Le 53,876.56 2. Le 49,945.60, Additional interest = Le 1,945.60
3. Le 504,672.82 4. A. Le 878,460.00 b. Le 278,460.00

Lesson Title: Compound interest – Part 2
Lesson Number: PHM3-L064

1. Le 53,876.56 2. Le 2,782.22
3. a. Le 629,856.00 b. Le 129,856.00
4. a. Le 477,613.60 b. Le 27,613.60

Lesson Title: Profit and loss – Part 1
Lesson Number: PHM3-L065

1. 8% 2. Le 29,750.00 3. Le 367,346.94
4. Le 533,333.33 5. Le 1,650,000.00 6. 80%
7. Le 171,900.00

Lesson Title: Profit and loss – Part 2
Lesson Number: PHM3-L066

1. 27% 2. 2%
3. a. Le 750,000.00 b. Le 150,000.00
4. a. Le 400,000.00 b. Le 300,000.00

Lesson Title: Hire purchase
Lesson Number: PHM3-L067

1. Le 1,080,000.00 2. a. Le 7,200,000.00 b. Le 1,380,000.00
3. Le 52,500.00 4. a. Le 1,780,000.00 b. Le 682,333.33

Lesson Title: Discount
Lesson Number: PHM3-L068

1. Le 412,500.00 2. $18\frac{3}{4}\%$
3. a. Le 4,410,000.00 b. Le 1,590,000.00
4. a. Le 332,120.00 b. Le 47,880.00

5. Le 616,000.00 6. 42.9%

Lesson Title: Depreciation	
Lesson Number: PHM3-L069	

1. a. Le 27,127,305.60 b. 45.2%
2. a. Le 1,009,800.00 b. 33%
3. a. Le 14,467,896.32 b. 10.5%
4. Le 1,953,125.00
5. a. Le 4,500,000.00 b. Le 3,280,500.00

Lesson Title: Financial partnerships	
Lesson Number: PHM3-L070	

1. Umaru's share = Le 288,281.25
   Victor's share = Le 191,718.75
2. a. Victoria's share = Le 282,240.00
      Hannah's share = Le 430,080.00
      Janet's share = Le 207,360.00
   b. 56%
3. a. i. Le 1,000,000.00 ii. 35% b. Le 94,500.00

Lesson Title: Foreign exchange	
Lesson Number: PHM3-L071	

1. a. GH¢4,487.18 b. €786.52 c. $886.08
2. a Le 600,000.00 b. Le 2,850,000.00 c. Le 50,000.00
3. Le 240,000.00 4. Le 150,000.00
5. $1,898.73, GH¢5,128.21

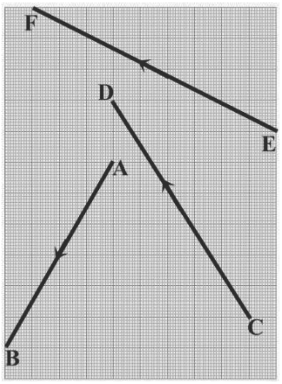
Lesson Title: Additional practice with applications of percentages	
Lesson Number: PHM3-L072	

1. Le 1,356,000.00 2. Le 13,650.00 3. 11.1%
4. Le 855,000.00
5. a. Le 3,500,000.00 b. Le 70,000.00 c. Le 490,000.00

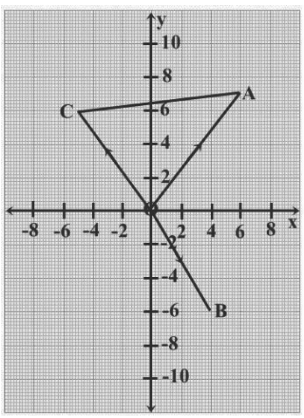
Lesson Title: Introduction to vectors and scalars
Lesson Number: PHM3-L073

1. $\overrightarrow{PQ} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$ $\overrightarrow{RP} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ $\overrightarrow{SR} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$ $\overrightarrow{FG} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}$

2.



3. a.



b. $\overrightarrow{AC} = \begin{pmatrix} -11 \\ -1 \end{pmatrix}$ c. $\overrightarrow{CA} = \begin{pmatrix} 11 \\ 1 \end{pmatrix}$

4. a. $\overrightarrow{AB} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$ b. $\overrightarrow{BC} = \begin{pmatrix} -9 \\ -15 \end{pmatrix}$ c. $\overrightarrow{AC} = \begin{pmatrix} -1 \\ -13 \end{pmatrix}$

Lesson Title: Basic vector properties
Lesson Number: PHM3-L074

1. a. $\overrightarrow{QP} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}$ b. $\overrightarrow{SR} = \begin{pmatrix} -7 \\ 3 \end{pmatrix}$ c. $\overrightarrow{YX} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ d. $\overrightarrow{BA} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$
2. $a = 2,\ b = -2$ 3. $x = 3,\ y = 5$
4. a. $\overrightarrow{BA} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}$ b. $\overrightarrow{YX} = \begin{pmatrix} 9 \\ -7 \end{pmatrix}$ c. $\overrightarrow{SR} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$
5. $a = 1,\ b = -7$ 6. $x = -9,\ y = -5$

Lesson Title: Addition and subtraction of vectors
Lesson Number: PHM3-L075

1. a. $\begin{pmatrix} 7 \\ -1 \end{pmatrix}$ b. $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ c. $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ d. $\begin{pmatrix} -1 \\ 11 \end{pmatrix}$ e. $\begin{pmatrix} 7 \\ -8 \end{pmatrix}$

2. a. $\begin{pmatrix} 8 \\ 1 \end{pmatrix}$ b. $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$ c. $\begin{pmatrix} 4 \\ -5 \end{pmatrix}$ d. $\begin{pmatrix} 12 \\ 7 \end{pmatrix}$

3. a. $\begin{pmatrix} 11 \\ 11 \end{pmatrix}$ b. $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$ c. $\begin{pmatrix} 12 \\ 4 \end{pmatrix}$ d. $\begin{pmatrix} -12 \\ -4 \end{pmatrix}$

4. a. $\begin{pmatrix} -1 \\ -7 \end{pmatrix}$ b. $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ c. $\begin{pmatrix} -7 \\ 4 \end{pmatrix}$ d. $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$

Lesson Title: Multiplication of vectors by scalars
Lesson Number: PHM3-L076

1. a. $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ b. $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ c. $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ d. $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$

   e. $\begin{pmatrix} 1 \\ 11 \end{pmatrix}$ f. $\begin{pmatrix} -3 \\ 10 \end{pmatrix}$ g. $\begin{pmatrix} 8 \\ -9 \end{pmatrix}$

2. a. $\begin{pmatrix} 14 \\ -3 \end{pmatrix}$ b. $\begin{pmatrix} -1 \\ -11 \end{pmatrix}$ c. $\begin{pmatrix} -6 \\ 12 \end{pmatrix}$ d. $\begin{pmatrix} 34 \\ 21 \end{pmatrix}$

3. a. $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$ b. $\begin{pmatrix} -16 \\ 16 \end{pmatrix}$ c. $\begin{pmatrix} -17 \\ 4 \end{pmatrix}$

Lesson Title: Position vectors
Lesson Number: PHM3-L077

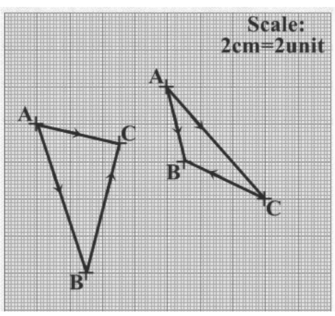
1. a. $\overrightarrow{OX} = \begin{pmatrix} 7 \\ -5 \end{pmatrix}$, $\overrightarrow{OY} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ b. $\begin{pmatrix} -9 \\ 8 \end{pmatrix}$ c. $\begin{pmatrix} 9 \\ -8 \end{pmatrix}$

2. a. $\overrightarrow{AB} = \begin{pmatrix} -4 \\ -7 \end{pmatrix}$, $\overrightarrow{BA} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$ b. $\overrightarrow{AB} = \begin{pmatrix} -9 \\ -2 \end{pmatrix}$, $\overrightarrow{BA} = \begin{pmatrix} 9 \\ 2 \end{pmatrix}$

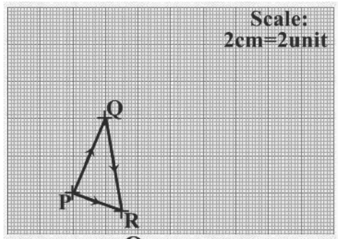
3. T(7, 2) 4. B(7, 12) 5. F(-7, -4)

Lesson Title: Triangle law of vector addition
Lesson Number: PHM3-L078

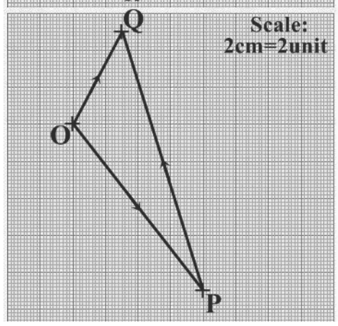
1. a. $\overrightarrow{AC} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$ b. $\overrightarrow{AC} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$



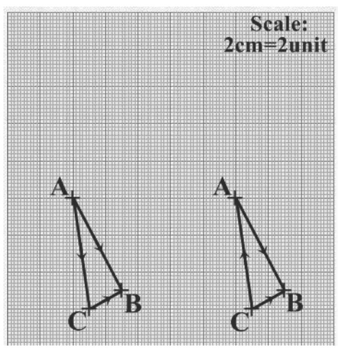
2. $\overrightarrow{QR} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$



3. $\overrightarrow{PQ} = \begin{pmatrix} -5 \\ 14 \end{pmatrix}$



4. $\overrightarrow{AC} = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$ $\overrightarrow{CA} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$



Lesson Title: Mid-point of a line segment
Lesson Number: PHM3-L079

1. M(1.5, 4.5) 2. Y(7, -12) 3. $\overrightarrow{KT} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ 4. $\overrightarrow{OM} = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$.

5. E(9, 9) 6. $\overrightarrow{PC} = \begin{pmatrix} 4.5 \\ 0.5 \end{pmatrix}$.

Lesson Title: Magnitude of a vector
Lesson Number: PHM3-L080

1. a. 8.6 b. 4.5 2. a. $4\sqrt{10}$ b. 12.65

3. $\overrightarrow{XY} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$ $\overrightarrow{YZ} = \begin{pmatrix} -10 \\ 0 \end{pmatrix}$ $\overrightarrow{ZX} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}$ $|\overrightarrow{XY}| = |\overrightarrow{YZ}| = 10$ ΔXYZ is Isosceles

4. $y = 24$ 5. $x = 5$

Lesson Title: Direction of a vector
Lesson Number: PHM3-L081

1. a. $108°$ b. $207°$ 2. a. Q(2,-2) b. $323°$
3. a. $x = -1$ and $y = 3$ b. $\binom{1}{5}$ c. $\binom{-1}{-5}$
4. a. $\binom{-1}{6}$ b. $174°$

Lesson Title: Parallel and perpendicular vectors
Lesson Number: PHM2-L082

1. a. not parallel b. parallel c. parallel
   d. not parallel e. parallel
2. $k = 10$
3. a. not perpendicular b. perpendicular c. perpendicular
   d. not perpendicular e. perpendicular f. not perpendicular
   g. not perpendicular
4. $\overrightarrow{CD} = -2$ m

Lesson Title: Parallelogram law of vector Addition
Lesson Number: PHM3-L083

1. $m = 8$ and $n = 2$
2. $\overrightarrow{AB} = \binom{4}{0}$ $\overrightarrow{DC} = \binom{5-p}{3-q}$, $\overrightarrow{BD} = \binom{p-2}{q-1}$, $p = 1$ and $q = 3$
3. a. $\overrightarrow{AB} = \binom{4}{-4}$ $\overrightarrow{DC} = \binom{5-y}{-y}$ b. $x = 1$ and $y = 4$
4. a. $\overrightarrow{PQ} = \binom{-2}{-3}$ $\overrightarrow{SR} = \binom{6-x}{2-y}$ $\overrightarrow{QS} = \binom{x-1}{y-1}$ b. $x = 8$ and $y = 5$

Lesson Title: Application of vectors – Part 1
Lesson Number: PHM3-L084

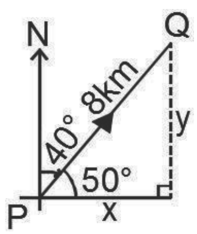
1. Q (10, 2) 2. C (4, 8)
3. a. C (9, 3) b. $\overrightarrow{AC} = \binom{6}{-1}$
4. a. B (3, -1) C (4, -2) D (3, -4) b. $\overrightarrow{MC} = \binom{3}{-2.5}$
5. a. $\overrightarrow{AB} = \binom{-5}{12}$ b. 13 units

Lesson Title: Application of vectors – Part 2
Lesson Number: PHM3-L085

1. a. 320 km b. 264°
2. a. 13 km b. 206°
3. a. 157 km b. 198°
4. a. 5 km b. 37°
5. a. 19 km b. 222°

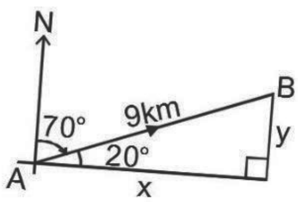
Lesson Title: Application of vectors –Part 3
Lesson Number: PHM3-L086

1. a.



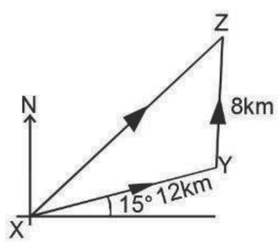
b. $\overrightarrow{PQ}=\binom{5.1}{6.1}$

2. a.



b. $\overrightarrow{AB}=\binom{8.5}{3.1}$

3. a.



b. $\overrightarrow{XY}=\binom{11.6}{3.1}$ $\overrightarrow{YZ}=\binom{0}{8}$ $\overrightarrow{XZ}=\binom{11.6}{11.1}$

c. The cyclist is at a distance 16.1 km from X at a direction $044°$.

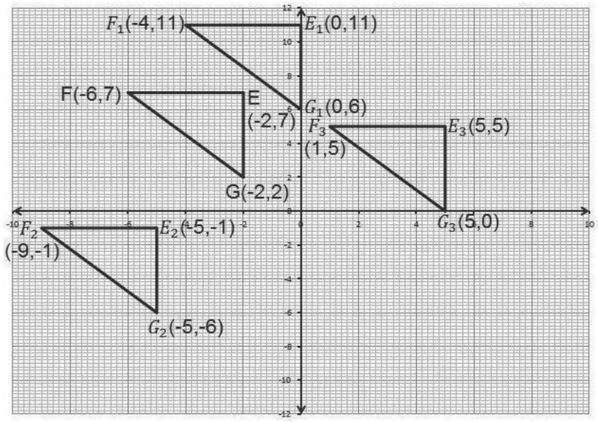
4. a. $\binom{3}{-1.2}$ b. 3.23 m/s at 111.8°

Lesson Title: Translation – Part 1
Lesson Number: PHM3-L087

1. a. A' (5, 5) b. B' (1, 8) c. C' (1, 1) d. D' (5, 0)
2. a. A' (4, 2) b. B' (-3, -1) c. C' (0, -8) d. D' (-4, -7)
3. A(3,4) 4. $\binom{-5}{3}$ 5. $S(-4,-8)$ 6. (7, -6)

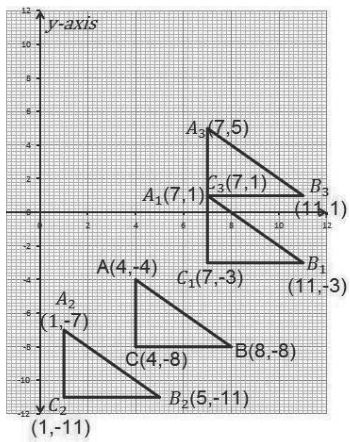
Lesson Title: Translation – Part 2
Lesson Number: PHM3-L088

1. a.



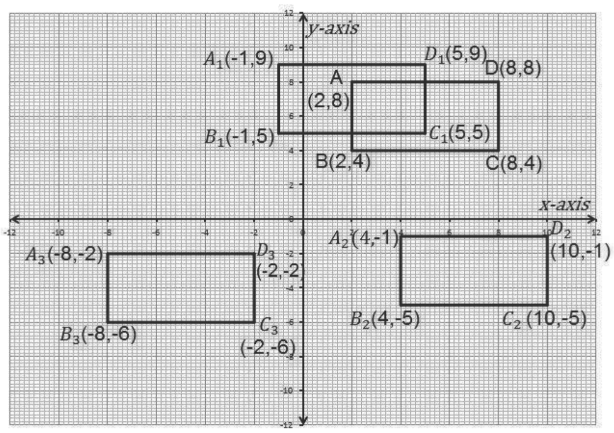
b. $E_1(0, 11)$, $F_1\ (-4, 11)$, $G_1(0, 6)$
c. $E_2(-5, -1)$, $F_2\ (-9, -1)$, $G_2(-5, -6)$
d. $E_3(5\ ,5)$, $F_3\ (1,5)$, $G_3(5,0)$

2. a.



b. $A_1(7,1)$, $B_1\ (11, -3)$, $C_1(7, -3)$
c. $A_2(1, -7)$, $B_2(5, -11)$, $C_2(1, -11)$
d. $A_3(7,5)$, $B_3\ (11,1)$, $C_3(7,1)$

3. a.



b. $A_1(-1,9)$, $B_1\ (-1,5)$, $C_1(5, 5)$, $D_1(5, 9)$
c. $A_2(4, -1)$, $B_2\ (4, -5)$, $C_2(10, -\ 5)$, $D_2(10, -1)$
d. $A_2(-8, -2)$, $B_2\ (-8, -6)$, $C_2(-2, -\ 6)$, $D_2(-2, -2)$

Lesson Title: Reflection – Part 1
Lesson Number: PHM3-L089

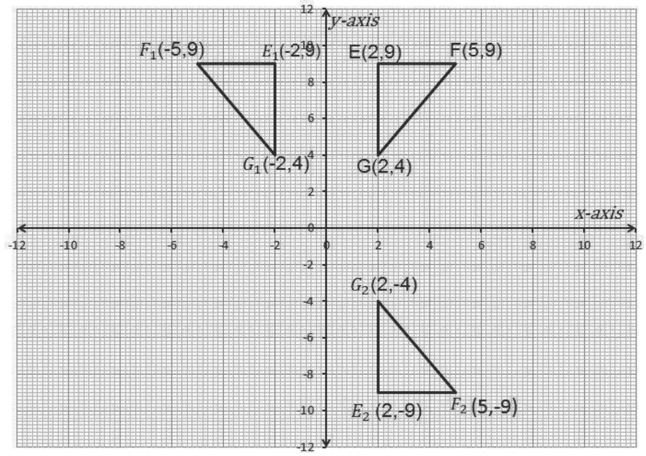
1. A_1(1,-4) , $B_1(2, -3)$ $C_1(3, -5)$ D_1(4,-6)

2. P_1(-1, 4) , $Q_1(-2, 3)$, $R_1(-3\ ,5)$ S_1(-4,6)

3. a. A_1(1, 10)   b. A_1(9, -6)   c. A_1(-9, -6)

d. A_1(-3, -6)   e. A_1(1, 4)

4. a.



b. $E_1(-2{,}9)$ $F_1\ (-5{,}9)$ G_1(-2,4)

c. $E_2\ (2,-9)$ $F_2\ (5,-9)$ $G_2(2,-4)$

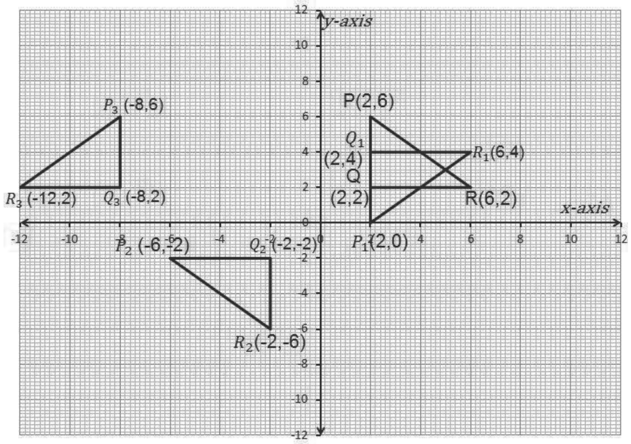
Lesson Title: Reflection – Part 2
Lesson Number: PHM3-L090

1. $K_1(-8,-4)$

2. $A_1(-4,-2\ )$ $B_1\ (-3,-1)$ $C_1(-4,-5)$

3. X_1(3, 1) $Y_1(5{,}4)$ $Z_1(7{,}3)$

4. a.

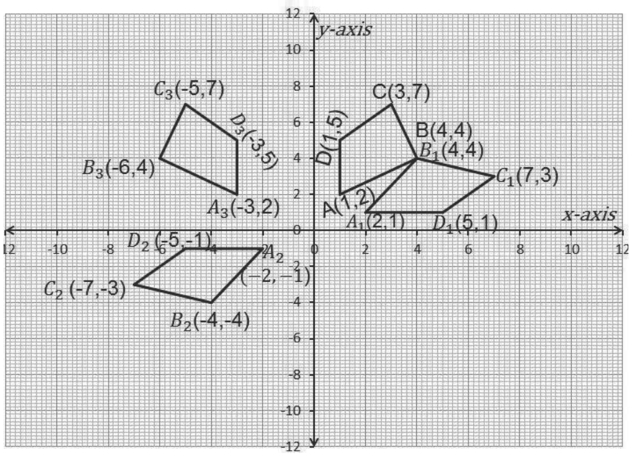


b. $P_1(2{,}0)$ $Q_1\ (2{,}4)$ R_1(6, 4)

c. $P_2(-6,-2)$ $Q_2\ (-2,-2)$$R_2$(-2,-6)

d. $P_3(-8,6)$ $Q_3\ (-8{,}2)$ R_3(-12, 2)

5. a.



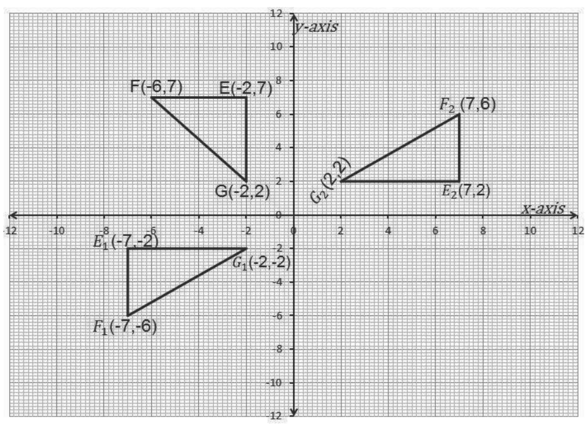
b. $A_1(2,1)$ B_1 $(4,4)$ $C_1(7,3)$ $D_1(5,1)$
c. $A_2(-2,-1)$ B_2 $(-4,-4)$
$C_2(-7,-3)$ $D_2(-5,-1)$
d. $A_3(-3,2)$ B_3 $(-6,4)$
$C_3(-5,7)$ $D_3(-3,5)$

Lesson Title: Rotation – Part 1
Lesson Number: PHM3-L091

1. Q_1(6,-3)

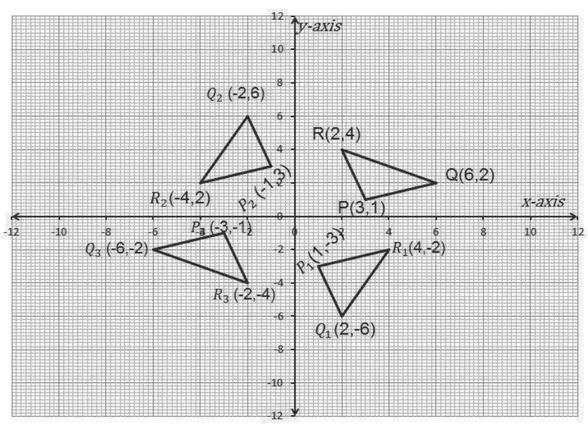
2. A_1(-4,3) B_1 $(-3,2)$ C_1(-4,2)

3. a.



b. $E_1(-7,-2)$ $F_1(-7,-6)$ $G_1(-2,-2)$
c. $E_2(7,2)$ F_2 $(7,6)$ $G_2(2,2)$

4. a.

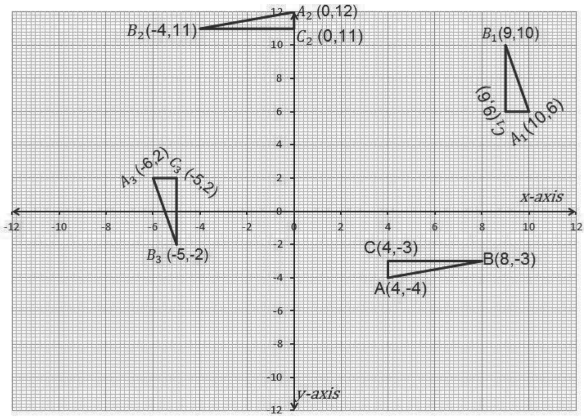


b. $P_1(1,-3)$ Q_1 $(2,-6)$ $R_1(4,-2)$
c. $P_2(-1,3)$ Q_2 $(-2,6)$ $R_2(-4,2)$
d. $P_3(-3,-1)$ Q_3 $(-6,-2)$ $R_3(-2,-4$

Lesson Title: Rotation – Part 2
Lesson Number: PHM3-L092

1. a. $(y - b + a, -x + a + b)$ b. $R_1(-4, -7)$
2. a. $(-y + b + a, x - a + b)$ b. $P_1(13, -2)$
3. a. $(-x + 2a\ , -y + 2b)$ b. $Q_1(-10, -3)$
4. a. $A_1(3, -8)$ $B_1\ (2, -7)$ $C_1(3, -7)$

   b. $A_1(5, -10)$ $B_1\ (6, -9)$ $C_1(6, -10)$
5. a.



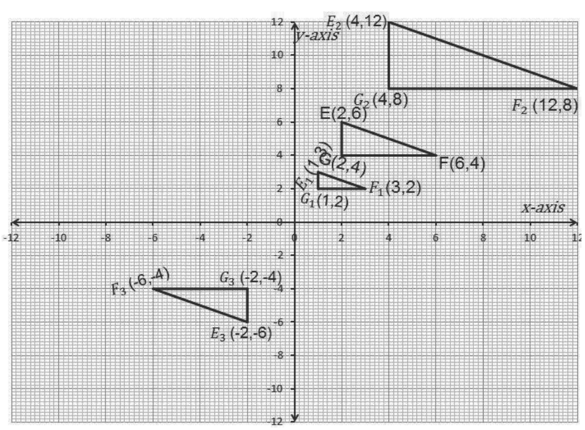
   b. $A_1(10{,}6)$ $B_1\ (9{,}10)$ $C_1(9{,}6)$

   c. $A_2(0{,}12)$ $B_2\ (-4{,}11)$ $C_2(0{,}11)$

   d. $A_3(-6{,}2)$ $B_3\ (-5, -2)$ $C_3(-5{,}2)$

Lesson Title: Enlargement – Part 1
Lesson Number: PHM3-L093

1. a. $D_1(-12{,}6)$ b. $D_1(-22, -6)$
2. a.

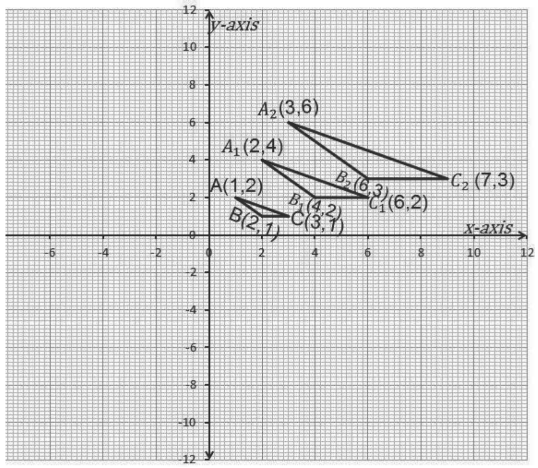


   b. $E_1(1{,}3)$ $F_1\ (3{,}2)$ $G_1(1{,}2)$

   c. $E_2(4{,}12)$ $F_2\ (12{,}8)$ $G_2(4{,}8)$

   d. $E_3(-2, -6)$ $F_3\ (-6, -4)$ $G_3(-2, -4)$

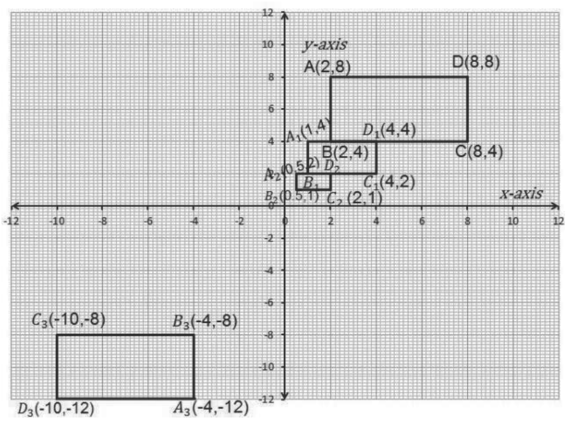
3. a.



b. $A_1(2,4)$ $B_1\,(4\,,2)$ $C_1(6,2)$

c. $A_2(3,6)$ $B_2\,(6\,,3)$ $C_2(9,3)$

4. a.



b. $A_1(1,4)$ $B_1\,(1,2)$ $C_1(4,2)$ $D_1(4,4)$

c. $A_2(0.5,2)$ $B_2\,(0.5,1)$ $C_2(2,1)$ $D_2(2,2)$

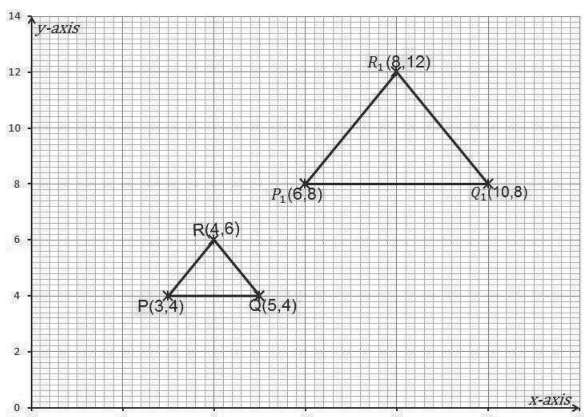
d. $A_3(-4,-12)$ $B_3\,(-4,-8)$
$C_3(-10,-8)$ $D_3(-10,-12)$

Lesson Title: Enlargement – Part 2
Lesson Number: PHM3-L094

1. $A_1(-3,-2)$ $B_1(4,-1)$ $C_1(-5,2)$

2. a. 2

b.



c.

Length of triangle		Length of image	
PQ	2 cm	P_1Q_1	4 cm
QR	2 cm	Q_1R_1	4 cm
PR	2 cm	P_1R_1	4 cm

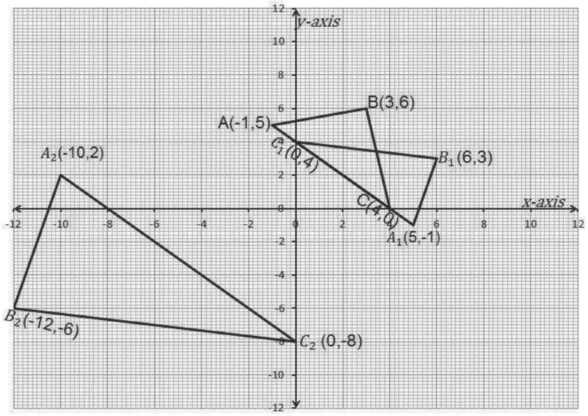
d. Scale factor of enlargement=2

e. $\text{Ratio} = 4 = 2^2$

3. 135 cm^2 4. a. 216 cm^3 b. 6 cm

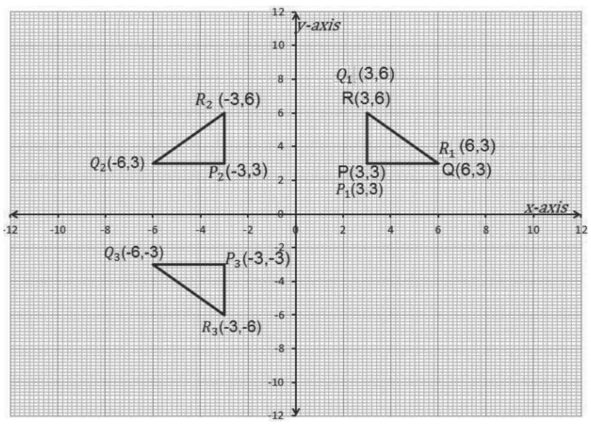
Lesson Title: Combination of Transformations	
Lesson Number: PHM3-L095	

1\. a.



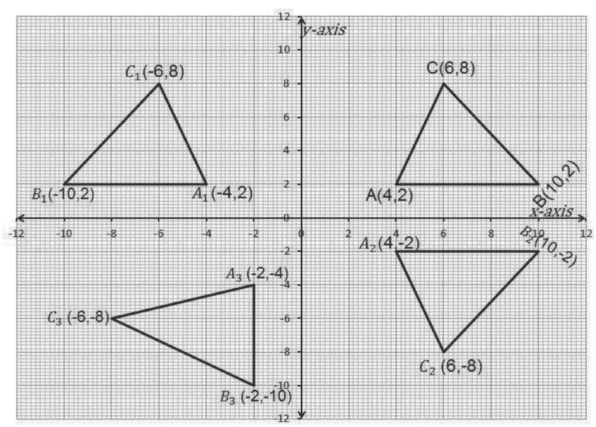
b. $A_1(-4,2)$ B_1 $(-10,2)$ $C_1(-6,8)$
c. $A_2(4,-2)$ B_2 $(10,-2)$ $C_2(6,-8)$
d. $A_3(-2,-4)$ B_3 $(-2,-10)$
$C_3(-8,-6)$

2\. a.



b. $P_1(3{,}3\ )$ Q_1 $(3{,}6)$ $R_1(6{,}3)$
c. $P_2(-3{,}3\ )$ Q_2 $(-6{,}3)$ $R_2(-3{,}6)$
d. $P_3(-3,-3\ )$ Q_3 $(-6,-3)$
$R_3(-3,-6)$
e. The single transformation that maps ΔPQR→ $\Delta P_3Q_3\,R_3$ is a rotation through 180° clockwise or anti-clockwise about the origin.

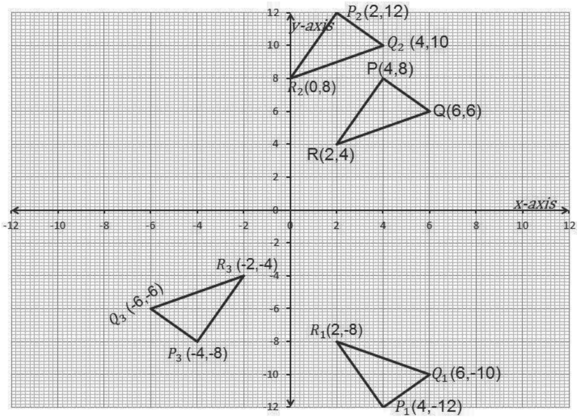
3\. a.



b. $A_1(5,-1)$ B_1 $(6,3)$ $C_1(0,4)$
c.
$A_2(-10,2)$ B_2 $(-12,-6)$ $C_2(0,-8)$

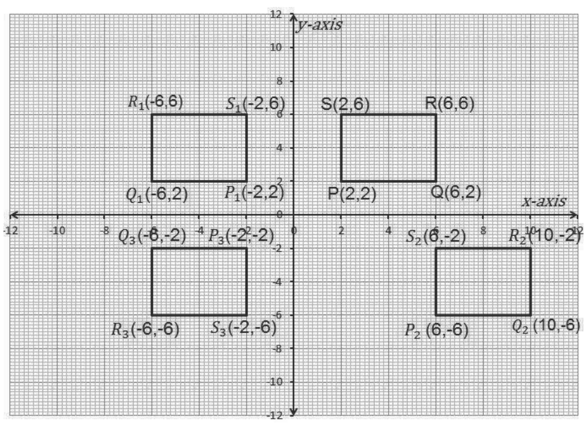
Lesson Title: Application of transformations
Lesson Number: PHM3-L096

1. a.
   b.



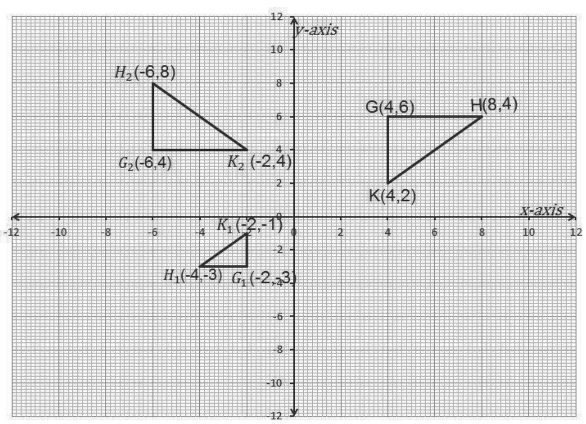
b. i. Q(6,6) R(2,4)
ii. $P_1(4,-12)$ $Q_1\,(6,-10)$ $R_1(2,-8)$
iii. $P_2(2,12)$ $Q_2\,(4,10)$ $R_2(0,8)$
iv. $P_3(-4,-8)$ $Q_3\,(-6,-6)$ $R_3(-2,-4)$
c. $\overrightarrow{Q_2\,Q_3}=\begin{pmatrix}-10\\-16\end{pmatrix}$

2. a.
   b. i.



b. ii. $P_1(-2,2)$ $Q_1\,(-6,2)$ $R_1(-6,6)$ $S_1(-2,6)$
iii. $P_2(6,-6)$ $Q_2\,(10,-6)$ $R_2(10,-2)$ $S_2(6,-2)$
iv. $P_3(-2,-2)$ $Q_3\,(-6,-2)$ $R_3(-6,-6)$ $S_3(-2,-6)$
c. $\overrightarrow{P_2\,P_3}.=\begin{pmatrix}-8\\4\end{pmatrix}$

3. a.



b. i. H(8,6) K(4,2)
ii. $G_1(-2,-3)$ $H_1\,(-2,-1)$ $K_1(-4,-3)$
iii. $G_2(-6,4)$ $H_2\,(-2,4)$ $K_2(-6,8)$
c. i. $\overrightarrow{HH_1}=\begin{pmatrix}-12\\-9\end{pmatrix}$
ii. The single transformation T under which $K_1.\rightarrow K_2$ is a reflection in the line $y=1$ or $y-1=0$.

Appendix I: Sines of Angles

Sines of Angles (*x* in degrees)

$x \rightarrow \sin x$

x	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	ADD Differences 1	2	3	4	5	6	7	8	9
0	0·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	2	3	5	7	9	10	12	14	16
1	0·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	2	3	5	7	9	10	12	14	16
2	0·0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	2	3	5	7	9	10	12	14	16
3	0·0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	2	3	5	7	9	10	12	14	16
4	0·0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	2	3	5	7	9	10	12	14	16
5	0·0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	2	3	5	7	9	10	12	14	16
6	0·1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	2	3	5	7	9	10	12	14	16
7	0·1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	2	3	5	7	9	10	12	14	16
8	0·1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	2	3	5	7	9	10	12	14	16
9	0·1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	2	3	5	7	9	10	12	14	15
10	0·1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	2	3	5	7	9	10	12	14	15
11	0·1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	2	3	5	7	9	10	12	14	15
12	0·2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	2	3	5	7	9	10	12	14	15
13	0·2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	2	3	5	7	8	10	12	14	15
14	0·2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	2	3	5	7	8	10	12	14	15
15	0·2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	2	3	5	7	8	10	12	13	15
16	0·2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	2	3	5	7	8	10	12	13	15
17	0·2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	2	3	5	7	8	10	12	13	15
18	0·3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	2	3	5	7	8	10	12	13	15
19	0·3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	2	3	5	7	8	10	12	13	15
20	0·3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	2	3	5	7	8	10	11	13	15
21	0·3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	2	3	5	6	8	10	11	13	15
22	0·3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	2	3	5	6	8	10	11	13	15
23	0·3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	2	3	5	6	8	10	11	13	14
24	0·4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	2	3	5	6	8	10	11	13	14
25	0·4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	2	3	5	6	8	9	11	13	14
26	0·4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	2	3	5	5	8	9	11	12	14
27	0·4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	2	3	5	6	8	9	11	12	14
28	0·4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	2	3	5	6	8	9	11	12	14
29	0·4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	2	3	5	6	8	9	11	12	14
30	0·5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	2	3	5	6	8	9	11	12	14
31	0·5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	1	3	4	6	7	9	10	12	13
32	0·5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	1	3	4	6	7	9	10	12	13
33	0·5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	1	3	4	6	7	9	10	12	13
34	0·5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	1	3	4	6	7	9	10	12	13
35	0·5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	1	3	4	6	7	9	10	11	13
36	0·5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	1	3	4	6	7	8	10	11	13
37	0·6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	1	3	4	6	7	8	10	11	12
38	0·6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	1	3	4	5	7	8	10	11	12
39	0·6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	1	3	4	5	7	8	9	11	12
40	0·6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	1	3	4	5	7	8	9	11	12
41	0·6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	1	3	4	5	7	8	9	10	12
42	0·6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	1	3	4	5	6	8	9	10	12
43	0·6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	1	3	4	5	6	8	9	10	11
44	0·6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	1	2	4	5	6	7	9	10	11
x	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	2	3	4	5	6	7	8	9

x	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	ADD Differences 1	2	3	4	5	6	7	8	9
45	0·7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	1	2	4	5	6	7	9	10	11
46	0·7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	1	2	4	5	6	7	8	10	11
47	0·7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	1	2	4	5	6	7	8	9	11
48	0·7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	1	2	3	5	6	7	8	9	10
49	0·7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	1	2	3	5	6	7	8	9	10
50	0·7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	1	2	3	4	6	7	8	9	10
51	0·7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	1	2	3	4	5	7	8	9	10
52	0·7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	1	2	3	4	5	6	7	8	10
53	0·7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	1	2	3	4	5	6	7	8	9
54	0·8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	1	2	3	4	5	6	7	8	9
55	0·8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	1	2	3	4	5	6	7	8	9
56	0·8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	1	2	3	4	5	6	7	8	9
57	0·8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	1	2	3	4	5	6	7	8	8
58	0·8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	1	2	3	4	5	5	6	7	8
59	0·8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	2	3	4	4	5	6	7	8
60	0·8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1	2	3	3	4	5	6	7	8
61	0·8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	2	2	3	4	5	6	7	7
62	0·8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	2	2	3	4	5	6	6	7
63	0·8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	2	2	3	4	5	5	6	7
64	0·8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	2	2	3	4	5	5	6	7
65	0·9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	1	2	3	4	4	5	6	7
66	0·9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	1	2	3	3	4	5	6	6
67	0·9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	1	2	3	3	4	5	5	6
68	0·9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	1	2	3	3	4	4	5	6
69	0·9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	1	2	2	3	4	4	5	6
70	0·9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	1	2	2	3	3	4	5	5
71	0·9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	1	2	2	3	3	4	4	5
72	0·9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	1	2	2	3	3	4	4	5
73	0·9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	0	1	1	2	2	3	3	4	4
74	0·9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	0	1	1	2	2	3	3	4	4
75	0·9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	0	1	1	2	2	3	3	3	4
76	0·9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	0	1	1	2	2	2	3	3	4
77	0·9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	0	1	1	2	2	2	3	3	3
78	0·9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	0	1	1	1	2	2	2	3	3
79	0·9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	0	1	1	1	2	2	2	3	3
80	0·9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	1	1	2	2	2	3
81	0·9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	1	1	2	2	2	2
82	0·9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	0	1	1	1	1	2	2	2
83	0·9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	0	1	1	1	1	1	2	2
84	0·9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	0	1	1	1	1	1	1	2
85	0·9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0	0	0	1	1	1	1	1	1
86	0·9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	0	0	1	1	1	1	1
87	0·9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0	0	0	0	0	0	1	1	1
88	0·9994	9995	9995	9996	9996	9997	9997	9997	9998	9998									
89	0·9998	9999	9999	9999	9999	1·0000	1·0000	1·0000	1·0000	1·0000									
x	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	2	3	4	5	6	7	8	9

Appendix II: Cosines of Angles

											SUBTRACT Differences								
x	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	2	3	4	5	6	7	8	9
0	1·0000	1·000	1·000	1·000	1·000	1·000	·9999	·9999	·9999	·9999									
1	0·9998	·9998	·9998	·9997	·9997	·9997	·9996	·9996	·9995	·9995									
2	0·9994	·9993	·9993	·9992	·9991	·9990	·9990	·9989	·9988	·9987	0	0	0	0	0	0	1	1	1
3	0·9986	·9985	·9984	·9983	·9982	·9981	·9980	·9979	·9978	·9977	0	0	0	0	1	1	1	1	1
4	0·9976	·9974	·9973	·9972	·9971	·9969	·9968	·9966	·9965	·9963	0	0	0	1	1	1	1	1	1
5	0·9962	·9960	·9959	·9957	·9956	·9954	·9952	·9951	·9949	·9947	0	0	1	1	1	1	1	1	2
6	0·9945	·9943	·9942	·9940	·9938	·9936	·9934	·9932	·9930	·9928	0	0	1	1	1	1	1	2	2
7	0·9925	·9923	·9921	·9919	·9917	·9914	·9912	·9910	·9907	·9905	0	0	1	1	1	1	2	2	2
8	0·9903	·9900	·9898	·9895	·9893	·9890	·9888	·9885	·9882	·9880	0	1	1	1	1	2	2	2	2
9	0·9877	·9874	·9871	·9869	·9866	·9863	·9860	·9857	·9854	·9851	0	1	1	1	1	2	2	2	3
10	0·9848	·9845	·9842	·9839	·9836	·9833	·9829	·9826	·9823	·9820	0	1	1	1	2	2	2	3	3
11	0·9816	·9813	·9810	·9806	·9803	·9799	·9796	·9792	·9789	·9785	0	1	1	1	2	2	2	3	3
12	0·9781	·9778	·9774	·9770	·9767	·9763	·9759	·9755	·9751	·9748	0	1	1	2	2	2	3	3	3
13	0·9744	·9740	·9736	·9732	·9728	·9724	·9720	·9715	·9711	·9707	0	1	1	2	2	2	3	3	4
14	0·9703	·9699	·9694	·9690	·9686	·9681	·9677	·9673	·9668	·9664	0	1	1	2	2	3	3	3	4
15	0·9659	·9655	·9650	·9646	·9641	·9636	·9632	·9627	·9622	·9617	0	1	1	2	2	3	3	4	4
16	0·9613	·9608	·9603	·9598	·9593	·9588	·9583	·9578	·9573	·9568	0	1	1	2	2	3	3	4	4
17	0·9563	·9558	·9553	·9548	·9542	·9537	·9532	·9527	·9521	·9516	1	1	2	2	3	3	4	4	5
18	0·9511	·9505	·9500	·9494	·9489	·9483	·9478	·9472	·9466	·9461	1	1	2	2	3	3	4	4	5
19	0·9455	·9449	·9444	·9438	·9432	·9426	·9421	·9415	·9409	·9403	1	1	2	2	3	3	4	5	5
20	0·9397	·9391	·9385	·9379	·9373	·9367	·9361	·9354	·9348	·9342	1	1	2	2	3	4	4	5	6
21	0·9336	·9330	·9323	·9317	·9311	·9304	·9298	·9291	·9285	·9278	1	1	2	3	3	4	4	5	6
22	0·9272	·9265	·9259	·9252	·9245	·9239	·9232	·9225	·9219	·9212	1	1	2	3	3	4	5	5	6
23	0·9205	·9198	·9191	·9184	·9178	·9171	·9164	·9157	·9150	·9143	1	1	2	3	3	4	5	6	6
24	0·9135	·9128	·9121	·9114	·9107	·9100	·9092	·9085	·9078	·9070	1	1	2	3	4	4	5	6	7
25	0·9063	·9056	·9048	·9041	·9033	·9026	·9018	·9011	·9003	·8996	1	2	2	3	4	5	5	6	7
26	0·8988	·8980	·8973	·8965	·8957	·8949	·8942	·8934	·8926	·8918	1	2	2	3	4	5	5	6	7
27	0·8910	·8902	·8894	·8886	·8878	·8870	·8862	·8854	·8846	·8838	1	2	2	3	4	5	6	6	7
28	0·8829	·8821	·8813	·8805	·8796	·8788	·8780	·8771	·8763	·8755	1	2	2	3	4	5	6	7	7
29	0·8746	·8738	·8729	·8721	·8712	·8704	·8695	·8686	·8678	·8669	1	2	3	3	4	5	6	7	8
30	0·8660	·8652	·8643	·8634	·8625	·8616	·8607	·8599	·8590	·8581	1	2	3	4	4	5	6	7	8
31	0·8572	·8563	·8554	·8545	·8536	·8526	·8517	·8508	·8499	·8490	1	2	3	4	5	5	6	7	8
32	0·8480	·8471	·8462	·8453	·8443	·8434	·8425	·8415	·8406	·8396	1	2	3	4	5	6	7	8	8
33	0·8387	·8377	·8368	·8358	·8348	·8339	·8329	·8320	·8310	·8300	1	2	3	4	5	6	7	8	9
34	0·8290	·8281	·8271	·8261	·8251	·8241	·8231	·8221	·8211	·8202	1	2	3	4	5	6	7	8	9
35	0·8192	·8181	·8171	·8161	·8151	·8141	·8131	·8121	·8111	·8100	1	2	3	4	5	6	7	8	9
36	0·8090	·8080	·8070	·8059	·8049	·8039	·8028	·8018	·8007	·7997	1	2	3	4	5	6	7	8	9
37	0·7986	·7976	·7965	·7955	·7944	·7934	·7923	·7912	·7902	·7891	1	2	3	4	5	6	7	8	10
38	0·7880	·7869	·7859	·7848	·7837	·7826	·7815	·7804	·7793	·7782	1	2	3	4	5	6	7	9	10
39	0·7771	·7760	·7749	·7738	·7727	·7716	·7705	·7694	·7683	·7672	1	2	3	4	6	7	8	9	10
40	0·7660	·7649	·7638	·7627	·7615	·7604	·7593	·7581	·7570	·7559	1	2	3	5	6	7	8	9	10
41	0·7547	·7536	·7524	·7513	·7501	·7490	·7478	·7466	·7455	·7443	1	2	3	5	6	7	8	9	10
42	0·7431	·7420	·7408	·7396	·7385	·7373	·7361	·7349	·7337	·7325	1	2	4	5	6	7	8	9	11
43	0·7314	·7302	·7290	·7278	·7266	·7254	·7242	·7230	·7218	·7206	1	2	4	5	6	7	8	10	11
44	0·7193	·7181	·7169	·7157	·7145	·7133	·7120	·7108	·7096	·7083	1	2	4	5	6	7	9	10	11
x	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	2	3	4	5	6	7	8	9

											SUBTRACT Differences								
x	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	2	3	4	5	6	7	8	9
45	0·7071	·7059	·7046	·7034	·7022	·7009	·6997	·6984	·6972	·6959	1	2	4	5	6	7	9	10	11
46	0·6947	·6934	·6921	·6909	·6896	·6884	·6871	·6858	·6845	·6833	1	3	4	5	6	8	9	10	11
47	0·6820	·6807	·6794	·6782	·6769	·6756	·6743	·6730	·6717	·6704	1	3	4	5	6	8	9	10	12
48	0·6691	·6678	·6665	·6652	·6639	·6626	·6613	·6600	·6587	·6574	1	3	4	5	7	8	9	10	12
49	0·6561	·6547	·6534	·6521	·6508	·6494	·6481	·6468	·6455	·6441	1	3	4	5	7	8	9	11	12
50	0·6428	·6414	·6401	·6388	·6374	·6361	·6347	·6334	·6320	·6307	1	3	4	5	7	8	9	11	12
51	0·6293	·6280	·6266	·6252	·6239	·6225	·6211	·6198	·6184	·6170	1	3	4	5	7	8	10	11	12
52	0·6157	·6143	·6129	·6115	·6101	·6088	·6074	·6060	·6046	·6032	1	3	4	5	7	8	10	11	12
53	0·6018	·6004	·5990	·5976	·5962	·5948	·5934	·5920	·5906	·5892	1	3	4	6	7	8	10	11	13
54	0·5878	·5864	·5850	·5835	·5821	·5807	·5793	·5779	·5764	·5750	1	3	4	6	7	9	10	11	13
55	0·5736	·5721	·5707	·5693	·5678	·5664	·5650	·5635	·5621	·5606	1	3	4	6	7	9	10	12	13
56	0·5592	·5577	·5563	·5548	·5534	·5519	·5505	·5490	·5476	·5461	1	3	4	6	7	9	10	12	13
57	0·5446	·5432	·5417	·5402	·5388	·5373	·5358	·5344	·5329	·5314	1	3	4	6	7	9	10	12	13
58	0·5299	·5284	·5270	·5255	·5240	·5225	·5210	·5195	·5180	·5165	1	3	4	6	7	9	10	12	13
59	0·5150	·5135	·5120	·5105	·5090	·5075	·5060	·5045	·5030	·5015	2	3	5	6	8	9	11	12	14
60	0·5000	·4985	·4970	·4955	·4939	·4924	·4909	·4894	·4879	·4863	2	3	5	6	8	9	11	12	14
61	0·4848	·4833	·4818	·4802	·4787	·4772	·4756	·4741	·4726	·4710	2	3	5	6	8	9	11	12	14
62	0·4695	·4679	·4664	·4648	·4633	·4617	·4602	·4586	·4571	·4555	2	3	5	6	8	9	11	12	14
63	0·4540	·4524	·4509	·4493	·4478	·4462	·4446	·4431	·4415	·4399	2	3	5	6	8	9	11	12	14
64	0·4384	·4368	·4352	·4337	·4321	·4305	·4289	·4274	·4258	·4242	2	3	5	6	8	9	11	13	14
65	0·4226	·4210	·4195	·4179	·4163	·4147	·4131	·4115	·4099	·4083	2	3	5	6	8	10	11	13	14
66	0·4067	·4051	·4035	·4019	·4003	·3987	·3971	·3955	·3939	·3923	2	3	5	6	8	10	11	13	14
67	0·3907	·3891	·3875	·3859	·3843	·3827	·3811	·3795	·3778	·3762	2	3	5	6	8	10	11	13	15
68	0·3746	·3730	·3714	·3697	·3681	·3665	·3649	·3633	·3616	·3600	2	3	5	6	8	10	11	13	15
69	0·3584	·3567	·3551	·3535	·3518	·3502	·3486	·3469	·3453	·3437	2	3	5	6	8	10	11	13	15
70	0·3420	·3404	·3387	·3371	·3355	·3338	·3322	·3305	·3289	·3272	2	3	5	7	8	10	12	13	15
71	0·3256	·3239	·3223	·3206	·3190	·3173	·3156	·3140	·3123	·3107	2	3	5	7	8	10	12	13	15
72	0·3090	·3074	·3057	·3040	·3024	·3007	·2990	·2974	·2957	·2940	2	3	5	7	8	10	12	13	15
73	0·2924	·2907	·2890	·2874	·2857	·2840	·2823	·2807	·2790	·2773	2	3	5	7	8	10	12	13	15
74	0·2756	·2740	·2723	·2706	·2689	·2672	·2656	·2639	·2622	·2605	2	3	5	7	8	10	12	13	15
75	0·2588	·2571	·2554	·2538	·2521	·2504	·2487	·2470	·2453	·2436	2	3	5	7	8	10	12	14	15
76	0·2419	·2402	·2385	·2368	·2351	·2334	·2317	·2300	·2284	·2267	2	3	5	7	8	10	12	14	15
77	0·2250	·2233	·2215	·2198	·2181	·2164	·2147	·2130	·2113	·2096	2	3	5	7	9	10	12	13	15
78	0·2079	·2062	·2045	·2028	·2011	·1994	·1977	·1959	·1942	·1925	2	3	5	7	9	10	12	14	15
79	0·1908	·1891	·1874	·1857	·1840	·1822	·1805	·1788	·1771	·1754	2	3	5	7	9	10	12	14	15
80	0·1736	·1719	·1702	·1685	·1668	·1650	·1633	·1616	·1599	·1582	2	3	5	7	9	10	12	14	15
81	0·1564	·1547	·1530	·1513	·1495	·1478	·1461	·1444	·1426	·1409	2	3	5	7	9	10	12	14	16
82	0·1392	·1374	·1357	·1340	·1323	·1305	·1288	·1271	·1253	·1236	2	3	5	7	9	10	12	14	16
83	0·1219	·1201	·1184	·1167	·1149	·1132	·1115	·1097	·1080	·1063	2	3	5	7	9	10	12	14	16
84	0·1045	·1028	·1011	·0993	·0976	·0958	·0941	·0924	·0906	·0889	2	3	5	7	9	10	12	14	16
85	0·0872	·0854	·0837	·0819	·0802	·0785	·0767	·0750	·0732	·0715	2	3	5	7	9	10	12	14	16
86	0·0698	·0680	·0663	·0645	·0628	·0610	·0593	·0576	·0558	·0541	2	3	5	7	9	10	12	14	16
87	0·0523	·0506	·0488	·0471	·0454	·0436	·0419	·0401	·0384	·0366	2	3	5	7	9	10	12	14	16
88	0·0349	·0332	·0314	·0297	·0279	·0262	·0244	·0227	·0209	·0192	2	3	5	7	9	10	12	14	16
89	0·0175	·0157	·0140	·0122	·0105	·0087	·0070	·0052	·0035	·0017	2	3	5	7	9	10	12	14	16
x	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	2	3	4	5	6	7	8	9

Appendix III: Tangents of Angles

Tangents of Angles (*x* in degrees) $x \rightarrow$

tan x

											ADD Differences								
x	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	2	3	4	5	6	7	8	9
0	0·0000	·0017	·0035	·0052	·0070	·0087	·0105	·0122	·0140	·0157	2	3	5	7	9	10	12	14	16
1	0·0175	·0192	·0209	·0227	·0244	·0262	·0279	·0297	·0314	·0332	2	3	5	7	9	10	12	14	16
2	0·0349	·0367	·0384	·0402	·0419	·0437	·0454	·0472	·0489	·0507	2	3	5	7	9	10	12	14	16
3	0·0524	·0542	·0559	·0577	·0594	·0612	·0629	·0647	·0664	·0682	2	4	5	7	9	11	12	14	16
4	0·0699	·0717	·0734	·0752	·0769	·0787	·0805	·0822	·0840	·0857	2	4	5	7	9	11	12	14	16
5	0·0875	·0892	·0910	·0928	·0945	·0963	·0981	·0998	·1016	·1033	2	4	5	7	9	11	12	14	16
6	0·1051	·1069	·1086	·1104	·1122	·1139	·1157	·1175	·1192	·1210	2	4	5	7	9	11	12	14	16
7	0·1228	·1246	·1263	·1281	·1299	·1317	·1334	·1352	·1370	·1388	2	4	5	7	9	11	12	14	16
8	0·1405	·1423	·1441	·1459	·1477	·1495	·1512	·1530	·1548	·1566	2	4	5	7	9	11	12	14	16
9	0·1584	·1602	·1620	·1638	·1655	·1673	·1691	·1709	·1727	·1745	2	4	5	7	9	11	13	14	16
10	0·1763	·1781	·1799	·1817	·1835	·1853	·1871	·1890	·1908	·1926	2	4	5	7	9	11	13	14	16
11	0·1944	·1962	·1980	·1998	·2016	·2035	·2053	·2071	·2089	·2107	2	4	5	7	9	11	13	15	16
12	0·2126	·2144	·2162	·2180	·2199	·2217	·2235	·2254	·2272	·2290	2	4	5	7	9	11	13	15	16
13	0·2309	·2327	·2345	·2364	·2382	·2401	·2419	·2438	·2456	·2475	2	4	6	7	9	11	13	15	17
14	0·2493	·2512	·2530	·2549	·2568	·2586	·2605	·2623	·2642	·2661	2	4	6	7	9	11	13	15	17
15	0·2679	·2698	·2717	·2736	·2754	·2773	·2792	·2811	·2830	·2849	2	4	6	8	9	11	13	15	17
16	0·2867	·2886	·2905	·2924	·2943	·2962	·2981	·3000	·3019	·3038	2	4	6	8	9	11	13	15	17
17	0·3057	·3076	·3096	·3115	·3134	·3153	·3172	·3191	·3211	·3230	2	4	6	8	10	12	13	15	17
18	0·3249	·3269	·3288	·3307	·3327	·3346	·3365	·3385	·3404	·3424	2	4	6	8	10	12	14	16	17
19	0·3443	·3463	·3482	·3502	·3522	·3541	·3561	·3581	·3600	·3620	2	4	6	8	10	12	14	16	18
20	0·3640	·3659	·3679	·3699	·3719	·3739	·3759	·3779	·3799	·3819	2	4	6	8	10	12	14	16	18
21	0·3839	·3859	·3879	·3899	·3919	·3939	·3959	·3979	·4000	·4020	2	4	6	8	10	12	14	16	18
22	0·4040	·4061	·4081	·4101	·4122	·4142	·4163	·4183	·4204	·4224	2	4	6	8	10	12	14	16	18
23	0·4245	·4265	·4286	·4307	·4327	·4348	·4369	·4390	·4411	·4431	2	4	6	8	10	12	15	17	19
24	0·4452	·4473	·4494	·4515	·4536	·4557	·4578	·4599	·4621	·4642	2	4	6	8	11	13	15	17	19
25	0·4663	·4684	·4706	·4727	·4748	·4770	·4791	·4813	·4834	·4856	2	4	6	9	11	13	15	17	19
26	0·4877	·4899	·4921	·4942	·4964	·4986	·5008	·5029	·5051	·5073	2	4	7	9	11	13	15	17	20
27	0·5095	·5117	·5139	·5161	·5184	·5206	·5228	·5250	·5272	·5295	2	4	7	9	11	13	16	18	20
28	0·5317	·5340	·5362	·5384	·5407	·5430	·5452	·5475	·5498	·5520	2	5	7	9	11	14	16	18	20
29	0·5543	·5566	·5589	·5612	·5635	·5658	·5681	·5704	·5727	·5750	2	5	7	9	12	14	16	18	21
30	0·5774	·5797	·5820	·5844	·5867	·5890	·5914	·5938	·5961	·5985	2	5	7	9	12	14	16	19	21
31	0·6009	·6032	·6056	·6080	·6104	·6128	·6152	·6176	·6200	·6224	2	5	7	10	12	14	17	19	22
32	0·6249	·6273	·6297	·6322	·6346	·6371	·6395	·6420	·6445	·6469	2	5	7	10	12	15	17	20	22
33	0·6494	·6519	·6544	·6569	·6594	·6619	·6644	·6669	·6694	·6720	3	5	8	10	13	15	18	20	23
34	0·6745	·6771	·6796	·6822	·6847	·6873	·6899	·6924	·6950	·6976	3	5	8	10	13	15	18	21	23
35	0·7002	·7028	·7054	·7080	·7107	·7133	·7159	·7186	·7212	·7239	3	5	8	11	13	16	18	21	24
36	0·7265	·7292	·7319	·7346	·7373	·7400	·7427	·7454	·7481	·7508	3	5	8	11	14	16	19	22	24
37	0·7536	·7563	·7590	·7618	·7646	·7673	·7701	·7729	·7757	·7785	3	6	8	11	14	17	19	22	25
38	0·7813	·7841	·7869	·7898	·7926	·7954	·7983	·8012	·8040	·8069	3	6	9	11	14	17	20	23	26
39	0·8098	·8127	·8156	·8185	·8214	·8243	·8273	·8302	·8332	·8361	3	6	9	12	15	18	21	23	26
40	0·8391	·8421	·8451	·8481	·8511	·8541	·8571	·8601	·8632	·8662	3	6	9	12	15	18	21	24	27
41	0·8693	·8724	·8754	·8785	·8816	·8847	·8878	·8910	·8941	·8972	3	6	9	12	16	19	22	25	28
42	0·9004	·9036	·9067	·9099	·9131	·9163	·9195	·9228	·9260	·9293	3	6	10	13	16	19	22	26	29
43	0·9325	·9358	·9391	·9424	·9457	·9490	·9523	·9556	·9590	·9623	3	7	10	13	17	20	23	27	30
44	0·9657	·9691	·9725	·9759	·9793	·9827	·9861	·9896	·9930	·9965	3	7	10	14	17	21	24	27	31
x	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	2	3	4	5	6	7	8	9

											ADD Differences								
x	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	2	3	4	5	6	7	8	9
45	1·000	1·003	1·007	1·011	1·014	1·018	1·021	1·025	1·028	1·032	0	1	1	1	2	2	2	3	3
46	1·036	1·039	1·043	1·046	1·050	1·054	1·057	1·061	1·065	1·069	0	1	1	1	2	2	3	3	3
47	1·072	1·076	1·080	1·084	1·087	1·091	1·095	1·099	1·103	1·107	0	1	1	2	2	2	3	3	3
48	1·111	1·115	1·118	1·122	1·126	1·130	1·134	1·138	1·142	1·145	0	1	1	2	2	2	3	3	4
49	1·150	1·154	1·159	1·163	1·167	1·171	1·175	1·179	1·183	1·188	0	1	1	2	2	2	3	3	4
50	1·192	1·196	1·200	1·205	1·209	1·213	1·217	1·222	1·226	1·230	0	1	1	2	2	3	3	4	4
51	1·235	1·239	1·244	1·248	1·253	1·257	1·262	1·266	1·271	1·275	0	1	1	2	2	3	3	4	4
52	1·280	1·285	1·289	1·294	1·299	1·303	1·308	1·313	1·317	1·322	0	1	1	2	2	3	3	4	4
53	1·327	1·332	1·337	1·342	1·347	1·351	1·356	1·361	1·366	1·371	0	1	1	2	2	3	3	4	4
54	1·376	1·381	1·387	1·392	1·397	1·402	1·407	1·412	1·418	1·423	1	1	2	2	3	3	4	4	5
55	1·428	1·433	1·439	1·444	1·450	1·455	1·460	1·466	1·471	1·477	1	1	2	2	3	3	4	4	5
56	1·483	1·488	1·494	1·499	1·505	1·511	1·517	1·522	1·528	1·534	1	1	2	2	3	3	4	5	5
57	1·540	1·546	1·552	1·558	1·564	1·570	1·576	1·582	1·588	1·594	1	1	2	2	3	4	4	5	5
58	1·600	1·607	1·613	1·619	1·625	1·632	1·638	1·645	1·651	1·658	1	1	2	3	3	4	4	5	6
59	1·664	1·671	1·678	1·684	1·691	1·698	1·704	1·711	1·718	1·725	1	1	2	3	3	4	5	5	6
60	1·732	1·739	1·746	1·753	1·760	1·767	1·775	1·782	1·789	1·797	1	1	2	3	4	4	5	6	6
61	1·804	1·811	1·819	1·827	1·834	1·842	1·849	1·857	1·865	1·873	1	2	2	3	4	5	5	6	7
62	1·881	1·889	1·897	1·905	1·913	1·921	1·929	1·937	1·946	1·954	1	2	2	3	4	5	6	7	7
63	1·963	1·971	1·980	1·988	1·997	2·006	2·014	2·023	2·032	2·041	1	2	3	4	4	5	6	7	8
64	2·050	2·059	2·069	2·078	2·087	2·097	2·106	2·116	2·125	2·135	1	2	3	4	5	6	7	8	8
65	2·145	2·154	2·164	2·174	2·184	2·194	2·204	2·215	2·225	2·236	1	2	3	4	5	6	7	8	9
66	2·246	2·257	2·267	2·278	2·289	2·300	2·311	2·322	2·333	2·344	1	2	3	4	5	7	8	9	10
67	2·356	2·367	2·379	2·391	2·402	2·414	2·426	2·438	2·450	2·463	1	2	4	5	6	7	8	10	11
68	2·475	2·488	2·500	2·513	2·526	2·539	2·552	2·565	2·578	2·592	1	3	4	5	7	8	9	10	12
69	2·605	2·619	2·633	2·646	2·660	2·675	2·689	2·703	2·718	2·733	1	3	4	6	7	9	10	11	13
70	2·747	2·762	2·778	2·793	2·808	2·824	2·840	2·856	2·872	2·888	2	3	5	6	8	9	11	13	14
71	2·904	2·921	2·937	2·954	2·971	2·989	3·006	3·024	3·042	3·060	2	3	5	7	9	10	12	14	16
72	3·078	3·096	3·115	3·133	3·152	3·172	3·191	3·211	3·230	3·251	2	4	6	8	10	12	14	15	17
73	3·271	3·291	3·312	3·333	3·354	3·376	3·398	3·420	3·442	3·465	2	4	6	9	11	13	15	17	19
74	3·487	3·511	3·534	3·558	3·582	3·606	3·630	3·655	3·681	3·706	2	5	7	10	12	15	17	20	22
75	3·732	3·758	3·785	3·812	3·839	3·867	3·895	3·923	3·952	3·981	3	6	8	11	14	17	20	22	25
76	4·011	4·041	4·071	4·102	4·134	4·165	4·198	4·230	4·264	4·297	3	6	10	13	16	19	22	26	29
77	4·331	4·366	4·402	4·437	4·474	4·511	4·548	4·586	4·625	4·666									
78	4·705	4·745	4·787	4·829	4·872	4·915	4·959	5·005	5·050	5·097									
79	5·145	5·193	5·242	5·292	5·343	5·396	5·449	5·503	5·558	5·614									
80	5·671	5·730	5·789	5·850	5·912	5·976	6·041	6·107	6·174	6·243	differences unreliable:								
81	6·314	6·386	6·460	6·535	6·612	6·691	6·772	6·855	6·940	7·026									
82	7·115	7·207	7·300	7·396	7·495	7·596	7·700	7·806	7·916	8·028									
83	8·144	8·264	8·386	8·513	8·643	8·777	8·915	9·058	9·205	9·357									
84	9·514	9·677	9·845	10·02	10·20	10·39	10·58	10·78	10·99	11·20									
85	11·43	11·66	11·91	12·16	12·43	12·71	13·00	13·30	13·62	13·95									
86	14·30	14·67	15·06	15·46	15·89	16·35	16·83	17·34	17·89	18·46									
87	19·08	19·74	20·45	21·20	22·02	22·90	23·86	24·90	26·03	27·27									
88	28·64	30·14	31·82	33·69	35·80	38·19	40·92	44·07	47·74	52·08									
89	57·29	63·66	71·62	81·85	95·46	114·6	143·2	191·0	286·5	573·0									
x	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	2	3	4	5	6	7	8	9

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