VERSION 1 CAPS

GRADE 11 MATHEMATICS

WRITTEN BY VOLUNTEERS

EVERYTHING SCIENCE BY

SIYAVULA

TECHNOLOGY-POWERED LEARNING

TEACHERS' GUIDE

basic education
Department:
Basic Education
REPUBLIC OF SOUTH AFRICA

METROPOLITAN

EVERYTHING MATHS

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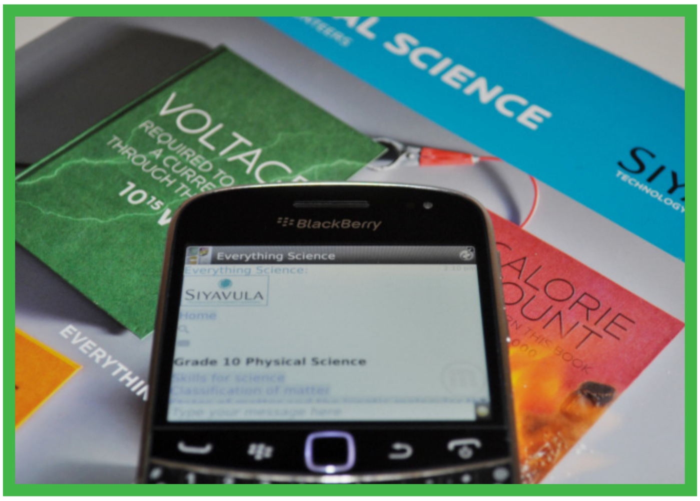

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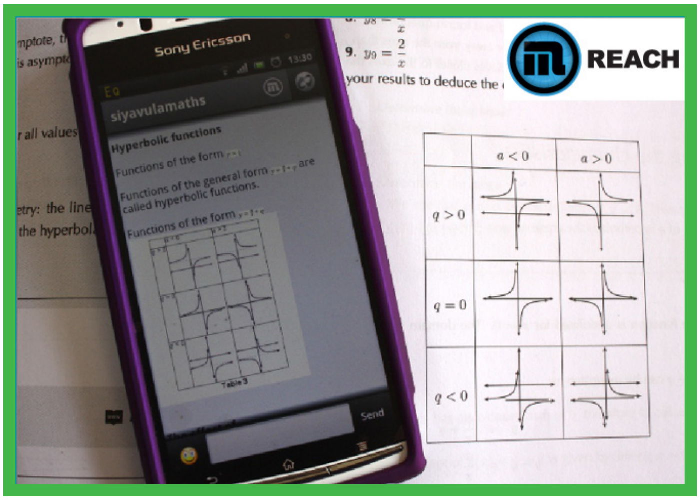

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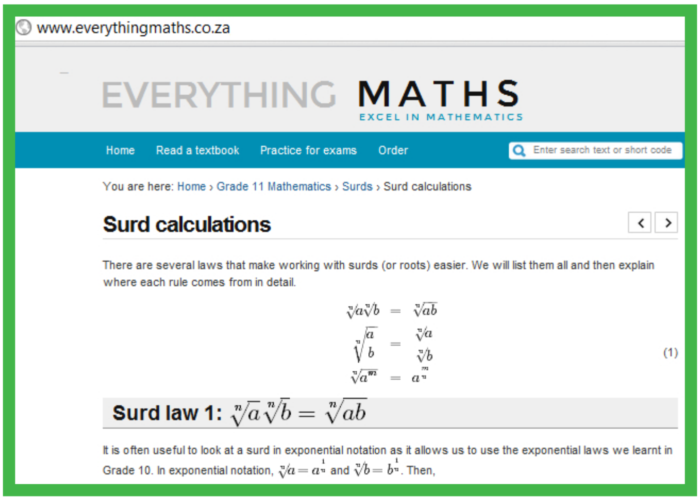

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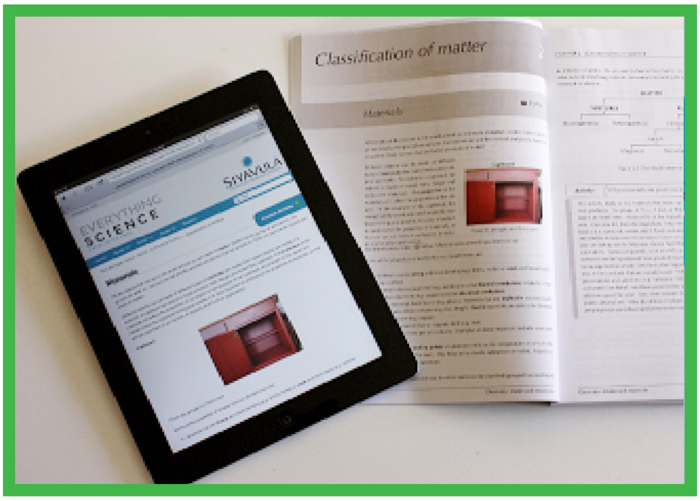

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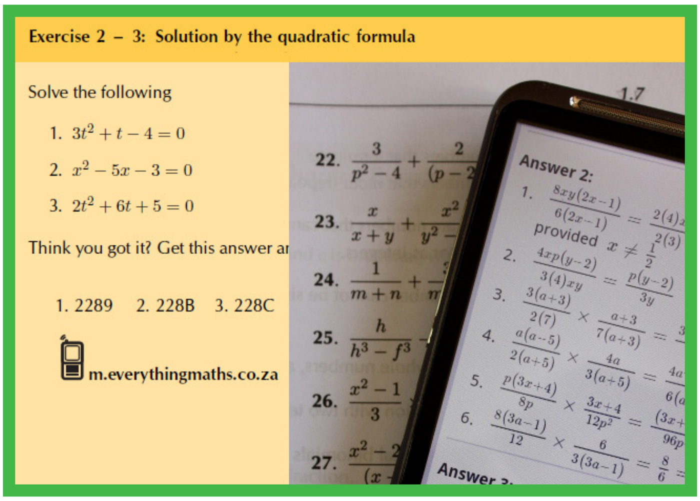

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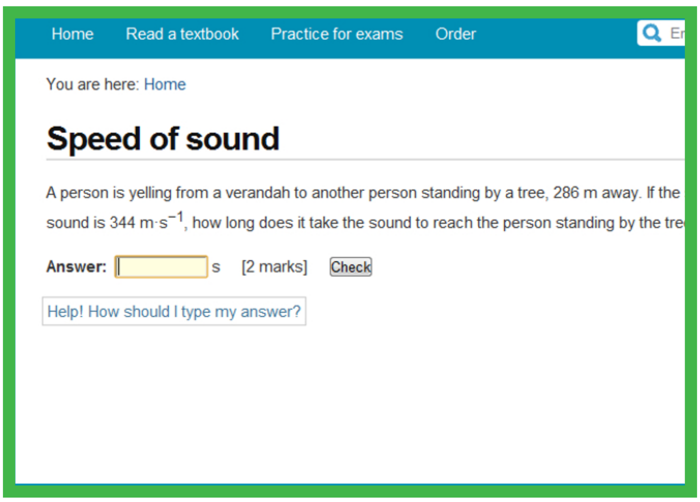

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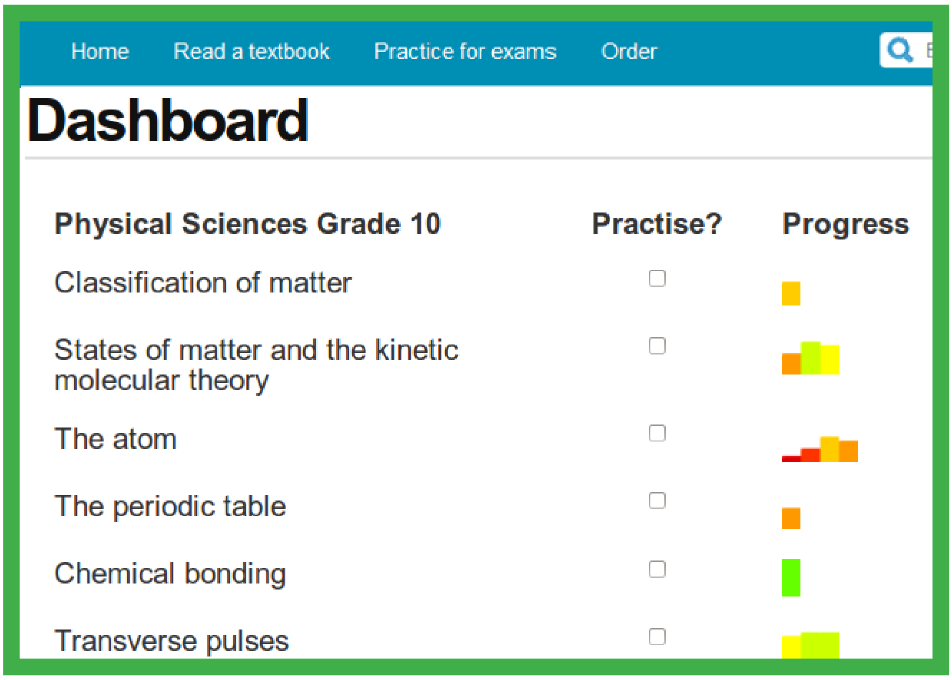

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Mathematics is commonly thought of as being about numbers but mathematics is actually a language! Mathematics is the language that nature speaks to us in. As we learn to understand and speak this language, we can discover many of nature's secrets. Just as understanding someone's language is necessary to learn more about them, mathematics is required to learn about all aspects of the world – whether it is physical sciences, life sciences or even finance and economics.

The great writers and poets of the world have the ability to draw on words and put them together in ways that can tell beautiful or inspiring stories. In a similar way, one can draw on mathematics to explain and create new things. Many of the modern technologies that have enriched our lives are greatly dependent on mathematics. DVDs, Google searches, bank cards with PIN numbers are just some examples. And just as words were not created specifically to tell a story but their existence enabled stories to be told, so the mathematics used to create these technologies was not developed for its own sake, but was available to be drawn on when the time for its application was right.

There is in fact not an area of life that is not affected by mathematics. Many of the most sought after careers depend on the use of mathematics. Civil engineers use mathematics to determine how to best design new structures; economists use mathematics to describe and predict how the economy will react to certain changes; investors use mathematics to price certain types of shares or calculate how risky particular investments are; software developers use mathematics for many of the algorithms (such as Google searches and data security) that make programmes useful.

But, even in our daily lives mathematics is everywhere – in our use of distance, time and money. Mathematics is even present in art, design and music as it informs proportions and musical tones. The greater our ability to understand mathematics, the greater our ability to appreciate beauty and everything in nature. Far from being just a cold and abstract discipline, mathematics embodies logic, symmetry, harmony and technological progress. More than any other language, mathematics is everywhere and universal in its application.

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Contents

CHAPTER 1

Mathematics - Teachers guide

1.1 Blog posts

General blogs

- Educator's Monthly - Education News and Resources (http://www.teachersmonthly.com)
  - "We eat, breathe and live education! "
  - "Perhaps the most remarkable yet overlooked aspect of the South African teaching community is its enthusiastic, passionate spirit. Every day, thousands of talented, hard-working educators gain new insight from their work and come up with brilliant, inventive and exciting ideas. Educator's Monthly aims to bring educators closer and help them share knowledge and resources.
  - Our aim is twofold . . .
    - To keep South African educators updated and informed.
    - To give educators the opportunity to express their views and cultivate their interests."
- Head Thoughts – Personal Reflections of a School Headmaster (http://headthoughts.co.za/)
  - blog by Arthur Preston
  - "Arthur is currently the headmaster of a growing independent school in Worcester, in the Western Cape province of South Africa. His approach to primary education is progressive and is leading the school through an era of new development and change."

Maths blog

- CEO: Circumspect Education Officer - Educating The Future
  - blog by Robyn Clark
  - "Mathematics teacher and inspirer."
  - http://clarkformaths.tumblr.com/
- dy/dan - Be less helpful
  - blog by Dan Meyer
  - "I'm Dan Meyer. I taught high school math between 2004 and 2010 and I am currently studying at Stanford University on a doctoral fellowship. My specific interests include curriculum design (answering the question, "how we design the ideal learning experience for students?") and teacher education (answering the questions, "how do teachers learn?" and "how do we retain more teachers?" and "how do we teach teachers to teach?")."
  - http://blog.mrmeyer.com

- Without Geometry, Life is Pointless - Musings on Math, Education, Teaching, and Research
  - blog by Avery
  - "I've been teaching some permutation (or is that combination?) of math and science to third through twelfth graders in private and public schools for 11 years. I'm also pursuing my EdD in education and will be both teaching and conducting research in my classroom this year."
  - http://mathteacherorstudent.blogspot.com/
- Overthinking my teaching - The Mathematics I Encounter in Classrooms
  - blog by Christopher Danielson
  - "I think a lot about my math teaching. Perhaps too much. This is my outlet. I hope you find it interesting and that you'll let me know how it's going."
  - http://christopherdanielson.wordpress.com
- A Recursive Process - Math Teacher Seeking Patterns
  - blog by Dan
  - "I am a High School math teacher in upstate NY. I currently teach Geometry, Computer Programming (Alice and Java), and two half year courses: Applied and Consumer Math. This year brings a new 21st century classroom (still not entirely sure what that entails) and a change over to standards based grades."
  - http://dandersod.wordpress.com
- Think Thank Thunk – Dealing with the Fear of Being a Boring Teacher
  - blog by Shawn Cornally
  - "I am Mr. Cornally. I desperately want to be a good teacher. I teach Physics, Calculus, Programming, Geology, and Bioethics. Warning: I have problem with using colons. I proof read, albeit poorly."
  - http://101studiostreet.com/wordpress/

1.2 Overview

Before 1994 there existed a number of education departments and subsequent curriculum according to the segregation that was so evident during the apartheid years. As a result, the curriculum itself became one of the political icons of freedom or suppression. Since then the government and political leaders have sought to try and develop one curriculum that is aligned with our national agenda of democratic freedom and equality for all, in fore-grounding the knowledge, skills and values our country believes our learners need to acquire and apply, in order to participate meaningfully in society as citizens of a free country. The National Curriculum Statement (NCS) of Grades R – 12 (DBE, 2012) therefore serves the purposes of:

- equipping learners, irrespective of their socio-economic background, race, gender, physical ability or intellectual ability, with the knowledge, skills and values necessary for self-fulfilment, and meaningful participation in society as citizens of a free country;

- providing access to higher education;
- facilitating the transition of learners from education institutions to the workplace; and
- providing employers with a sufficient profile of a learner's competencies.

Although elevated to the status of political icon, the curriculum remains a tool that requires the skill of an educator in interpreting and operationalising this tool within the classroom. The curriculum itself cannot accomplish the purposes outlined above without the community of curriculum specialists, material developers, educators and assessors contributing to and supporting the process, of the intended curriculum becoming the implemented curriculum. A curriculum can succeed or fail, depending on its implementation, despite its intended principles or potential on paper. It is therefore important that stakeholders of the curriculum are familiar with and aligned to the following principles that the NCS is based on:

Principle	**Implementation**
Social Transformation	Redressing imbalances of the past. Providing equal opportunities for all.
Active and Critical Learning	Encouraging an active and critical approach to learning. Avoiding excessive rote and uncritical learning of given truths.
High Knowledge and Skills	Learners achieve minimum standards of knowledge and skills specified for each grade in each subject.
Progression	Content and context shows progression from simple to complex.
Social and Environmental Justice and Human Rights	These practices as defined in the Constitution are infused into the teaching and learning of each of the subjects.
Valuing Indigenous Knowledge Systems	Acknowledging the rich history and heritage of this country.
Credibility, Quality and Efficiency	Providing an education that is globally comparable in quality.

This guide is intended to add value and insight to the existing National Curriculum for Grade 10 Mathematics, in line with its purposes and principles. It is hoped that this will assist you as the educator in optimising the implementation of the intended curriculum.

Curriculum requirements and objectives

The main objectives of the curriculum relate to the learners that emerge from our educational system. While educators are the most important stakeholders in the implementation of the intended curriculum, the quality of learner coming through this curriculum will be evidence of the actual attained curriculum from what was intended and then implemented.

These purposes and principles aim to produce learners that are able to:

- identify and solve problems and make decisions using critical and creative thinking;
- work effectively as individuals and with others as members of a team;
- organise and manage themselves and their activities responsibly and effectively;
- collect, analyse, organise and critically evaluate information;
- communicate effectively using visual, symbolic and/or language skills in various modes;
- use science and technology effectively and critically showing responsibility towards the environment and the health of others; and
- demonstrate an understanding of the world as a set of related systems by recognising that problem solving contexts do not exist in isolation.

The above points can be summarised as an independent learner who can think critically and analytically, while also being able to work effectively with members of a team and identify and solve problems through effective decision making. This is also the outcome of what educational research terms the "reformed" approach rather than the "traditional" approach many educators are more accustomed to. Traditional practices have their role and cannot be totally abandoned in favour of only reform practices. However, in order to produce more independent and mathematical thinkers, the reform ideology needs to be more embraced by educators within their instructional behaviour. Here is a table that can guide you to identify your dominant instructional practice and try to assist you in adjusting it (if necessary) to be more balanced and in line with the reform approach being suggested by the NCS.

Traditional Versus Reform Practices	
Values	**Traditional** – values content, correctness of learners' responses and mathematical validity of methods. **Reform** – values finding patterns, making connections, communicating mathematically and problem-solving.
Teaching Methods	**Traditional** – expository, transmission, lots of drill and practice, step by step mastery of algorithms. **Reform** – hands-on guided discovery methods, exploration, modelling. High level reasoning processes are central.
Grouping Learners	**Traditional** – dominantly same grouping approaches. **Reform** – dominantly mixed grouping and abilities.

The subject of mathematics, by the nature of the discipline, provides ample opportunities to meet the reformed objectives. In doing so, the definition of mathematics needs to be understood and embraced by educators involved in the teaching and the learning of the subject. In research it has been well documented that, as educators, our conceptions of what mathematics is, has an influence on our approach to the teaching and learning of the subject.

Three possible views of mathematics can be presented. The instrumentalist view of mathematics assumes the stance that mathematics is an accumulation of facts, rules and skills that need to be used as a means to an end, without there necessarily being any relation between these components. The Platonist view of mathematics sees the subject as a static but unified body of certain knowledge, in which mathematics is discovered rather than created. The problem solving view of mathematics is a dynamic, continually expanding and evolving field of human creation and invention that is in itself a cultural product. Thus mathematics is viewed as a process of enquiry, not a finished product. The results remain constantly open to revision. It is suggested that a hierarchical order exists within these three views, placing the instrumentalist view at the lowest level and the problem solving view at the highest.

According to the NCS:

Mathematics is the study of quantity, structure, space and change. Mathematicians seek out patterns, formulate new conjectures, and establish axiomatic systems by rigorous deduction from appropriately chosen axioms and definitions. Mathematics is a distinctly human activity practised by all cultures, for thousands of years. Mathematical problem solving enables us to understand the world (physical, social and economic) around us, and, most of all, to teach us to think creatively.

This corresponds well to the problem solving view of mathematics and may challenge some of our instrumentalist or Platonistic views of mathematics as a static body of knowledge of accumulated facts, rules and skills to be learnt and applied. The NCS is trying to discourage such an approach and encourage mathematics educators to dynamically and creatively involve their learners as mathematicians engaged in a process of study, understanding, reasoning, problem solving and communicating mathematically.

Below is a check list that can guide you in actively designing your lessons in an attempt to embrace the definition of mathematics from the NCS and move towards a problem solving conception of the subject. Adopting such an approach to the teaching and learning of mathematics will in turn contribute to the intended curriculum being properly implemented and attained through the quality of learners coming out of the education system.

Practice	**Example**
Learners engage in solving contextual problems related to their lives that require them to interpret a problem and then find a suitable mathematical solution.	Learners are asked to work out which bus service is the cheapest given the fares they charge and the distance they want to travel.
Learners engage in solving problems of a purely mathematical nature, which require higher order thinking and application of knowledge (non-routine problems).	Learners are required to draw a graph; they have not yet been given a specific technique on how to draw (for example a parabola), but have learnt to use the table method to draw straight-line graphs.
Learners are given opportunities to negotiate meaning.	Learners discuss their understanding of concepts and strategies for solving problems with each other and the educator.
Learners are shown and required to represent situations in various but equivalent ways (mathematical modelling).	Learners represent data using a graph, a table and a formula to represent the same data.
Learners individually do mathematical investigations in class, guided by the educator where necessary.	Each learner is given a paper containing the mathematical problem (for instance to find the number of prime numbers less than 50) that needs to be investigated and the solution needs to be written up. Learners work independently.

Learners work together as a group/team to investigate or solve a mathematical problem.	A group is given the task of working together to solve a problem that requires them investigating patterns and working through data to make conjectures and find a formula for the pattern.
Learners do drill and practice exercises to consolidate the learning of concepts and to master various skills.	Completing an exercise requiring routine procedures.
Learners are given opportunities to see the interrelatedness of the mathematics and to see how the different outcomes are related and connected.	While learners work through geometry problems, they are encouraged to make use of algebra.
Learners are required to pose problems for their educator and peer learners.	Learners are asked to make up an algebraic word problem (for which they also know the solution) for the person sitting next to them to solve.

Overview of topics

Summary of topics and their relevance:

1. Functions – linear, quadratic, exponential, rational	**Relevance**
Extend Gr10 work on the relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between therse representations (tables, graphs, words and formulae). Include linear and quadratic polynomial functions, exponential functions, some rational functions and trigonometric functins.) Generate as many graphs as necessary, initially by means of point-by-point plotting, supported by available technology, to make and test conjectures and hence generalise the effects of the parameter which results in a horizontal shift and that which results in a horizontal stretch and/or reflection about the y-axis. Problem solving and graph work involving the prescribed functions. Average gradient between two points.	Functions form a core part of learners' mathematical understanding and reasoning processes in algebra. This is also an excellent opportunity for contextual mathematical modelling questions.
2. Number Patterns, Sequences and Series	**Relevance**
Investigate number patterns leading to those where there is a constant second difference between consecutive terms, and the general term is therefore quadratic.	Much of mathematics revolves around the identification of patterns.
3. Finance, Growth and Decay	**Relevance**
Use simple and compound decay formulae to solve problems (including straight line depreciation and depreciation on a reducing balance.) Link to work on functions. The effect of different periods of compounding growth and decay (including effective and nominal interest rates).	The mathematics of finance is very relevant to daily and long-term financial decisions learners will need to make in terms of investing, taking loans, saving and understanding exchange rates and their influence more globally.
4. Algebra	**Relevance**

Take note that there exist numbers other than those on the real number line, the so-called non-real numbers. It is possible to square certain non-real numbers and obtain negative real numbers as answers. Nature of roots. a. Apply the laws of exponents to expressions involving rational exponents. b. Add, subtract, multiply and divide simple surds Revise factorisation Solve: quadratic equations, quadratic inequalities in one variable (and interpret the solution graphically) and equations in two unknonws, one of which is linear, the other quadratic. (either algebraically or graphically)	Algebra provides the basis for mathematics learners to move from numerical calculations to generalising operations, simplifying expressions, solving equations and using graphs and inequalities in solving contextual problems.
5. Differential Calculus	**Relevance**
Not covered in grade 11.	
6. Probability	**Relevance**
a. Dependent and independent events. b. Venn diagrams or contingency tables and tree diagrams as aids to solving probability problems (where events are not necessarily independent).	This topic is helpful in developing good logical reasoning in learners and for educating them in terms of real-life issues such as gambling and the possible pitfalls thereof.
7. Euclidean Geometry and Measurement	**Relevance**
a. Investigate and prove theorems of the geometry of circles assuming results from earlier grades, together with one other result concerning tangets and radii of circles. b. Solve circle geometry problems, providing reasons for statements when required. c. Prove riders Revise grade 10 work.	The thinking processes and mathematical skills of proving conjectures and identifying false conjectures is more the relevance here than the actual content studied. The surface area and volume content studied in real-life contexts of designing kitchens, tiling and painting rooms, designing packages, etc. is relevant to the current and future lives of learners.
8. Trigonometry	**Relevance**
a. Derive and use the identities: $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta = 1$. b. Derive the reduction formulae. c. Determine the general solution and/or specific solutions of trigonometric equations. d. Establish the sine, cosine and area rules Solve problems in two dimensions	Trigonometry has several uses within society, including within navigation, music, geographical locations and building design and construction.
9. Analytical Geometry	**Relevance**
Use a Cartesian co-ordinate system to derive and apply: the equation of a line through two given points, the equation of a line through one point and parallel or perpendicular to a given line and the inclination of a line.	This section provides a further application point for learners' algebraic and trigonometric interaction with the Cartesian plane. Artists and design and layout industries often draw on the content and thought processes of this mathematical topic.
10. Statistics	**Relevance**

a. Represent measures of central tendency and dispersion in univariate numerical data by: using ogives and calculating the variance and standard deviation of sets of data manually (for small sets of data) and using calculators (for larger sets of data) and representing results graphically. b. Represent skewed data in box and whisker diagrams and frequency polygons. Identify outliers.	Citizens are daily confronted with interpreting data presented from the media. Often this data may be biased or misrepresented within a certain context. In any type of research, data collection and handling is a core feature. This topic also educates learners to become more socially and politically educated with regards to the media.

Mathematics educators also need to ensure that the following important specific aims and general principles are applied in mathematics activities across all grades:

- Calculators should only be used to perform standard numerical computations and verify calculations done by hand.
- Real-life problems should be incorporated into all sections to keep mathematical modelling as an important focal point of the curriculum.
- Investigations give learners the opportunity to develop their ability to be more methodical, to generalise and to make and justify and/or prove conjectures.
- Appropriate approximation and rounding skills should be taught and continuously included and encouraged in activities.
- The history of mathematics should be incorporated into projects and tasks where possible, to illustrate the human aspect and developing nature of mathematics.
- Contextual problems should include issues relating to health, social, economic, cultural, scientific, political and environmental issues where possible.
- Conceptual understanding of when and why should also feature in problem types.
- Mixed ability teaching requires educators to challenge able learners and provide remedial support where necessary.
- Misconceptions exposed by assessment need to be dealt with and rectified by questions designed by educators.
- Problem solving and cognitive development should be central to all mathematics teaching and learning so that learners can apply the knowledge effectively.

Allocation of teaching time:

Time allocation for Mathematics per week: 4 hours and 30 minutes e.g. six forty-five minute periods per week.

Term	Topic	No. of weeks
Term 1	Exponents and surds	3
	Equations and inequalities	3
	Number patterns	2
	Analytical geometry	3
Term 2	Functions	4
	Trigonometry (reduction formulae, graphs, equations)	4
	Mid year exams	3
Term 3	Measurement	1
	Euclidean geometry	3
	Trigonometry (sine, cosine and area rules)	2
	Finance, growth and decay	2
	Probability	2
Term 4	Statistics	3
	Revisions	3
	Final exams	3

Please see page 19 of the Curriculum and Assessment Policy Statement for the sequencing and pacing of topics.

1.3 Assessment

"Educator assessment is part of everyday teaching and learning in the classroom. Educators discuss with learners, guide their work, ask and answer questions, observe, help, encourage and challenge. In addition, they mark and review written and other kinds of work. Through these activities they are continually finding out about their learners' capabilities and achievements. This knowledge then informs plans for future work. It is this continuous process that makes up educator assessment. It should not be seen as a separate activity necessarily requiring the use of extra tasks or tests."

As the quote above suggests, assessment should be incorporated as part of the classroom practice, rather than as a separate activity. Research during the past ten years indicates that learners get a sense of what they do and do not know, what they might do about this and how they feel about it, from frequent and regular classroom assessment and educator feedback. The educator's perceptions of and approach to assessment (both formal and informal assessment) can have an influence on the classroom culture that is created with regard to the learners' expectations of and performance in assessment tasks. Literature on classroom assessment distinguishes between two different purposes of assessment; assessment of learning and assessment for learning.

Assessment of learning tends to be a more formal assessment and assesses how much learners have learnt or understood at a particular point in the annual teaching plan. The NCS provides comprehensive guidelines on the types of and amount of formal assessment that needs to take place within the teaching year to make up the school-based assessment mark. The school-based assessment mark contributes 25% of the final percentage of a learner's promotion mark, while the end-of-year examination constitutes the other 75% of the annual promotion mark. Learners are expected to have 7 formal assessment tasks for their school-based assessment mark. The number of tasks and their weighting in the Grade 11 Mathematics curriculum is summarised below:

		Tasks	Weight (percent)
School-Based Assessment	Term 1	Project/Investigation Test	20 10
	Term 2	Assignment/Test Examination	10 30
	Term 3	Test Test	10 10
	Term 4	Test	10
School-Based Assessment Mark			100
School-Based Assessment Mark (as a percent of Promotion Mark)			25%
End-of-Year Examination			75%
Promotion Mark			75%

The following provides a brief explanation of each of the assessment tasks included in the assessment programme above.

Tests

All mathematics educators are familiar with this form of formal assessment. Tests include a variety of items/questions covering the topics that have been taught prior to the test. The new NCS also stipulates that mathematics tests should include questions that cover the following four types of cognitive levels in the stipulated weightings:

Cognitive levels	Description	Weighting (percent)
Knowledge	Estimation and appropriate rounding of numbers. Proofs of prescribed theorems. Derivation of formulae. Straight recall. Identification and direct use of formula on information sheet (no changing of the subject). Use of mathematical facts. Appropriate use of mathematical vocabulary.	20
Routine procedures	Perform well known procedures. Simple applications and calculations. Derivation from given information. Identification and use (including changing the subject) of correct formula. Questions generally similar to those done in class.	35

Complex procedures	Problems involve complex calculations and/or higher reasoning. There is often not an obvious route to the solution. Problems need not be based on real world context. Could involve making significant connections between different representations. Require conceptual understanding.	25
Problem solving	Unseen, non-routine problems (which are not necessarily difficult). Higher order understanding and processes are often involved. Might require the ability to break the problem down into its constituent parts.	15

The breakdown of the tests over the four terms is summarised from the NCS assessment programme as follows:

Term 1: One test of at least 50 marks, and one hour or two/three tests of at least 40 minutes each.

Term 2: Either one test (of at least 50 marks) or an assignment.

Term 3: Two tests, each of at least 50 marks and one hour.

Term 4: One test of at least 50 marks.

Projects/Investigations

Investigations and projects consist of open-ended questions that initiate and expand thought processes. Acquiring and developing problem-solving skills are an essential part of doing investigations and projects. These tasks provide learners with the opportunity to investigate, gather information, tabulate results, make conjectures and justify or prove these conjectures. Examples of investigations and projects and possible marking rubrics are provided in the next section on assessment support. The NCS assessment programme indicates that only one project or investigation (of at least 50 marks) should be included per year. Although the project/investigation is scheduled in the assessment programme for the first term, it could also be done in the second term.

Assignments

The NCS includes the following tasks as good examples of assignments:

- Open book test

- Translation task
- Error spotting and correction
- Shorter investigation
- Journal entry
- Mind-map (also known as a metacog)
- Olympiad (first round)
- Mathematics tutorial on an entire topic
- Mathematics tutorial on more complex/problem solving questions

The NCS assessment programme requires one assignment in term 2 (of at least 50 marks) which could also be a combination of some of the suggested examples above. More information on these suggested examples of assignments and possible rubrics are provided in the following section on assessment support.

Examinations

Educators are also all familiar with this summative form of assessment that is usually completed twice a year: mid-year examinations and end-of-year examinations. These are similar to the tests but cover a wider range of topics completed prior to each examination. The NCS stipulates that each examination should also cover the four cognitive levels according to their recommended weightings as summarised in the section above on tests. The following table summarises the requirements and information from the NCS for the two examinations.

Examination	Marks	Breakdown	Content and Mark distribution
Mid-Year Exam	100 50 + 50	One paper: 2 hours **or** Two papers: each of 1 hour	Topics completed
End-of-Year Exam	100+	Paper 1: 2 hours	Algebraic expressions, equations and inequalities (± 45) Patterns and sequences (± 25) Finance, growth and decay (± 15) Functions and graphs (± 45) Probability (± 20)
	100	Paper 2: 2 hours	Statistics (± 20) Analytical geometry (± 30) Trigonometry (± 50) Euclidean geometry and measurement (± 50)

In the annual teaching plan summary of the NCS in Mathematics for Grade 10, the pace setter section provides a detailed model of the suggested topics to be covered each week of each term and the accompanying formal assessment.

Assessment **for** learning tends to be more informal and focuses on using assessment in and of daily classroom activities that can include:

1. Marking homework
2. Baseline assessments
3. Diagnostic assessments
4. Group work
5. Class discussions
6. Oral presentations
7. Self-assessment
8. Peer-assessment

These activities are expanded on in the next section on assessment support and suggested marking rubrics are provided. Where formal assessment tends to restrict the learner to written assessment tasks, the informal assessment is necessary to evaluate and encourage the progress of the learners in their verbal mathematical reasoning and communication skills. It also provides a less formal assessment environment that allows learners to openly and honestly assess themselves and each other, taking responsibility for their own learning, without the heavy weighting of the performance (or mark) component. The assessment for learning tasks should be included in the classroom activities at least once a week (as part of a lesson) to ensure that the educator is able to continuously evaluate the learners' understanding of the topics covered as well as the effectiveness, and identify any possible deficiencies in his or her own teaching of the topics.

Assessment support

A selection of explanations, examples and suggested marking rubrics for the assessment of learning (formal) and the assessment for learning (informal) forms of assessment discussed in the preceding section are provided in this section.

Baseline assessment

Baseline assessment is a means of establishing:

- What prior knowledge a learner possesses
- What the extent of knowledge is that they have regarding a specific learning area?
- The level they demonstrate regarding various skills and applications
- The learner's level of understanding of various learning areas

It is helpful to educators in order to assist them in taking learners from their individual point of departure to a more advanced level and to thus make progress. This also helps avoid large "gaps" developing in the learners' knowledge as the learner moves through the education system. Outcomes-based education is a more learner-centered approach than we are used to in South Africa, and therefore the emphasis should now be on the level of each individual learner rather than that of the whole class.

The baseline assessments also act as a gauge to enable learners to take more responsibility for their own learning and to view their own progress. In the traditional assessment system, the weaker learners often drop from a 40% average in the first term to a 30% average in the fourth term due to an increase in workload, thus demonstrating no obvious progress. Baseline assessment, however, allows for an initial assigning of levels which can be improved upon as the learner progresses through a section of work and shows greater knowledge, understanding and skill in that area.

Diagnostic assessments

These are used to specifically find out if any learning difficulties or problems exist within a section of work in order to provide the learner with appropriate additional help and guidance. The assessment helps the educator and the learner identify problem areas, misunderstandings, misconceptions and incorrect use and interpretation of notation.

Some points to keep in mind:

- Try not to test too many concepts within one diagnostic assessment.
- Be selective in the type of questions you choose.
- Diagnostic assessments need to be designed with a certain structure in mind. As an educator, you should decide exactly what outcomes you will be assessing and structure the content of the assessment accordingly.
- The assessment is marked differently to other tests in that the mark is not the focus but rather the type of mistakes the learner has made.

An example of an understanding rubric for educators to record results is provided below:

0: indicates that the learner has not grasped the concept at all and that there appears to be a fundamental mathematical problem.

1: indicates that the learner has gained some idea of the content, but is not demonstrating an understanding of the notation and concept.

2: indicates evidence of some understanding by the learner but further consolidation is still required.

3: indicates clear evidence that the learner has understood the concept and is using the notation correctly.

Calculator worksheet - diagnostic skills assessment

1. Calculate:

   a) $242 + 63 =$

   b) $2 - 36 \times (114 + 25) =$

   c) $\sqrt{144 + 25} =$

d) $\sqrt[4]{729} =$

e) $-312 + 6 + 879 - 321 + 18\,901 =$

2. Calculate:

a) $\frac{2}{7} + \frac{1}{3} =$

b) $2\frac{1}{5} - \frac{2}{9} =$

c) $-2\frac{5}{6} + \frac{3}{8} =$

d) $4 - \frac{3}{4} \times \frac{5}{7} =$

e) $\left(\frac{9}{10} - \frac{8}{9}\right) \div \frac{3}{5} =$

f) $2 \times \left(\frac{4}{5}\right)^2 - \left(\frac{19}{25}\right) =$

g) $\sqrt{\frac{9}{4} - \frac{4}{16}} =$

Self-Assesment Rubric:

Name:

Question	Answer	yes	no	If no, write down sequence of keys pressed
1a				
1b				
1c				
1d				
1e				
Subtotal				
2a				
2b				
2c				
2d				
2e				
Subtotal				
Total				

Educator Assessment Rubric:

Type of skill	Competent	Needs practice	Problem
Raising to a power			
Finding a root			
Calculations with Fractions			
Brackets and order of operations			
Estimation and mental control			

Guidelines for Calculator Skills Assessment:

Type of skill	Sub-Division	Questions
Raising to a Power	Squaring and cubing Higher order powers	1a, 2f 1b
Finding a Root	Square and cube roots Higher order roots	1c, 2g 1d
Calculations with Fractions	Basic operations Mixed numbers Negative numbers Squaring fractions Square rooting fractions	2a, 2d 2b, 2c 1e, 2c 2f 2g
Brackets and Order of Operations	Correct use of brackets or order of operations	1b, 1c, 2e, 2f, 2g
Brackets and Order of Operations	Estimation and Mental Control	All

Suggested guideline to allocation of overall levels

Level 1

- Learner is able to do basic operations on calculator.
- Learner is able to do simple calculations involving fractions.
- Learner does not display sufficient mental estimation and control techniques.

Level 2

- Learner is able to do basic operations on calculator.
- Learner is able to square and cube whole numbers as well as find square and cube roots of numbers.
- Learner is able to do simple calculations involving fractions as well as correctly execute calculations involving mixed numbers.
- Learner displays some degree of mental estimation awareness.

Level 3

- Learner is able to do basic operations on calculator.
- Learner is able to square and cube rational numbers as well as find square and cube roots of numbers.
- Learner is also able to calculate higher order powers and roots.
- Learner is able to do simple calculations involving fractions as well as correctly execute calculations involving mixed numbers.
- Learner works correctly with negative numbers.
- Learner is able to use brackets in certain calculations but has still not fully understood the order of operations that the calculator has been programmed to execute, hence the need for brackets.

- Learner is able to identify possible errors and problems in their calculations but needs assistance solving the problem.

Level 4

- Learner is able to do basic operations on calculator.
- Learner is able to square and cube rational numbers as well as find square and cube roots.
- Learner is also able to calculate higher order powers and roots.
- Learner is able to do simple calculations involving fractions as well as correctly execute calculations involving mixed numbers.
- Learner works correctly with negative numbers.
- Learner is able to work with brackets correctly and understands the need and use of brackets and the "= key" in certain calculations due to the nature of a scientific calculator.
- Learner is able to identify possible errors and problems in their calculations and to find solutions to these in order to arrive at a "more viable" answer.

Other short diagnostic tests

These are short tests that assess small quantities of recall knowledge and application ability on a day-to-day basis. Such tests could include questions on one or a combination of the following:

- Definitions
- Theorems
- Riders (geometry)
- Formulae
- Applications
- Combination questions

Exercises

This entails any work from the textbook or other source that is given to the learner, by the educator, to complete either in class or at home. Educators should encourage learners not to copy each other's work and be vigilant when controlling this work. It is suggested that such work be marked/controlled by a check list (below) to speed up the process for the educator.

The marks obtained by the learner for a specific piece of work need not be based on correct and/or incorrect answers but preferably on the following:

1. the effort of the learner to produce answers.
2. the quality of the corrections of work that was previously incorrect.
3. the ability of the learner to explain the content of some selected examples (whether in writing or orally).

The following rubric can be used to assess exercises done in class or as homework:

Criteria	Performance indicators		
Work Done	2 All the work	1 Partially completed	0 No work done
Work Neatly Done	2 Work neatly done	1 Some work not neatly done	0 Messy and muddled
Corrections Done	2 All corrections done consistently	1 At least half of the corrections done	0 No corrections done
Correct Mathematical Method	2 Consistently	1 Sometimes	0 Never
Understanding of Mathematical Techniques and Processes	2 Can explain concepts and processes precisely	1 Explanations are ambiguous or not focused	0 Explanations are confusing or irrelevant

Journal entries

A journal entry is an attempt by a learner to express in the written word what is happening in Mathematics. It is important to be able to articulate a mathematical problem, and its solution in the written word.

This can be done in a number of different ways:

- Today in Maths we learnt...
- Write a letter to a friend, who has been sick, explaining what was done in class today.
- Explain the thought process behind trying to solve a particular maths problem, e.g. sketch the graph of $y = x^2 - 2x^2 + 1$ and explain how to sketch such a graph.
- Give a solution to a problem, decide whether it is correct and if not, explain the possible difficulties experienced by the person who wrote the incorrect solution.

A journal is an invaluable tool that enables the educator to identify any mathematical misconceptions of the learners. The marking of this kind of exercise can be seen as subjective but a marking rubric can simplify the task.

The following rubric can be used to mark journal entries. The learners must be given the marking rubric before the task is done.

Task	Competent (2 marks)	Still developing (1 mark)	Not yet developed (0 marks)
Completion in time limit?			
Correctness of the explanation?			
Correct and relevant use of mathematical language?			
Has the concept been interpret correctly?			

Translations

Translations assess the learner's ability to translate from words into mathematical notation or to give an explanation of mathematical concepts in words. Often when learners can use mathematical language and notation correctly, they demonstrate a greater understanding of the concepts.

For example:

Write the letter of the correct expression next to the matching number:

x increased by 10	a)	xy
The product of x and y	b)	x^2
The sum of a certain number and	c)	x^2
double that number	d)	$29x$
Half of a certain number multiplied by itself	e)	$\frac{1}{2} \times 2$
Two less than x	f)	$x + x + 2$
A certain number multiplied by itself	g)	x^2

Group work

One of the principles in the NCS is to produce learners who are able to work effectively within a group. Learners generally find this difficult to do. Learners need to be encouraged to work within small groups. Very often it is while learning under peer assistance that a better understanding of concepts and processes is reached. Clever learners usually battle with this sort of task, and yet it is important that they learn how to assist and communicate effectively with other learners.

Mind maps or metacogs

A metacog or "mind map" is a useful tool. It helps to associate ideas and make connections that would otherwise be too unrelated to be linked. A metacog can be used at the beginning or end of a section of work in order to give learners an overall perspective of the work covered, or as a way of recalling a section already completed. It must be emphasised that it is not a summary. Whichever way you use it, it is a way in which a learner is given the opportunity of doing research in a particular field and can show that he/she has an understanding of the required section.

This is an open book form of assessment and learners may use any material they feel will assist them. It is suggested that this activity be practised, using other topics, before a test metacog is submitted for portfolio assessment purposes.

On completion of the metacog, learners must be able to answer insightful questions on the metacog. This is what sets it apart from being just a summary of a section of work. Learners must refer to their metacog when answering the questions, but may not refer to any reference material. Below are some guidelines to give to learners to adhere to when constructing a metacog as well as two examples to help you get learners started. A marking rubric is also provided. This should be made available to learners before they start constructing their metacogs. On the next page is a model question for a metacog, accompanied by some sample questions that can be asked within the context of doing a metacog about analytical geometry.

A basic metacog is drawn in the following way:

- Write the title/topic of the subject in the centre of the page and draw a circle around it.
- For the first main heading of the subject, draw a line out from the circle in any direction, and write the heading above or below the line.
- For sub-headings of the main heading, draw lines out from the first line for each subheading and label each one.
- For individual facts, draw lines out from the appropriate heading line.

Metacogs are one's own property. Once a person understands how to assemble the basic structure they can develop their own coding and conventions to take things further, for example to show linkages between facts. The following suggestions may assist educators and learners to enhance the effectiveness of their metacogs:

- Use single words or simple phrases for information. Excess words just clutter the metacog and take extra time to write down.
- Print words – joined up or indistinct writing can be more difficult to read and less attractive to look at.
- Use colour to separate different ideas – this will help your mind separate ideas where it is necessary, and helps visualisation of the metacog for easy recall. Colour also helps to show organisation.
- Use symbols and images where applicable. If a symbol means something to you, and conveys more information than words, use it. Pictures also help you to remember information.
- Use shapes, circles and boundaries to connect information – these are additional tools to help show the grouping of information.

Use the concept of analytical geometry as your topic and construct a mind map (or metacog) containing all the information (including terminology, definitions, formulae and examples) that you know about the topic of analytical geometry.

Possible questions to ask the learner on completion of their metacog:

- Briefly explain to me what the mathematics topic of analytical geometry entails.
- Identify and explain the distance formula, the derivation and use thereof for me on your metacog.
- How does the calculation of gradient in analytical geometry differ (or not) from the approach used to calculate gradient in working with functions?

Here is a suggested simple rubric for marking a metacog:

Task	**Competent (2 Marks)**	**Still Developing (1 Mark)**	**Not Yet Developed 1 Mark)**
Completion in Time Limit			
Main Headings			
Correct Theory (Formulae, Definitions, Terminology etc.)			
Explanation			
Readability			

10 marks for the questions, which are marked using the following scale:

0 - no attempt or a totally incorrect attempt has been made

1 - a correct attempt was made, but the learner did not get the correct answer

2 - a correct attempt was made and the answer is correct

Investigations

Investigations consist of open-ended questions that initiate and expand thought processes. Acquiring and developing problem-solving skills are an essential part of doing investigations.

It is suggested that 2 – 3 hours be allowed for this task. During the first 30 – 45 minutes learners could be encouraged to talk about the problem, clarify points of confusion, and discuss initial conjectures with others. The final written-up version should be done individually though and should be approximately four pages.

Assessing investigations may include feedback/ presentations from groups or individuals on the results keeping the following in mind:

- following of a logical sequence in solving the problems
- pre-knowledge required to solve the problem
- correct usage of mathematical language and notation
- purposefulness of solution
- quality of the written and oral presentation

Some examples of suggested marking rubrics are included on the next few pages, followed by a selection of topics for possible investigations.

The following guidelines should be provided to learners before they begin an investigation:

General Instructions Provided to Learners

- You may choose any one of the projects/investigations given (see model question on investigations)
- You should follow the instructions that accompany each task as these describe the way in which the final product must be presented.
- You may discuss the problem in groups to clarify issues, but each individual must write-up their own version.
- Copying from fellow learners will cause the task to be disqualified.
- Your educator is a resource to you, and though they will not provide you with answers / solutions, they may be approached for hints.

The investigation is to be handed in on the due date, indicated to you by your educator. It should have as a minimum:

- A description of the problem.
- A discussion of the way you set about dealing with the problem.
- A description of the final result with an appropriate justification of its validity.
- Some personal reflections that include mathematical or other lessons learnt, as well as the feelings experienced whilst engaging in the problem.
- The written-up version should be attractively and neatly presented on about four A4 pages.
- Whilst the use of technology is encouraged in the presentation, the mathematical content and processes must remain the major focus.

Below is an example of a possible rubric to use when marking investigations:

Level of Performance	Criteria
4	• Contains a complete response. • Clear, coherent, unambiguous and elegant explanation. • Includes clear and simple diagrams where appropriate. • Shows understanding of the question's mathematical ideas and processes. • Identifies all the important elements of the question. • Includes examples and counter examples. • Gives strong supporting arguments. • Goes beyond the requirements of the problem.

3	• Contains a complete response. • Explanation less elegant, less complete. • Shows understanding of the question's mathematical ideas and processes. • Identifies all the important elements of the question. • Does not go beyond the requirements of the problem.
2	• Contains an incomplete response. • Explanation is not logical and clear. • Shows some understanding of the question's mathematical ideas and processes. • Identifies some of the important elements of the question. • Presents arguments, but incomplete. • Includes diagrams, but inappropriate or unclear.
1	• Contains an incomplete response. • Omits significant parts or all of the question and response. • Contains major errors. • Uses inappropriate strategies.
0	• No visible response or attempt

Orals

An oral assessment involves the learner explaining to the class as a whole, a group or the educator his or her understanding of a concept, a problem or answering specific questions. The focus here is on the correct use of mathematical language by the learner and the conciseness and logical progression of their explanation as well as their communication skills.

Orals can be done in a number of ways:

- A learner explains the solution of a homework problem chosen by the educator.
- The educator asks the learner a specific question or set of questions to ascertain that the learner understands, and assesses the learner on their explanation.
- The educator observes a group of learners interacting and assesses the learners on their contributions and explanations within the group.
- A group is given a mark as a whole, according to the answer given to a question by any member of a group.

An example of a marking rubric for an oral:

1 - the learner has understood the question and attempts to answer it

2 - the learner uses correct mathematical language

2 - the explanation of the learner follows a logical progression

2 - the learner's explanation is concise and accurate

2 - the learner shows an understanding of the concept being explained

1 - the learner demonstrates good communication skills

Maximum mark = 10

An example of a peer-assessment rubric for an oral:

My name:

Name of person I am assessing:

Criteria	Mark Awarded	Maximum Mark
Correct Answer		2
Clarity of Explanation		3
Correctness of Explanation		3
Evidence of Understanding		2
Total		10

CHAPTER 1

Exponents and surds

- Discuss the number system; explain the difference between real and non-real numbers.
- Encourage learners not to use calculators in this chapter.
- Common misconception: π (irrational) $\approx \frac{22}{7}$ (rational).
- Explain that the square root of a negative number is non-real.
- Discuss raising a negative number to even and odd powers.
- Explain that surds are a special notation or way of expressing rational exponents.
- Key strategy in manipulation of exponential expressions: express base in terms of its prime factors.
- Emphasize the principle of equivalence and using the additive inverse in the simplification of equations (and not "simply taking term to the other side").
- Rationalising the denominators is a useful tool for working with special angles in Trigonometry.
- Learners should leave their final answers as mixed fractions.
- Answers should always be written with positive exponents.

1.1 Revision

Exercise 1 – 1: The number system

Use the list of words below to describe each of the following numbers (in some cases multiple words will be applicable):

- Natural ($\mathbb{N}$)
- Whole ($\mathbb{N}_0$)
- Integer ($\mathbb{Z}$)
- Rational ($\mathbb{Q}$)
- Irrational ($\mathbb{Q}'$)
- Real ($\mathbb{R}$)
- Non-real ($\mathbb{R}'$)

1. $\sqrt{7}$

Solution:

$\mathbb{R}; \mathbb{Q}'$

2. $0{,}01$

Solution:

$\mathbb{R}; \mathbb{Q}$

3. $16\frac{2}{5}$

Solution:

$\mathbb{R}; \mathbb{Q}$

4. $\sqrt{6\frac{1}{4}}$

Solution:

$\mathbb{R}; \mathbb{Q}$

5. 0

Solution:

$\mathbb{R}; \mathbb{Q}; \mathbb{Z}; \mathbb{N}_0$

6. 2π

Solution:

$\mathbb{R}'\mathbb{Q}'$

7. $-5{,}3\dot{8}$

Solution:

$\mathbb{R}; \mathbb{Q}$

8. $\frac{1-\sqrt{2}}{2}$

Solution:

$\mathbb{R}; \mathbb{Q}'$

9. $-\sqrt{-3}$

Solution:

$\mathbb{R}'$

10. $(\pi)^2$

Solution:

$\mathbb{R}; \mathbb{Q}'$

11. $-\frac{9}{11}$

Solution:

$\mathbb{R}; \mathbb{Q}$

12. $\sqrt[3]{-8}$

Solution:

$\mathbb{R}; \mathbb{Q}; \mathbb{Z}$

13. $\frac{22}{7}$

Solution:

$\mathbb{R}; \mathbb{Q}$

14. $2{,}45897\ldots$

Solution:

$\mathbb{R}; \mathbb{Q}'$

15. $0{,}\overline{65}$

Solution:

$\mathbb{R}; \mathbb{Q}$

16. $\sqrt[5]{-32}$

Solution:

$\mathbb{R}; \mathbb{Q}; \mathbb{Z}$

Laws of exponents

Exercise 1 – 2: Laws of exponents

Simplify the following:

1. $4 \times 4^{2a} \times 4^2 \times 4^a$

Solution:

$$
\begin{aligned}
4 \times 4^{2a} \times 4^{2} \times 4^{a} &= 4^{1+2a+2+a} \\
&= 4^{3a+3}
\end{aligned}
$$

2. $\frac{3^2}{2^{-3}}$

Solution:

$$
\begin{aligned}
\frac{3^2}{2^{-3}} &= 3^2 \times 2^3 \\
&= 9 \times 8 \\
&= 72
\end{aligned}
$$

3. $(3p^5)^2$

Solution:

$$
\begin{aligned}
(3p^5)^2 &= 3^2 \times p^{10} \\
&= 9p^{10}
\end{aligned}
$$

4. $\frac{k^2 k^{3x-4}}{k^x}$

Solution:

$$
\begin{aligned}
\frac{k^2 k^{3x-4}}{k^x} &= \frac{k^{3x-2}}{k^x} \\
&= k^{3x-2-(x)} \\
&= k^{2x-2}
\end{aligned}
$$

5. $(5^{z-1})^2 + 5^z$

Solution:

$$
(5^{z-1})^2 + 5^z = 5^{2z-2} + 5^z
$$

6. $(\frac{1}{4})^0$

Solution:

$$
\left(\frac{1}{4}\right)^0 = 1
$$

7. $(x^2)^5$

Solution:

$$
\left(x^2\right)^5 = x^{10}
$$

8. $\left(\frac{a}{b}\right)^{-2}$

Solution:

$$\left(\frac{a}{b}\right)^{-2} = \frac{a^{-2}}{b^{-2}}$$
$$= \frac{b^2}{a^2}$$

9. $(m+n)^{-1}$

Solution:

$$(m+n)^{-1} = \frac{1}{m+n}$$

10. $2(p^t)^s$

Solution:

$$2\left(p^t\right)^s = 2p^{ts}$$

11. $\frac{1}{\left(\frac{1}{a}\right)^{-1}}$

Solution:

$$\frac{1}{\left(\frac{1}{a}\right)^{-1}} = \frac{1}{a}$$

12. $\frac{k^0}{k^{-1}}$

Solution:

$$\frac{k^0}{k^{-1}} = k$$

13. $\frac{-2}{-2^{-a}}$

Solution:

$$\frac{-2}{-2^{-a}} = 2 \times 2^a$$
$$= 2^{a+1}$$

14. $\frac{-h}{(-h)^{-3}}$

Solution:

$$\frac{-h}{(-h)^{-3}} = -h\,(-h)^3$$
$$= -h\left(-h^3\right)$$
$$= h^4$$

15. $\left(\frac{a^2b^3}{c^3d}\right)^2$

Solution:

$$\left(\frac{a^2b^3}{c^3d}\right)^2 = \frac{a^4b^6}{c^6d^2}$$

16. $10^7(7^0) \times 10^{-6}(-6)^0 - 6$

Solution:

$$\begin{aligned} 10^7\left(7^0\right) \times 10^{-6}\left(-6\right)^0 - 6 &= 10^7(1) \times 10^{-6}(1) - 6 \\ &= 10^1 - 6 \\ &= 4 \end{aligned}$$

17. $m^3n^2 \div nm^2 \times \frac{mn}{2}$

Solution:

$$\begin{aligned} m^3n^2 \div nm^2 \times \frac{mn}{2} &= m^3n^2 \times \frac{1}{m^2n} \times \frac{mn}{2} \\ &= \frac{m^3n^2}{m^2n} \times \frac{mn}{2} \\ &= \frac{m^2n^2}{2} \end{aligned}$$

18. $(2^{-2} - 5^{-1})^{-2}$

Solution:

$$\begin{aligned} \left(2^{-2} - 5^{-1}\right)^{-2} &= \left(\frac{1}{4} - \frac{1}{5}\right)^{-2} \\ &= \left(\frac{1}{20}\right)^{-2} \\ &= 20^2 \\ &= 400 \end{aligned}$$

19. $(y^2)^{-3} \div \left(\frac{x^2}{y^3}\right)^{-1}$

Solution:

$$\begin{aligned} \left(y^2\right)^{-3} \div \left(\frac{x^2}{y^3}\right)^{-1} \div \frac{y^{-2}}{x^{-2}} &= \frac{1}{y^6} \times \frac{x^2}{y^3} \times \frac{y^2}{x^2} \\ &= \frac{1}{y^7} \end{aligned}$$

20. $\frac{2^{c-5}}{2^{c-8}}$

Solution:

$$\begin{aligned}\frac{2^{c-5}}{2^{c-8}} &= 2^{(c-5)-(c-8)} \\ &= 2^{c-5-c+8} \\ &= 2^3 \\ &= 8\end{aligned}$$

21. $\frac{2^{9a} \times 4^{6a} \times 2^2}{8^{5a}}$

Solution:

$$\begin{aligned}\frac{2^{9a} \times 4^{6a} \times 2^2}{8^{5a}} &= \frac{2^{9a} \times 2^{12a} \times 2^2}{2^{15a}} \\ &= \frac{2^{9a+12a+2}}{2^{15a}} \\ &= 2^{21a+2-15a} \\ &= 2^{6a+2}\end{aligned}$$

22. $\frac{20t^5p^{10}}{10t^4p^9}$

Solution:

$$\begin{aligned}\frac{20t^5p^{10}}{10t^4p^9} &= 2t^{5-4}p^{10-9} \\ &= 2pt\end{aligned}$$

23. $\left(\frac{9q^{-2s}}{q^{-3s}y^{-4a-1}}\right)^2$

Solution:

$$\begin{aligned}\left(\frac{9q^{-2s}}{q^{-3s}y^{-4a-1}}\right)^2 &= \frac{\left(9q^{-2s}\right)^2}{\left(q^{-3s}y^{-4a-1}\right)^2} \\ &= \frac{81q^{-4s}}{q^{-6s}y^{-8a-2}} \\ &= \frac{81q^{6s}}{q^{4s}y^{-(8a+2)}} \\ &= 81q^{2s}y^{8a+2}\end{aligned}$$

Exercise 1 – 3: Rational exponents and surds

1. Simplify the following and write answers with positive exponents:

a) $\sqrt{49}$

b) $\sqrt{36^{-1}}$

c) $\sqrt[3]{6^{-2}}$

d) $\sqrt[3]{-\frac{64}{27}}$

e) $\sqrt[4]{(16x^4)^3}$

Solution:

a) $\sqrt{49} = 7$

b)

$$\begin{aligned} 36^{-\frac{1}{2}} &= \frac{1}{36^{\frac{1}{2}}} \\ &= \frac{1}{\sqrt{36}} \\ &= \frac{1}{6} \end{aligned}$$

c)

$$\begin{aligned} 6^{-\frac{2}{3}} &= \frac{1}{6^{\frac{2}{3}}} \\ &= \frac{1}{\sqrt[3]{6^2}} \\ &= \frac{1}{\sqrt[3]{36}} \end{aligned}$$

d)

$$\begin{aligned} \sqrt[3]{-\frac{64}{27}} &= \sqrt[3]{-\frac{4^3}{3^3}} \\ &= \sqrt[3]{\left(-\frac{4}{3}\right)^3} \\ &= -\frac{4}{3} \end{aligned}$$

e)

$$\begin{aligned} \sqrt[4]{(16x^4)^3} &= \left((16x^4)^3\right)^{\frac{1}{4}} \\ &= \left(2^4x^4\right)^{\frac{3}{4}} \\ &= 2^3x^3 \\ &= 8x^3 \end{aligned}$$

2. Simplify:

a) $s^{\frac{1}{2}} \div s^{\frac{1}{3}}$

b) $\left(64m^6\right)^{\frac{2}{3}}$

c) $\dfrac{12m^{\frac{7}{9}}}{8m^{-\frac{11}{9}}}$

d) $(5x)^0 + 5x^0 - (0{,}25)^{-0{,}5} + 8^{\frac{2}{3}}$

Solution:

a)

$$\begin{aligned} s^{\frac{1}{2}} \div s^{\frac{1}{3}} &= s^{\frac{1}{2}} \times \frac{1}{s^{\frac{1}{3}}} \\ &= s^{\frac{1}{2}-\frac{1}{3}} \\ &= s^{\frac{3}{6}-\frac{2}{6}} \\ &= s^{\frac{1}{6}} \end{aligned}$$

b)

$$\begin{aligned} \left(64m^6\right)^{\frac{2}{3}} &= \left(2^6\right)^{\frac{2}{3}} \left(m^6\right)^{\frac{2}{3}} \\ &= \left(2^4\right)\left(m^4\right) \\ &= 16m^4 \end{aligned}$$

c)

$$\begin{aligned} \frac{12m^{\frac{7}{9}}}{8m^{-\frac{11}{9}}} &= \frac{3}{2}m^{\frac{7}{9}-(-\frac{11}{9})} \\ &= \frac{3}{2}m^{\frac{18}{9}} \\ &= \frac{3}{2}m^2 \end{aligned}$$

d)

$$\begin{aligned} (5x)^0 + 5x^0 - (0{,}25)^{-0{,}5} + 8^{\frac{2}{3}} &= (1) + 5(1) - \left(\frac{1}{4}\right)^{-\frac{1}{2}} + \left(2^3\right)^{\frac{2}{3}} \\ &= 6 - 4^{\frac{1}{2}} + 4 \\ &= 10 - 2 \\ &= 8 \end{aligned}$$

3. Use the laws to re-write the following expression as a power of x:

$x\sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}}$

Solution:

$$\begin{aligned} x\sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}} &= x \times x^{\frac{1}{2}} \times x^{\frac{1}{4}} \times x^{\frac{1}{8}} \times x^{\frac{1}{16}} \\ &= x^{\frac{16}{16}} \times x^{\frac{8}{16}} \times x^{\frac{4}{16}} \times x^{\frac{2}{16}} \times x^{\frac{1}{16}} \\ &= x^{\frac{31}{16}} \end{aligned}$$

Exercise 1 – 4: Simplification of surds

1. Simplify the following and write answers with positive exponents:

   a) $\sqrt[3]{16} \times \sqrt[3]{4}$
   b) $\sqrt{a^2b^3} \times \sqrt{b^5c^4}$
   c) $\frac{\sqrt{12}}{\sqrt{3}}$
   d) $\sqrt{x^2y^{13}} \div \sqrt{y^5}$

   Solution:

   a) $\sqrt[3]{16} \times \sqrt[3]{4} = \sqrt[3]{16 \times 4} = \sqrt[3]{64} = 4$
   b) $\sqrt{a^2b^3} \times \sqrt{b^5c^4} = \sqrt{a^2b^8c^4} = ab^4c^2$
   c) $\frac{\sqrt{12}}{\sqrt{3}} = \sqrt{\frac{12}{3}} = \sqrt{4} = 2$
   d) $\sqrt{x^2y^{13}} \div \sqrt{y^5} = \sqrt{\frac{x^2y^{13}}{y^5}} = \sqrt{x^2y^8} = xy^4$

2. Simplify the following:

   a) $\left(\frac{1}{a} - \frac{1}{b}\right)^{-1}$
   b) $\frac{b-a}{a^{\frac{1}{2}} - b^{\frac{1}{2}}}$

   Solution:

   a)

$$\begin{aligned}\left(\frac{1}{a} - \frac{1}{b}\right)^{-1} &= \left(\frac{b-a}{ab}\right)^{-1} \\ &= \frac{ab}{b-a}\end{aligned}$$

   b)

$$\begin{aligned}\frac{b-a}{a^{\frac{1}{2}} - b^{\frac{1}{2}}} &= -\frac{a-b}{a^{\frac{1}{2}} - b^{\frac{1}{2}}} \\ &= -\frac{\left(a^{\frac{1}{2}}\right)^2 - \left(b^{\frac{1}{2}}\right)^2}{a^{\frac{1}{2}} - b^{\frac{1}{2}}} \\ &= -\frac{\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right)\left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)}{a^{\frac{1}{2}} - b^{\frac{1}{2}}} \\ &= -\left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)\end{aligned}$$

Exercise 1 – 5: Rationalising the denominator

Rationalise the denominator in each of the following:

1. $\frac{10}{\sqrt{5}}$

   Solution:

$$\begin{aligned}\frac{10}{\sqrt{5}} &= \frac{10}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}\\ &= \frac{10\sqrt{5}}{5}\\ &= 2\sqrt{5}\end{aligned}$$

2. $\frac{3}{\sqrt{6}}$

   Solution:

$$\begin{aligned}\frac{3}{\sqrt{6}} &= \frac{3}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}\\ &= \frac{3\sqrt{6}}{6}\\ &= \frac{\sqrt{6}}{2}\end{aligned}$$

3. $\frac{2}{\sqrt{3}} \div \frac{\sqrt{2}}{3}$

   Solution:

$$\begin{aligned}\frac{2}{\sqrt{3}} \div \frac{\sqrt{2}}{3} &= \frac{2}{\sqrt{3}} \div \frac{\sqrt{2}}{3}\\ &= \frac{2}{\sqrt{3}} \times \frac{3}{\sqrt{2}}\\ &= \frac{6}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}\\ &= \frac{6\sqrt{6}}{6}\\ &= \sqrt{6}\end{aligned}$$

4. $\frac{3}{\sqrt{5} - 1}$

   Solution:

$$\begin{aligned}\frac{3}{\sqrt{5}-1} &= \frac{3}{\sqrt{5}-1} \times \frac{\sqrt{5}+1}{\sqrt{5}+1} \\ &= \frac{3\sqrt{5}+3}{5-1} \\ &= \frac{3\sqrt{5}+3}{4}\end{aligned}$$

5. $\dfrac{x}{\sqrt{y}}$

Solution:

$$\begin{aligned}\frac{x}{\sqrt{y}} &= \frac{x}{\sqrt{y}} \times \frac{\sqrt{y}}{\sqrt{y}} \\ &= \frac{x\sqrt{y}}{y}\end{aligned}$$

6. $\dfrac{\sqrt{3}+\sqrt{7}}{\sqrt{2}}$

Solution:

$$\begin{aligned}\frac{\sqrt{3}+\sqrt{7}}{\sqrt{2}} &= \frac{\sqrt{3}+\sqrt{7}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{3}\sqrt{2}+\sqrt{7}\sqrt{2}}{2} \\ &= \frac{\sqrt{6}+\sqrt{14}}{2}\end{aligned}$$

7. $\dfrac{3\sqrt{p}-4}{\sqrt{p}}$

Solution:

$$\begin{aligned}\frac{3\sqrt{p}-4}{\sqrt{p}} &= \frac{3\sqrt{p}-4}{\sqrt{p}} \times \frac{\sqrt{p}}{\sqrt{p}} \\ &= \frac{3\left(\sqrt{p}\right)^2-4\left(\sqrt{p}\right)}{p} \\ &= \frac{3p-4\sqrt{p}}{p}\end{aligned}$$

8. $\dfrac{t-4}{\sqrt{t}+2}$

Solution:

$$\begin{aligned}\frac{t-4}{\sqrt{t}+2} &= \frac{t-4}{\sqrt{t}+2} \times \frac{\sqrt{t}-2}{\sqrt{t}-2} \\ &= \frac{(t-4)\left(\sqrt{t}-2\right)}{t-4} \\ &= \sqrt{t}-2\end{aligned}$$

9. $(1+\sqrt{m})^{-1}$

Solution:

$$\begin{aligned}(1+\sqrt{m})^{-1} &= \frac{1}{1+\sqrt{m}} \times \frac{1-\sqrt{m}}{1-\sqrt{m}} \\ &= \frac{1-\sqrt{m}}{1-m}\end{aligned}$$

10. $a\left(\sqrt{a} \div \sqrt{b}\right)^{-1}$

Solution:

$$\begin{aligned}a\left(\sqrt{a} \div \sqrt{b}\right)^{-1} &= a\left(\sqrt{a} \times \frac{1}{\sqrt{b}}\right)^{-1} \\ &= a\left(\frac{\sqrt{a}}{\sqrt{b}}\right)^{-1} \\ &= a\frac{\sqrt{b}}{\sqrt{a}} \times \frac{\sqrt{a}}{\sqrt{a}} \\ &= \frac{a\sqrt{ab}}{a} \\ &= \sqrt{ab}\end{aligned}$$

1.3 Solving surd equations

Exercise 1 – 6: Solving surd equations

Solve for the unknown variable (remember to check that the solution is valid):

1. $2^{x+1} - 32 = 0$

Solution:

$$\begin{aligned}2^{x+1} - 32 &= 0 \\ 2^{x+1} &= 32 \\ 2^{x+1} &= 2^5 \\ \therefore x + 1 &= 5 \\ x &= 4\end{aligned}$$

2. $125\left(3^p\right) = 27\left(5^p\right)$

Solution:

$$
\begin{aligned}
125\left(3^p\right) &= 27\left(5^p\right)\\
\frac{5^p}{3^p} &= \frac{125}{27}\\
\left(\frac{5}{3}\right)^p &= \left(\frac{5}{3}\right)^3\\
\therefore p &= 3
\end{aligned}
$$

3. $2y^{\frac{1}{2}} - 3y^{\frac{1}{4}} + 1 = 0$

Solution:

$$
\begin{aligned}
2y^{\frac{1}{2}} - 3y^{\frac{1}{4}} + 1 &= 0\\
\left(2y^{\frac{1}{4}} - 1\right)\left(y^{\frac{1}{4}} - 1\right) &= 0\\
\text{Therefore } 2y^{\frac{1}{4}} - 1 &= 0\\
y^{\frac{1}{4}} - \frac{1}{2} &= 0\\
y^{\frac{1}{4}} &= \frac{1}{2}\\
y^{\frac{4}{4}} &= \left(\frac{1}{2}\right)^4\\
\therefore y &= \frac{1}{16}\\
&\text{or}\\
y^{\frac{1}{4}} - 1 &= 0\\
y^{\frac{1}{4}} &= 1\\
\therefore y &= 1
\end{aligned}
$$

4. $t - 1 = \sqrt{7 - t}$

Solution:

$$
\begin{aligned}
t - 1 &= \sqrt{7-t}\\
(t-1)^2 &= \left(\sqrt{7-t}\right)^2\\
t^2 - 2t + 1 &= 7 - t\\
t^2 - t - 6 &= 0\\
(t-3)(t+2) &= 0\\
\therefore t &= 3 \text{ or } t = -2\\
\text{Check RHS for } t = 3: &= \sqrt{7-3}\\
&= \sqrt{4}\\
&= 2\\
&= \text{LHS} \therefore \text{ valid solution}\\
\text{Check RHS for } t = -2: &= \sqrt{7-(-2)}\\
&= \sqrt{9}\\
&= 3\\
&\neq \text{LHS} \therefore \text{ not valid solution}
\end{aligned}
$$

5. $2z - 7\sqrt{z} + 3 = 0$

Solution:

$$
\begin{aligned}
2z - 7z^{\frac{1}{2}} + 3 &= 0 \\
\left(z^{\frac{1}{2}} - 3\right)\left(2z^{\frac{1}{2}} - 1\right) &= 0 \\
\text{Therefore } z^{\frac{1}{2}} - 3 &= 0 \\
\left(z^{\frac{1}{2}}\right)^2 &= 3^2 \\
\therefore z &= 9 \\
&\text{or} \\
2z^{\frac{1}{2}} - 1 &= 0 \\
\left(z^{\frac{1}{2}}\right)^2 &= \left(\frac{1}{2}\right)^2 \\
\therefore z &= \frac{1}{4}
\end{aligned}
$$

6. $x^{\frac{1}{3}}(x^{\frac{1}{3}} + 1) = 6$

Solution:

$$
\begin{aligned}
x^{\frac{1}{3}}\left(x^{\frac{1}{3}} + 1\right) &= 6 \\
x^{\frac{2}{3}} + x^{\frac{1}{3}} &= 6 \\
\left(x^{\frac{1}{3}} - 2\right)\left(x^{\frac{1}{3}} + 3\right) &= 0 \\
\text{Therefore } x^{\frac{1}{3}} - 2 &= 0 \\
x^{\frac{1}{3}} &= 2 \\
\therefore x &= 8 \\
&\text{or} \\
x^{\frac{1}{3}} + 3 &= 0 \\
x^{\frac{1}{3}} &= -3 \\
\therefore x &= -27
\end{aligned}
$$

7. $2^{4n} - \frac{1}{\sqrt[4]{16}} = 0$

Solution:

$$
\begin{aligned}
2^{4n} - \frac{1}{\sqrt[4]{16}} &= 0 \\
2^{4n} &= \frac{1}{16^{\frac{1}{4}}} \\
2^{4n} &= \frac{1}{(2^4)^{\frac{1}{4}}} \\
2^{4n} &= \frac{1}{2} \\
2^{4n} &= 2^{-1} \\
\therefore 4n &= -1 \\
\therefore n &= -\frac{1}{4}
\end{aligned}
$$

8. $\sqrt{31 - 10d} = 4 - d$

Solution:

$$
\begin{aligned}
\sqrt{31 - 10d} &= d - 4 \\
\left(\sqrt{31 - 10d}\right)^2 &= (d - 4)^2 \\
31 - 10d &= d^2 - 8d + 16 \\
0 &= d^2 + 2d - 15 \\
0 &= (d - 3)(d + 5) \\
\text{Therefore } d &= 3 \\
\text{or} & \\
d &= -5 \\
\text{Check LHS for } d = 3: &= \sqrt{31 - 30} \\
&= \sqrt{1} \\
&= 1 \\
&= \text{RHS} \therefore \text{ valid solution} \\
\text{Check LHS for } t = -5: &= \sqrt{31 - (-50)} \\
&= \sqrt{81} \\
&= 9 \\
&= \text{RHS} \therefore \text{ valid solution}
\end{aligned}
$$

9. $y - 10\sqrt{y} + 9 = 0$

Solution:

$$
\begin{aligned}
y - 10\sqrt{y} + 9 &= 0 \\
y - 10y^{\frac{1}{2}} + 9 &= 0 \\
\left(y^{\frac{1}{2}} - 1\right)\left(y^{\frac{1}{2}} - 9\right) &= 0 \\
\text{Therefore } y^{\frac{1}{2}} - 1 &= 0 \\
y^{\frac{1}{2}} &= 1 \\
\left(y^{\frac{1}{2}}\right)^2 &= (1)^2 \\
\therefore y &= 1 \\
&\text{or} \\
y^{\frac{1}{2}} - 9 &= 0 \\
y^{\frac{1}{2}} &= 9 \\
\left(y^{\frac{1}{2}}\right)^2 &= (9)^2 \\
\therefore y &= 81
\end{aligned}
$$

10. $f = 2 + \sqrt{19 - 2f}$

Solution:

$$
\begin{aligned}
f &= 2 + \sqrt{19 - 2f} \\
f - 2 &= \sqrt{19 - 2f} \\
(f - 2)^2 &= \left(\sqrt{19 - 2f}\right)^2 \\
f^2 - 4f + 4 &= 19 - 2f \\
f^2 - 2f - 15 &= 0 \\
(f - 5)(f + 3) &= 0 \\
\text{Therefore } f - 5 &= 0 \\
\therefore f &= 5 \\
&\text{or} \\
f + 3 &= 0 \\
\therefore f &= -3 \\
\text{Check RHS for } f = 5: &= 2 + \sqrt{19 - 10} \\
&= 2 + \sqrt{9} \\
&= 5 \\
&= \text{LHS} \therefore \text{ valid solution} \\
\text{Check RHS for } f = -3: &= 2 + \sqrt{19 + 6} \\
&= 2 + \sqrt{25} \\
&= 7 \\
&\neq \text{LHS} \therefore \text{ not valid solution}
\end{aligned}
$$

Exercise 1 – 7: Applications of exponentials

1. Nqobani invests R 5530 into an account which pays out a lump sum at the end of 6 years. If he gets R 9622,20 at the end of the period, what compound interest rate did the bank offer him? Give answer correct to one decimal place.

   Solution:

$$\begin{aligned} A &= 9622{,}20 \\ P &= 5530 \\ n &= 6 \end{aligned}$$

$$\begin{aligned} A &= P(1+i)^n \\ 9622{,}20 &= 5530(1+i)^6 \\ \frac{9622{,}20}{5530} &= (1+i)^6 \\ \sqrt[6]{\frac{9622{,}20}{5530}} &= 1+i \\ \sqrt[6]{\frac{9622{,}20}{5530}} - 1 &= i \\ \therefore i &= 0{,}096709\ldots \\ &= 9{,}7\% \end{aligned}$$

2. The current population of Johannesburg is 3 885 840 and the average rate of population growth in South Africa is 0,7% p.a. What can city planners expect the population of Johannesburg to be in 13 years time?

   Solution:

$$\begin{aligned} P &= 3\ 885\ 840 \\ i &= 0{,}007 \\ n &= 13 \end{aligned}$$

$$\begin{aligned} A &= P(1+i)^n \\ &= 3\ 885\ 840(1+0{,}007)^{13} \\ &= 4\ 254\ 691 \end{aligned}$$

3. Abiona places 3 books in a stack on her desk. The next day she counts the books in the stack and then adds the same number of books to the top of the stack. After how many days will she have a stack of 192 books?

Solution:

$3; 6; 12; 24; 48; \ldots$

$$\begin{aligned} 3 \times 2^{n-1} &= 192 \\ 2^{n-1} &= 64 \\ &= 2^6 \\ \therefore n - 1 &= 6 \\ \therefore n &= 7 \end{aligned}$$

4. A type of mould has a very high exponential growth rate of 40% every hour. If there are initially 45 individual mould cells in the population, determine how many there will be in 19 hours.

Solution:

$$\begin{aligned} \text{Population } &= \text{Initial population } \times (1 + \text{growth percentage })^{\text{time period in hours}} \\ &= 45(1 + 0{,}4)^{19} \\ &= 26\ 893 \end{aligned}$$

1.5 Summary

Exercise 1 – 8: End of chapter exercises

1. Simplify as far as possible:

   a) $8^{-\frac{2}{3}}$

   b) $\sqrt{16} + 8^{-\frac{2}{3}}$

Solution:

a)

$$\begin{aligned} 8^{-\frac{2}{3}} &= \left(2^3\right)^{-\frac{2}{3}} \\ &= 2^{-2} \\ &= \frac{1}{4} \end{aligned}$$

b)

$$\begin{aligned} \sqrt{16} + 8^{-\frac{2}{3}} &= 4 + \left(2^3\right)^{-\frac{2}{3}} \\ &= 4 + \frac{1}{4} \\ &= 4\frac{1}{4} \end{aligned}$$

2. Simplify:

a) $\left(x^3\right)^{\frac{4}{3}}$

b) $\left(s^2\right)^{\frac{1}{2}}$

c) $\left(m^5\right)^{\frac{5}{3}}$

d) $\left(-m^2\right)^{\frac{4}{3}}$

e) $-\left(m^2\right)^{\frac{4}{3}}$

f) $\left(3y^{\frac{4}{3}}\right)^4$

Solution:

a)

$$\left(x^3\right)^{\frac{4}{3}} = x^4$$

b)

$$\left(s^2\right)^{\frac{1}{2}} = s$$

c)

$$\left(m^5\right)^{\frac{5}{3}} = m^{\frac{25}{3}}$$

d)

$$\left(-m^2\right)^{\frac{4}{3}} = m^{\frac{8}{3}}$$

e)

$$-\left(m^2\right)^{\frac{4}{3}} = -m^{\frac{8}{3}}$$

f)

$$\left(3y^{\frac{4}{3}}\right)^4 = 81y^{\frac{16}{3}}$$

3. Simplify the following:

a) $\dfrac{3a^{-2}b^{15}c^{-5}}{(a^{-4}b^3c)^{\frac{-5}{2}}}$

b) $\left(9a^6b^4\right)^{\frac{1}{2}}$

c) $\left(a^{\frac{3}{2}}b^{\frac{3}{4}}\right)^{16}$

d) $x^3\sqrt{x}$

e) $\sqrt[3]{x^4b^5}$

Solution:

a)

$$\begin{aligned}\frac{3a^{-2}b^{15}c^{-5}}{(a^{-4}b^3c)^{-\frac{5}{2}}} &= \frac{3a^{-2}b^{15}c^{-5}}{a^{10}b^{-\frac{15}{2}}c^{-\frac{5}{2}}}\\ &= \frac{3b^{15+\frac{15}{2}}}{a^{12}c^{5-\frac{5}{2}}}\\ &= \frac{3b^{\frac{45}{2}}}{a^{12}c^{\frac{5}{2}}}\end{aligned}$$

b)

$$\left(9a^6b^4\right)^{\frac{1}{2}} = 3a^3b^2$$

c)

$$\left(a^{\frac{3}{2}}b^{\frac{3}{4}}\right)^{16} = a^{24}b^{12}$$

d)

$$\begin{aligned}x^3\sqrt{x} &= x^3 \times x^{\frac{1}{2}}\\ &= x^{\frac{6}{2}} \times x^{\frac{1}{2}}\\ &= x^{\frac{7}{2}}\end{aligned}$$

e)

$$\begin{aligned}\sqrt[3]{x^4b^5} &= \left(x^4b^5\right)^{\frac{1}{3}}\\ &= x^{\frac{4}{3}}b^{\frac{5}{3}}\end{aligned}$$

4. Re-write the following expression as a power of x:

   $\dfrac{x\sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}}}{x^2}$

   Solution:

$$\begin{aligned}\frac{x\sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}}}{x^2} &= \frac{x \times x^{\frac{1}{2}} \times x^{\frac{1}{4}} \times x^{\frac{1}{8}} \times x^{\frac{1}{16}}}{x^2}\\ &= \frac{x^{\frac{31}{16}}}{x^{\frac{32}{16}}}\\ &= \frac{1}{x^{\frac{1}{16}}}\end{aligned}$$

5. Expand:

   $\left(\sqrt{x}-\sqrt{2}\right)\left(\sqrt{x}+\sqrt{2}\right)$

   Solution:

$$\begin{aligned}\left(\sqrt{x}-\sqrt{2}\right)\left(\sqrt{x}+\sqrt{2}\right) &= \left(\sqrt{x}\right)^2 - \left(\sqrt{2}\right)^2\\ &= x-2\end{aligned}$$

6. Rationalise the denominator:

   $\dfrac{10}{\sqrt{x}-\frac{1}{x}}$

   Solution:

$$\begin{aligned}\frac{10}{\sqrt{x}-1} &= \frac{10}{\sqrt{x}-1} \times \frac{\sqrt{x}+1}{\sqrt{x}+1}\\ &= \frac{10\left(\sqrt{x}+1\right)}{\left(\sqrt{x}\right)^2-1}\\ &= \frac{10\sqrt{x}+10}{x-1}\end{aligned}$$

7. Write as a single term with a rational denominator:

   $\dfrac{3}{2\sqrt{x}}+\sqrt{x}$

   Solution:

$$\begin{aligned}\frac{3}{2\sqrt{x}}+\sqrt{x} &= \frac{3+2\sqrt{x}\sqrt{x}}{2\sqrt{x}}\\ &= \frac{3+2x}{2\sqrt{x}} \times \frac{\sqrt{x}}{\sqrt{x}}\\ &= \frac{3\sqrt{x}+2x\sqrt{x}}{2x}\end{aligned}$$

8. Write in simplest surd form:

a) $\sqrt{72}$

b) $\sqrt{45}+\sqrt{80}$

c) $\dfrac{\sqrt{48}}{\sqrt{12}}$

d) $\dfrac{\sqrt{18}\div\sqrt{72}}{\sqrt{8}}$

e) $\dfrac{4}{(\sqrt{8}\div\sqrt{2})}$

f) $\dfrac{16}{(\sqrt{20}\div\sqrt{12})}$

Solution:

a)

$$\begin{aligned}\sqrt{72} &= \sqrt{8\times 9}\\ &= \sqrt{8}\times\sqrt{9}\\ &= 2\sqrt{2}\times 3\\ &= 6\sqrt{2}\end{aligned}$$

b)

$$\begin{aligned}\sqrt{45}+\sqrt{80} &= \sqrt{5\times 9}+\sqrt{5\times 16}\\ &= 3\sqrt{5}+4\sqrt{5}\\ &= 7\sqrt{5}\end{aligned}$$

c)

$$\begin{aligned}\frac{\sqrt{48}}{\sqrt{12}} &= \sqrt{\frac{48}{12}}\\ &= \sqrt{4}\\ &= 2\end{aligned}$$

$$\begin{aligned}\text{or } \frac{\sqrt{48}}{\sqrt{12}} &= \frac{\sqrt{3\times 16}}{\sqrt{3\times 4}}\\ &= \frac{4\sqrt{3}}{2\sqrt{3}}\\ &= \frac{4}{2}\\ &= 2\end{aligned}$$

d)

$$\begin{aligned}\frac{\sqrt{18}\div\sqrt{72}}{\sqrt{8}} &= \frac{\sqrt{\frac{18}{72}}}{\sqrt{8}}\\ &= \frac{\sqrt{\frac{1}{4}}}{2\sqrt{2}}\\ &= \frac{\frac{1}{2}}{2\sqrt{2}}\\ &= \frac{1}{2}\times\frac{1}{2\sqrt{2}}\\ &= \frac{1}{4\sqrt{2}}\end{aligned}$$

e)

$$\begin{aligned}\frac{4}{\sqrt{8}\div\sqrt{2}} &= \frac{4}{2\sqrt{2}\div\sqrt{2}}\\ &= \frac{4}{2}\\ &= 2\end{aligned}$$

f)

$$\begin{aligned}\frac{16}{\sqrt{20}\div\sqrt{12}} &= \frac{16}{2\sqrt{5}\div 2\sqrt{3}}\\ &= \frac{16}{\frac{2\sqrt{5}}{2\sqrt{3}}}\\ &= 16\times\frac{\sqrt{3}}{\sqrt{5}}\\ &= 16\times\frac{\sqrt{3}}{\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}\\ &= \frac{16\sqrt{15}}{5}\end{aligned}$$

9. Expand and simplify:

a) $\left(2+\sqrt{2}\right)^2$

b) $\left(2+\sqrt{2}\right)\left(1+\sqrt{8}\right)$

c) $\left(1+\sqrt{3}\right)\left(1+\sqrt{8}+\sqrt{3}\right)$

Solution:

a)

$$\begin{aligned}\left(2+\sqrt{2}\right)^2 &= \left(2+\sqrt{2}\right)\left(2+\sqrt{2}\right)\\ &= 4+4\sqrt{2}+\left(\sqrt{2}\right)^2\\ &= 4+4\sqrt{2}+2\\ &= 6+4\sqrt{2}\end{aligned}$$

b)

$$\begin{aligned}\left(2+\sqrt{2}\right)\left(1+\sqrt{8}\right) &= \left(2+\sqrt{2}\right)\left(1+2\sqrt{2}\right)\\ &= 2+4\sqrt{2}+\sqrt{2}+2\left(\sqrt{2}\right)^2\\ &= 2+5\sqrt{2}+2(2)\\ &= 6+5\sqrt{2}\end{aligned}$$

c)

$$\begin{aligned}\left(1+\sqrt{3}\right)\left(1+\sqrt{8}+\sqrt{3}\right) &= 1+\sqrt{8}+\sqrt{3}+\sqrt{3}+\sqrt{8\times 3}+\left(\sqrt{3}\right)^2\\ &= 1+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}+3\\ &= 4+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}\end{aligned}$$

10. Simplify, without use of a calculator:

a) $\sqrt{5}\left(\sqrt{45}+2\sqrt{80}\right)$

b) $\dfrac{\sqrt{98}-\sqrt{8}}{\sqrt{50}}$

Solution:

a)

$$\begin{aligned}\sqrt{5}\left(\sqrt{45}+2\sqrt{80}\right) &= \sqrt{5}\left(\sqrt{9\times 5}+2\sqrt{16\times 5}\right)\\ &= \sqrt{5}\left(3\sqrt{5}+8\sqrt{5}\right)\\ &= 3\times 5+8\times 5\\ &= 15+40\\ &= 55\end{aligned}$$

b)

$$\begin{aligned}\frac{\sqrt{98}-\sqrt{8}}{\sqrt{50}} &= \frac{\sqrt{49\times 2}-\sqrt{4\times 2}}{\sqrt{25\times 2}}\\ &= \frac{7\sqrt{2}-2\sqrt{2}}{5\sqrt{2}}\\ &= \frac{5\sqrt{2}}{5\sqrt{2}}\\ &= 1\end{aligned}$$

11. Simplify:
$\sqrt{98x^6}+\sqrt{128x^6}$

Solution:

$$\begin{aligned}\sqrt{98x^6}+\sqrt{128x^6} &= \sqrt{2\times 49x^6}+\sqrt{2\times 64x^6}\\ &= 7x^3\sqrt{2}+8x^3\sqrt{2}\\ &= 15\sqrt{2}x^3\end{aligned}$$

12. Rationalise the denominator:

a) $\dfrac{\sqrt{5}+2}{\sqrt{5}}$

b) $\dfrac{y-4}{\sqrt{y}-2}$

c) $\dfrac{2x-20}{\sqrt{x}-\sqrt{10}}$

Solution:

a)

$$\begin{aligned}\frac{\sqrt{5}+2}{\sqrt{5}} &= \frac{\sqrt{5}+2}{\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}\\ &= \frac{5+2\sqrt{5}}{5}\\ &= 1+\frac{2\sqrt{5}}{5}\end{aligned}$$

b)

$$\begin{aligned}\frac{\sqrt{y}-4}{\sqrt{y}-2} &= \frac{y-4}{\sqrt{y}-2} \times \frac{\sqrt{y}+2}{\sqrt{y}+2} \\ &= \frac{y\sqrt{y}+2y-4\sqrt{y}-8}{y-4} \\ &= \frac{2y+y\sqrt{y}-4\sqrt{y}-8}{y-4}\end{aligned}$$

c)

$$\begin{aligned}\frac{2x-20}{\sqrt{x}-\sqrt{10}} &= \frac{2x-20}{\sqrt{x}-\sqrt{10}} \times \frac{\sqrt{x}+\sqrt{10}}{\sqrt{x}+\sqrt{10}} \\ &= \frac{2(x-10)\left(\sqrt{x}+\sqrt{10}\right)}{x-10} \\ &= 2\sqrt{x}+2\sqrt{10}\end{aligned}$$

13. Evaluate without using a calculator: $\left(2-\frac{\sqrt{7}}{2}\right)^{\frac{1}{2}} \times \left(2+\frac{\sqrt{7}}{2}\right)^{\frac{1}{2}}$

Solution:

$$\begin{aligned}\left(2-\frac{\sqrt{7}}{2}\right)^{\frac{1}{2}}\left(2+\frac{\sqrt{7}}{2}\right)^{\frac{1}{2}} &= \left(4-\frac{7}{4}\right)^{\frac{1}{2}} \\ &= \sqrt{\frac{16}{4}-\frac{7}{4}} \\ &= \sqrt{\frac{9}{4}} \\ &= \frac{3}{2}\end{aligned}$$

14. Prove (without the use of a calculator):

$$\sqrt{\frac{8}{3}}+5\sqrt{\frac{5}{3}}-\sqrt{\frac{1}{6}} = \frac{10\sqrt{15}+3\sqrt{6}}{6}$$

Solution:

$$\begin{aligned}\text{LHS} &= \sqrt{\frac{8}{3}}+5\sqrt{\frac{5}{3}}-\sqrt{\frac{1}{6}} \\ &= \sqrt{\frac{8}{3}} \times \frac{\sqrt{3}}{\sqrt{3}}+5\sqrt{\frac{5}{3}} \times \frac{\sqrt{3}}{\sqrt{3}}-\sqrt{\frac{1}{6}} \times \frac{\sqrt{6}}{\sqrt{6}} \\ &= \frac{\sqrt{8}\sqrt{3}}{3}+5\frac{\sqrt{5}\sqrt{3}}{3}-\frac{\sqrt{6}}{6} \\ &= \frac{2\sqrt{24}}{6}+\frac{10\sqrt{15}}{6}-\frac{\sqrt{6}}{6} \\ &= \frac{4\sqrt{6}+10\sqrt{15}-\sqrt{6}}{6} \\ &= \frac{10\sqrt{15}+3\sqrt{6}}{6} \\ &= \text{RHS}\end{aligned}$$

15. Simplify completely by showing all your steps (do not use a calculator):

$$3^{-\frac{1}{2}}\left[\sqrt{12}+\sqrt[3]{\left(3\sqrt{3}\right)}\right]$$

Solution:

$$\begin{aligned}3^{-\frac{1}{2}}\left(\sqrt{12}+\sqrt[3]{3\sqrt{3}}\right) &= \frac{1}{\sqrt{3}}\left(2\sqrt{3}+\sqrt[3]{3\times 3^{\frac{1}{2}}}\right)\\ &= \frac{1}{\sqrt{3}}\left(2\sqrt{3}+\left(3^{\frac{3}{2}}\right)^{\frac{1}{3}}\right)\\ &= \frac{1}{\sqrt{3}}\left(2\sqrt{3}+3^{\frac{1}{2}}\right)\\ &= \frac{1}{\sqrt{3}}\left(3\sqrt{3}\right)\\ &= 3\end{aligned}$$

16. Fill in the blank surd-form number on the right hand side of the equal sign which will make the following a true statement: $-3\sqrt{6}\times -2\sqrt{24} = -\sqrt{18}\times ...$

Solution:

$$\begin{aligned}\text{LHS} &= -3\sqrt{6}\times -2\sqrt{24}\\ &= 6\sqrt{6}\sqrt{4\times 6}\\ &= 12\sqrt{6}\sqrt{6}\\ &= 12(6)\\ &= 72\\ \text{So then if LHS} &= \text{RHS}\\ \text{RHS} &= 72\\ &= \sqrt{5184}\\ &= -\sqrt{18}\times -\sqrt{288}\end{aligned}$$

17. Solve for the unknown variable:

a) $3^{x-1}-27=0$
b) $8^x-\frac{1}{\sqrt[3]{8}}=0$
c) $27(4^x)=(64)3^x$
d) $\sqrt{2x-5}=2-x$
e) $2x^{\frac{2}{3}}+3x^{\frac{1}{3}}-2=0$

Solution:

a)

$$\begin{aligned}3^{x-1}-27&=0\\ 3^{x-1}&=27\\ 3^{x-1}&=3^3\\ x-1&=3\\ \therefore x&=4\end{aligned}$$

b)

$$
\begin{aligned}
8^x - \frac{1}{\sqrt[3]{8}} &= 0 \\
2^{3x} &= \frac{1}{\sqrt[3]{2^3}} \\
2^{3x} &= \frac{1}{2} \\
2^{3x} &= 2^{-1} \\
\therefore x &= -\frac{1}{3}
\end{aligned}
$$

or

$$
\begin{aligned}
8^x - \frac{1}{\sqrt[3]{8}} &= 0 \\
8^x &= \frac{1}{\sqrt[3]{8}} \\
8^x &= 8^{-\frac{1}{3}} \\
\therefore x &= -\frac{1}{3}
\end{aligned}
$$

c)

$$
\begin{aligned}
27(4^x) &= (64)3^x \\
\frac{27}{64} &= \frac{3^x}{4^x} \\
\frac{3^3}{4^3} &= \left(\frac{3}{4}\right)^x \\
\left(\frac{3}{4}\right)^3 &= \left(\frac{3}{4}\right)^x \\
\therefore x &= 3
\end{aligned}
$$

d)

$$
\begin{aligned}
\sqrt{2x-5} &= 2-x \\
\left(\sqrt{2x-5}\right)^2 &= (2-x)^2 \\
2x-5 &= 4-4x+x^2 \\
0 &= x^2-6x+9 \\
0 &= (x-3)(x-3) \\
\therefore x &= 3
\end{aligned}
$$

$$
\begin{aligned}
\text{Check solution: LHS } &= \sqrt{2(3)-5} \\
&= \sqrt{6-5} \\
&= \sqrt{1} \\
&= 1 \\
\text{Check solution: RHS } &= 2-3 \\
&= -1 \\
\text{RHS } &\neq \text{LHS} \\
&\therefore \text{No solution}
\end{aligned}
$$

e)

$$\begin{aligned}
2x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 2 &= 0 \\
\left(2x^{\frac{1}{3}} - 1\right)\left(x^{\frac{1}{3}} + 2\right) &= 0 \\
\therefore 2x^{\frac{1}{3}} - 1 &= 0 \\
2x^{\frac{1}{3}} &= 1 \\
x^{\frac{1}{3}} &= \frac{1}{2} \\
\left(x^{\frac{1}{3}}\right)^3 &= \left(\frac{1}{2}\right)^3 \\
\therefore x &= \frac{1}{8}
\end{aligned}$$

or

$$\begin{aligned}
x^{\frac{1}{3}} + 2 &= 0 \\
x^{\frac{1}{3}} &= -2 \\
\left(x^{\frac{1}{3}}\right)^3 &= (-2)^3 \\
x &= -8
\end{aligned}$$

Therefore $x = \frac{1}{8}$ or $x = -8$

18. a) Show that $\sqrt{\frac{3^{x+1} - 3^x}{3^{x-1}} + 3}$ is equal to 3

b) Hence solve $\sqrt{\frac{3^{x+1} - 3^x}{3^{x-1}} + 3} = \left(\frac{1}{3}\right)^{x-2}$

Solution:

a)

$$\begin{aligned}
\text{LHS} &= \sqrt{\frac{3^{x+1} - 3^x}{3^{x-1}} + 3} \\
&= \sqrt{\frac{3^{x+1}}{3^{x-1}} - \frac{3^x}{3^{x-1}} + 3} \\
&= \sqrt{3^{x+1-x+1} - 3^{x-x+1} + 3} \\
&= \sqrt{3^2 - 3^1 + 3} \\
&= \sqrt{3^2} \\
&= 3 \\
&= \text{RHS}
\end{aligned}$$

b)

$$\begin{aligned}
\sqrt{\frac{3^{x+1} - 3^x}{3^{x-1}} + 3} &= \left(\frac{1}{3}\right)^{x-2} \\
3 &= \left(\frac{1}{3}\right)^{x-2} \\
3 &= (3)^{-x+2} \\
1 &= -x + 2 \\
x &= 1
\end{aligned}$$

CHAPTER 2

Equations and inequalities

- Discuss terminology.
- Emphasize the golden rule of solving equations: what you do to the left-hand side (LHS), you must do to the right-hand side (RHS). The LHS must always equal the RHS.
- Emphasize using equivalence and the additive inverse in equations (and not "taking the term to the other side").
- Graphs have been included to help learners visualize answers. Encourage learners to draw quick sketches even if a graph is not required in the solution.
- Encourage learners to check that an expression has been correctly factorised by expanding the brackets to get back to the original expression.
- Encourage learners to check their answers (solutions) but substituting back into the original equation and making sure the solution satisfies the equation.
- Both methods for completing the square have been included:
  - Taking the square root of both sides of the equation. This is the preferred and easier method. Remember to include $\pm$ so as not to lose one solution.
  - Factorising: this method is used in Grade 12 for the equation of a circle.
- A common mistake made by learners is writing the quadratic formula as $x = -b \pm \frac{\sqrt{b^2-4ac}}{2a}$.
- For the nature of roots investigation, allow learners to discover the connection between answers obtained using the quadratic formula and the value of the discriminant.
- Word problems: encourage learners to develop their skills in interpreting word problems by drawing sketches and making tables of the given information.
- Remind learners not to treat an inequality as an equation.
- Emphasize that the critical values are not the solution to the inequality, they indicate the points where the signs changes. You need that to draw up a number line table of signs.
- Remind learners that when they multiply or divide an inequality by a negative number, the inequality sign changes direction.

2.1 Revision

Exercise 2 – 1: Solution by factorisation

Solve the following quadratic equations by factorisation. Answers may be left in surd form, where applicable.

1. $7t^2 + 14t = 0$

Solution:

$$\begin{aligned} 7t^2 + 14t &= 0 \\ 7t(t+2) &= 0 \\ t(t+2) &= 0 \\ t &= 0 \text{ or } t = -2 \end{aligned}$$

2. $12y^2 + 24y + 12 = 0$

Solution:

$$\begin{aligned} 12y^2 + 24y + 12 &= 0 \\ 12(y^2 + 2y + 1) &= 0 \\ y^2 + 2y + 1 &= 0 \\ (y+1)(y+1) &= 0 \\ y &= -1 \end{aligned}$$

3. $16s^2 = 400$

Solution:

$$\begin{aligned} 16s^2 - 400 &= 0 \\ 16(s^2 - 25) &= 0 \\ (s-5)(s+5) &= 0 \\ s &= \pm 5 \end{aligned}$$

4. $y^2 - 5y + 6 = 0$

Solution:

$$\begin{aligned} y^2 - 5y + 6 &= 0 \\ (y-3)(y-2) &= 0 \\ y &= 3 \text{ or } y = 2 \end{aligned}$$

5. $y^2 + 5y - 36 = 0$

Solution:

$$\begin{aligned} y^2 + 5y - 36 &= 0 \\ (y-4)(y+9) &= 0 \\ y &= 4 \text{ or } y = -9 \end{aligned}$$

6. $4 + p = \sqrt{p+6}$

Solution:

$$\begin{aligned} 4+p &= \sqrt{p+6} \\ (4+p)^2 &= p+6 \\ 16+8p+p^2 &= p+6 \\ p^2+7p+10 &= 0 \\ (p+5)(p+2) &= 0 \\ p &= -5 \text{ or } p = -2 \end{aligned}$$

Check $p = -5$:

$$\begin{aligned} \text{RHS} &= \sqrt{-5+6} \\ &= \sqrt{1} \\ &= 1 \\ \text{LHS} &= 4+(-5) \\ &= -1 \\ \therefore \text{LHS} &\neq \text{RHS} \end{aligned}$$

Solution $p = -5$ is not valid.

Check $p = -2$:

$$\begin{aligned} \text{RHS} &= \sqrt{-2+6} \\ &= \sqrt{4} \\ &= 2 \\ \text{LHS} &= 4+(-2) \\ &= 2 \\ \therefore \text{LHS} &= \text{RHS} \end{aligned}$$

Solution $p = -2$ is valid.

Final answer: $p = -2$.

7. $-y^2 - 11y - 24 = 0$

Solution:

$$\begin{aligned} -y^2 - 11y - 24 &= 0 \\ y^2 + 11y + 24 &= 0 \\ (y+3)(y+8) &= 0 \\ y &= -3 \text{ or } y = -8 \end{aligned}$$

8. $13y - 42 = y^2$

Solution:

$$\begin{aligned} 13y - 42 &= y^2 \\ y^2 - 13y + 42 &= 0 \\ (y-6)(y-7) &= 0 \\ y &= 6 \text{ or } y = 7 \end{aligned}$$

9. $(x-1)(x+10) = -24$

Solution:

$$\begin{aligned} (x-1)(x+10) &= -24 \\ x^2 + 9x - 10 &= -24 \\ x^2 + 9x + 14 &= 0 \\ (x+7)(x+2) &= 0 \\ x &= -7 \text{ or } x = -2 \end{aligned}$$

10. $y^2 - 5ky + 4k^2 = 0$

Solution:

$$\begin{aligned} y^2 - 5ky + 4k^2 &= 0 \\ (y-4k)(y-k) &= 0 \\ y &= 4k \text{ or } y = k \end{aligned}$$

11. $2y^2 - 61 = 101$

Solution:

$$\begin{aligned} 2y^2 - 61 &= 101 \\ 2y^2 - 162 &= 0 \\ 2(y^2 - 81) &= 0 \\ y^2 - 81 &= 0 \\ (y-9)(y+9) &= 0 \\ y &= 9 \text{ or } y = -9 \end{aligned}$$

12. $2y^2 - 10 = 0$

Solution:

$$\begin{aligned} 2y^2 - 10 &= 0 \\ 2(y^2 - 5) &= 0 \\ y^2 - 5 &= 0 \\ (y-\sqrt{5})(y+\sqrt{5}) &= 0 \\ y &= \sqrt{5} \text{ or } y = -\sqrt{5} \end{aligned}$$

13. $-8 + h^2 = 28$

Solution:

$$\begin{aligned} -8 + h^2 &= 28 \\ h^2 - 36 &= 0 \\ (h-6)(h+6) &= 0 \\ h &= \pm 6 \end{aligned}$$

14. $y^2 - 4 = 10$

Solution:

$$\begin{aligned} y^2 - 4 &= 10 \\ y^2 - 14 &= 0 \\ (y - \sqrt{14})(y + \sqrt{14}) &= 0 \\ y &= \sqrt{14} \text{ or } y = -\sqrt{14} \end{aligned}$$

15. $\sqrt{5 - 2p} - 4 = \frac{1}{2}p$

Solution:

$$\begin{aligned} \sqrt{5 - 2p} - 4 &= \frac{1}{2}p \\ \sqrt{5 - 2p} &= \frac{1}{2}p + 4 \\ \left(\sqrt{5 - 2p}\right)^2 &= \left(\frac{1}{2}p + 4\right)^2 \\ 5 - 2p &= \frac{p^2}{4} + 4p + 16 \\ 0 &= \frac{p^2}{4} + 6p + 11 \\ 0 &= p^2 + 24p + 44 \\ 0 &= (p + 2)(p + 22) \\ p &= -2 \text{ or } p = -22 \end{aligned}$$

Check $p = -2$:

$$\begin{aligned} \text{LHS} &= \sqrt{5 - 2(-2)} - 4 \\ &= \sqrt{9} - 4 \\ &= -1 \\ \text{RHS} &= \frac{1}{2}(-2) \\ &= -1 \\ \therefore \text{LHS} &= \text{RHS} \end{aligned}$$

Solution $p = -2$ is valid.

Check $p = -22$:

$$\begin{aligned} \text{LHS} &= \sqrt{5 - 2(-22)} - 4 \\ &= \sqrt{49} - 4 \\ &= 3 \\ \text{RHS} &= \frac{1}{2}(-22) \\ &= -11 \\ \therefore \text{LHS} &\neq \text{RHS} \end{aligned}$$

Solution $p = -22$ is not valid.

Final answer: $p = -2$.

16. $y^2 + 28 = 100$

Solution:

$$\begin{aligned} y^2 + 28 &= 100 \\ y^2 - 72 &= 0 \\ (y - \sqrt{72})(y + \sqrt{72}) &= 0 \\ y &= \pm\sqrt{72} \\ &= \pm\sqrt{2 \times 36} \\ &= \pm 6\sqrt{2} \end{aligned}$$

17. $f\,(2f + 1) = 15$

Solution:

$$\begin{aligned} f(2f + 1) &= 15 \\ 2f^2 + f - 15 &= 0 \\ (2f - 5)(f + 3) &= 0 \\ f &= \frac{5}{2} \text{ or } f = -3 \end{aligned}$$

18. $2x = \sqrt{21x - 5}$

Solution:

$$\begin{aligned} 2x &= \sqrt{21x - 5} \\ (2x)^2 &= \left(\sqrt{21x - 5}\right)^2 \\ 4x^2 &= 21x - 5 \\ 4x^2 - 21x + 5 &= 0 \\ (4x - 1)(x - 5) &= 0 \\ x &= \frac{1}{4} \text{ or } x = 5 \end{aligned}$$

Check $x = \frac{1}{4}$:

$$\begin{aligned} \text{LHS} &= \sqrt{21\left(\frac{1}{4}\right) - 5} \\ &= \sqrt{\frac{1}{4}} \\ &= \frac{1}{2} \\ \text{RHS} &= \frac{1}{4}(2) \\ &= \frac{1}{2} \\ \therefore \text{LHS} &= \text{RHS} \end{aligned}$$

Solution $x = \frac{1}{4}$ is valid.

Check $x = 5$:

$$\begin{aligned} \text{LHS} &= \sqrt{21(5) - 5} \\ &= \sqrt{100} \\ &= 10 \\ \text{RHS} &= 2(10) \\ &= 20 \\ \therefore \text{LHS} &\neq \text{RHS} \end{aligned}$$

Solution $x = 5$ is not valid.

Final answer: $x = \frac{1}{4}$.

19. $\dfrac{5y}{y-2} + \dfrac{3}{y} + 2 = \dfrac{-6}{y^2 - 2y}$

Solution:

Restrictions: $y \neq 0$, $y \neq 2$

$$\frac{5y}{y-2} + \frac{3}{y} + 2 = d\frac{-6}{y^2 - 2y}$$

Multiply both sides by $y(y - 2)$:

$$\begin{aligned} y(5y) + 3(y - 2) + 2y(y - 2) &= -6 \\ 5y^2 + 3y - 6 + 2y^2 - 4y &= -6 \\ 7y^2 - y &= 0 \\ y(7y - 1) &= 0 \\ y = 0 \text{ or } y &= \frac{1}{7} \end{aligned}$$

From the restriction $y \neq 0$, therefore $y = \frac{1}{7}$.

20. $\dfrac{x+9}{x^2-9} + \dfrac{1}{x+3} = \dfrac{2}{x-3}$

Solution:

Restrictions: $x \neq \pm 3$.

$$\frac{x+9}{(x+3)(x-3)} + \frac{1}{x+3}\frac{2}{x-3}$$

Multiply both sides by $(x + 3)(x - 3)$:

$$\begin{aligned} x + 9 + x - 3 &= 2(x + 3) \\ 2x + 6 &= 2x + 6 \\ 0x &= 0 \end{aligned}$$

Therefore $x \in \mathbb{R}$, $x \neq \pm 3$.

21. $\dfrac{y-2}{y+1} = \dfrac{2y+1}{y-7}$

Solution:

Restrictions: $y \neq -1$, $y \neq 7$

$$\begin{aligned}
\frac{y-2}{y+1} &= \frac{2y+1}{y-7} \\
(y-2)(y-7) &= (2y+1)(y+1) \\
y^2 - 9y + 14 &= 2y^2 + 3y + 1 \\
-y^2 - 12y + 13 &= 0 \\
y^2 + 12y - 13 &= 0 \\
(y+13)(y-1) &= 0 \\
y = -13 &\text{ or } y = 1
\end{aligned}$$

22. $1 + \frac{t-2}{t-1} = \frac{5}{t^2 - 4t + 3} + \frac{10}{3-t}$

Solution:

Restrictions: $t \neq 1$, $t \neq 3$

$$\begin{aligned}
1 + \frac{t-2}{t-1} &= \frac{5}{t^2 - 4t + 3} + \frac{10}{3-t} \\
1 + \frac{t-2}{t-1} &= \frac{5}{(t-3)(t-1)} - \frac{10}{t-3} \\
(t-3)(t-1) + (t-2)(t-3) &= 5 - 10(t-1) \\
t^2 - 4t + 3 + t^2 - 5t + 6 &= 5 - 10t + 10 \\
2t^2 - 9t + 9 &= -10t + 15 \\
2t^2 + t - 6 &= 0 \\
(2t-3)(t+2) &= 0 \\
\text{Therefore } t = \frac{3}{2} &\text{ or } t = -2
\end{aligned}$$

23. $\frac{4}{m+3} + \frac{4}{4-m^2} = \frac{5m-5}{m^2+m-6}$

Solution:

Restrictions: $m \neq -3$, $m \neq \pm 2$

$$\begin{aligned}
\frac{4}{m+3} + \frac{4}{4-m^2} &= \frac{5m-5}{m^2+m-6} \\
\frac{4}{m+3} - \frac{4}{m^2-4} &= \frac{5(m-1)}{(m+3)(m-2)} \\
4(m^2-4) - 4(m+3) &= 5(m-1)(m+2) \\
4m^2 - 16 - 4m - 12 &= 5(m^2 + m - 2) \\
4m^2 - 4m - 28 &= 5m^2 + 5m - 10 \\
m^2 + 9m + 18 &= 0 \\
(m+3)(m+6) &= 0 \\
m = -3 &\text{ or } m = -6 \\
\text{But } m \neq -3, &\text{ therefore } m = -6
\end{aligned}$$

24. $5\sqrt{5t+1} - 4 = 5t + 1$

Solution:

$$\begin{aligned}
5\sqrt{5t+1}-4 &= 5t+1 \\
5\sqrt{5t+1} &= 5t+5 \\
5\sqrt{5t+1} &= 5(t+1) \\
\sqrt{5t+1} &= t+1 \\
\left(\sqrt{5t+1}\right)^2 &= (t+1)^2 \\
5t+1 &= t^2+2t+1 \\
t^2-3t &= 0 \\
t(t-3) &= 0 \\
t &= 0 \text{ or } t = 3
\end{aligned}$$

Check $t = 0$:

$$\begin{aligned}
\text{LHS} &= 5\sqrt{5(0)+1}-4 \\
&= 1 \\
\text{RHS} &= 5(0)+1 \\
&= 1 \\
\therefore \text{LHS} &= \text{RHS}
\end{aligned}$$

Solution $t = 0$ is valid.

Check $t = 3$:

$$\begin{aligned}
\text{LHS} &= 5\sqrt{5(3)+1}-4 \\
&= 5\sqrt{16}-4 \\
&= 16 \\
\text{RHS} &= 5(3)+1 \\
&= 16 \\
\therefore \text{LHS} &= \text{RHS}
\end{aligned}$$

Solution $t = 3$ is valid.

Final answer: $t = 0$ or $t = 3$.

2.2 Completing the square

Exercise 2 – 2: Solution by completing the square

1. Solve the following equations by completing the square:

a) $x^2 + 10x - 2 = 0$
b) $x^2 + 4x + 3 = 0$
c) $p^2 - 5 = -8p$
d) $2(6x + x^2) = -4$
e) $x^2 + 5x + 9 = 0$
f) $t^2 + 30 = 2(10 - 8t)$
g) $3x^2 + 6x - 2 = 0$
h) $z^2 + 8z - 6 = 0$
i) $2z^2 = 11z$
j) $5 + 4z - z^2 = 0$

Solution:

a)

$$\begin{aligned} x^2 + 10x - 2 &= 0 \\ x^2 + 10x &= 2 \\ x^2 + 10x + 25 &= 2 + 25 \\ (x+5)^2 - 27 &= 0 \\ \left[(x+5) + \sqrt{27}\right]\left[(x+5) - \sqrt{27}\right] &= 0 \\ (x+5) = -\sqrt{27} &\text{ or } (x+5) = \sqrt{27} \\ x = -5 - 3\sqrt{3} &\text{ or } x = -5 + 3\sqrt{3} \end{aligned}$$

b)

$$\begin{aligned} x^2 + 4x + 3 &= 0 \\ x^2 + 4x &= -3 \\ x^2 + 4x + 4 &= -3 + 4 \\ (x+2)^2 &= 1 \\ (x+2) &= \pm\sqrt{1} \\ &= \pm 1 \\ x = -2 + 1 = -1 &\text{ or } x = -2 - 1 = -3 \end{aligned}$$

c)

$$\begin{aligned} p^2 - 5 &= -8p \\ p^2 + 8p - 5 &= 0 \\ p^2 + 8p &= 5 \\ p^2 + 8p + 16 &= 5 + 16 \\ (p+4)^2 &= 21 \\ (p+4) &= \pm\sqrt{21} \\ p &= -4 \pm \sqrt{21} \end{aligned}$$

d)

$$\begin{aligned} 2(6x + x^2) &= -4 \\ 2x^2 + 12x + 4 &= 0 \\ x^2 + 6x + 2 &= 0 \\ x^2 + 6x &= -2 \\ x^2 + 6x + 9 &= -2 + 9 \\ (x+3)^2 &= 7 \\ x + 3 &= \pm\sqrt{7} \\ x &= -3 \pm \sqrt{7} \end{aligned}$$

e)

$$
\begin{aligned}
x^2 + 5x + 9 &= 0 \\
x^2 + 5x &= -9 \\
x^2 + 5x + \frac{25}{4} &= -9 + \frac{25}{4} \\
\left(x + \frac{5}{2}\right)^2 &= -\frac{11}{4} \\
x + \frac{5}{2} &= \pm\sqrt{-\frac{11}{4}}
\end{aligned}
$$

No real solution

f)

$$
\begin{aligned}
t^2 + 30 &= 2(10 - 8t) \\
t^2 + 16t + 10 &= 0 \\
t^2 + 16t &= -10 \\
t^2 + 16t + 64 &= -10 + 64 \\
(t + 8)^2 &= 54 \\
t + 8 &= \pm\sqrt{54} \\
t &= -8 \pm \sqrt{9 \times 6} \\
\therefore t &= -8 \pm 3\sqrt{6}
\end{aligned}
$$

g)

$$
\begin{aligned}
3x^2 + 6x - 2 &= 0 \\
x^2 + 2x &= \frac{2}{3} \\
x^2 + 2x + 1 &= \frac{2}{3} + 1 \\
(x + 1)^2 &= \frac{5}{3} \\
x + 1 &= \pm\sqrt{\frac{5}{3}} \\
x &= -1 \pm \sqrt{\frac{5}{3}}
\end{aligned}
$$

h)

$$
\begin{aligned}
z^2 + 8z - 6 &= 0 \\
z^2 + 8z &= 6 \\
z^2 + 8z + 16 &= 6 + 16 \\
(z + 4)^2 &= 22 \\
z + 4 &= \pm\sqrt{22} \\
z &= -4 \pm \sqrt{22}
\end{aligned}
$$

i)

$$
\begin{aligned}
2z^2 &= 11z \\
2z^2 - 11z &= 0 \\
z^2 - \frac{11}{2}z &= 0 \\
z^2 - \frac{11}{2}z + \frac{121}{16} &= \frac{121}{16} \\
\left(z - \frac{11}{4}\right)^2 &= \frac{121}{16} \\
z - \frac{11}{4} &= \pm\frac{11}{4} \\
z = \frac{11}{4} + \frac{11}{4} = \frac{11}{2} &\text{ or } z = \frac{11}{4} - \frac{11}{4} = 0
\end{aligned}
$$

j)

$$
\begin{aligned}
5 + 4z - z^2 &= 0 \\
z^2 - 4z &= 5 \\
z^2 - 4z + 4 &= 5 + 4 \\
(z - 2)^2 &= 9 \\
z - 2 &= \pm\sqrt{9} \\
z = 2 + 3 = 5 &\text{ or } z = 2 - 3 = -1
\end{aligned}
$$

2. Solve for k in terms of a: $k^2 + 6k + a = 0$

Solution:

$$
\begin{aligned}
k^2 + 6k + a &= 0 \\
k^2 + 6k &= -a \\
k^2 + 6k + 9 &= 9 - a \\
(k + 3)^2 &= 9 - a \\
k + 3 &= \pm\sqrt{9 - a} \\
k &= -3 \pm \sqrt{9 - a}
\end{aligned}
$$

3. Solve for y in terms of p, q and r: $py^2 + qy + r = 0$

Solution:

$$
\begin{aligned}
py^2 + qy + r &= 0 \\
y^2 + \frac{q}{p}y + \frac{r}{p} &= 0 \\
y^2 + \frac{q}{p}y &= -\frac{r}{p} \\
y^2 + \frac{q}{p}y + \left(\frac{q}{2p}\right)^2 &= \left(\frac{q}{2p}\right)^2 - \frac{r}{p} \\
\left(y + \frac{q}{2p}\right)^2 &= \frac{q^2}{4p^2} - \frac{r}{p} \\
y + \frac{q}{2p} &= \pm\sqrt{\frac{q^2 - 4pr}{4p^2}} \\
y &= -\frac{q}{2p} \pm \frac{\sqrt{q^2 - 4pr}}{2p} \\
y &= \frac{-q \pm \sqrt{q^2 - 4pr}}{2p}
\end{aligned}
$$

2.3 Quadratic formula

Exercise 2 – 3: Solution by the quadratic formula

Solve the following using the quadratic formula.

1. $3t^2 + t - 4 = 0$

   Solution:

$$
\begin{aligned}
3t^2 + t - 4 &= 0 \\
t &= \frac{-(1) \pm \sqrt{1^2 - 4(3)(-4)}}{2(3)} \\
&= \frac{-1 \pm \sqrt{1 + 48}}{6} \\
&= \frac{-1 \pm \sqrt{49}}{6} \\
&= \frac{-1 \pm 7}{6}
\end{aligned}
$$

$$
t = \frac{-1 + 7}{6} = \frac{6}{6} = 1 \text{ or } t = \frac{-1 - 7}{6} = \frac{-8}{6} = \frac{-4}{3}
$$

2. $x^2 - 5x - 3 = 0$

   Solution:

$$
\begin{aligned}
x^2 - 5x - 3 &= 0 \\
x &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-3)}}{2(1)} \\
&= \frac{5 \pm \sqrt{25 + 12}}{2} \\
&= \frac{5 \pm \sqrt{37}}{2}
\end{aligned}
$$

therefore $x = \frac{5 + \sqrt{37}}{2}$ or $x = \frac{5 - \sqrt{37}}{2}$

3. $2t^2 + 6t + 5 = 0$

Solution:

$$
\begin{aligned}
2t^2 + 6t + 5 &= 0 \\
t &= \frac{-6 \pm \sqrt{(6)^2 - 4(2)(5)}}{2(2)} \\
&= \frac{-6 \pm \sqrt{36 - 40}}{4} \\
&= \frac{-6 \pm \sqrt{-4}}{4}
\end{aligned}
$$

No real solution

4. $2p(2p + 1) = 2$

Solution:

$$
\begin{aligned}
4p^2 + 2p - 2 &= 0 \\
2p^2 + p - 1 &= 0 \\
p &= \frac{-1 \pm \sqrt{1^2 - 4(2)(-1)}}{2(2)} \\
&= \frac{-1 \pm \sqrt{1 + 8}}{4} \\
&= \frac{-1 \pm \sqrt{9}}{4} \\
&= \frac{-1 \pm 3}{4}
\end{aligned}
$$

therefore $p = \frac{-1 + 3}{4} = \frac{1}{2}$ or $p = \frac{-1 - 3}{4} = -1$

5. $-3t^2 + 5t - 8 = 0$

Solution:

$$
\begin{aligned}
-3t^2 + 5t - 8 &= 0 \\
t &= \frac{-5 \pm \sqrt{5^2 - 4(-3)(-8)}}{2(-3)} \\
&= \frac{-5 \pm \sqrt{25 - 96}}{-6} \\
&= \frac{-5 \pm \sqrt{-71}}{-6}
\end{aligned}
$$

No real solution

6. $5t^2 + 3t - 3 = 0$

Solution:

$$
\begin{aligned}
5t^2 + 3t - 3 &= 0 \\
t &= \frac{-3 \pm \sqrt{3^2 - 4(5)(-3)}}{2(5)} \\
&= \frac{-3 \pm \sqrt{9 + 60}}{10} \\
&= \frac{-3 \pm \sqrt{69}}{10}
\end{aligned}
$$

therefore $t = \dfrac{-3 + \sqrt{69}}{10}$ or $t = \dfrac{-3 - \sqrt{69}}{10}$

7. $t^2 - 4t + 2 = 0$

Solution:

$$
\begin{aligned}
t^2 - 4t + 2 &= 0 \\
t &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} \\
&= \frac{4 \pm \sqrt{16 - 8}}{2} \\
&= \frac{4 \pm \sqrt{8}}{2} \\
&= 2 \pm \sqrt{2}
\end{aligned}
$$

8. $9(k^2 - 1) = 7k$

Solution:

$$
\begin{aligned}
9k^2 - 7k - 9 &= 0 \\
k &= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(9)(-9)}}{2(9)} \\
&= \frac{7 \pm \sqrt{49 + 324}}{18} \\
&= \frac{7 \pm \sqrt{373}}{18}
\end{aligned}
$$

$k = \dfrac{7 + \sqrt{373}}{18}$ or $k = \dfrac{7 - \sqrt{373}}{18}$

9. $3f - 2 = -2f^2$

Solution:

$$\begin{aligned} 2f^2 + 3f - 2 &= 0 \\ f &= \frac{-3 \pm \sqrt{3^2 - 4(2)(-2)}}{2(2)} \\ &= \frac{-3 \pm \sqrt{9 + 16}}{4} \\ &= \frac{-3 \pm \sqrt{25}}{4} \\ &= \frac{-3 \pm 5}{4} \\ \text{therefore } f = \frac{-3+5}{4} = \frac{1}{2} &\text{ or } f = \frac{-3-5}{4} = -2 \end{aligned}$$

10. $t^2 + t + 1 = 0$

Solution:

$$\begin{aligned} t^2 + t + 1 &= 0 \\ t &= \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2(1)} \\ &= \frac{-1 \pm \sqrt{-3}}{2} \end{aligned}$$

No real solution

2.4 Substitution

Exercise 2 – 4:

Solve the following quadratic equations by substitution:

1. $-24 = 10(x^2 + 5x) + (x^2 + 5x)^2$

Solution:

We notice that $x^2 + 5x$ is a repeated expression and we therefore let $k = x^2 + 5x$ so that the equation becomes

$$-24 = 10k + k^2$$

Now we can solve for k:

$$k^2 + 10k + 24 = 0$$
$$(k + 6)(k + 4) = 0$$
$$\text{Therefore } k = -6 \text{ or } k = -4$$

We now use these values for k to solve for the original variable x

For $k = -6$

$$x^2 + 5x = -6$$
$$x^2 + 5x + 6 = 0$$
$$(x + 2)(x + 3) = 0$$
$$\text{Therefore } x = -2 \text{ or } x = -3$$

For $k = -4$

$$x^2 + 5x = -4$$
$$x^2 + 5x + 4 = 0$$
$$(x + 1)(x + 4) = 0$$
$$\text{Therefore } x = -1 \text{ or } x = -4$$

The roots of the equation are $x = -1$, $x = -4$, $x = -2$ and $x = -3$.

2. $(x^2 - 2x)^2 - 8 = 7(x^2 - 2x)$

Solution:

We notice that $x^2 - 2x$ is a repeated expression and we therefore let $k = x^2 - 2x$ so that the equation becomes

$$k^2 - 8 = 7k$$

Now we can solve for k:

$$k^2 - 7k - 8 = 0$$
$$(k + 1)(k - 8) = 0$$
$$\text{Therefore } k = -1 \text{ or } k = 8$$

We now use these values for k to solve for the original variable x

For $k = -1$

$$x^2 - 2x = -1$$
$$x^2 - 2x + 1 = 0$$
$$(x - 1)(x - 1) = 0$$
$$\text{Therefore } x = 1$$

For $k = 8$

$$x^2 - 2x = 8$$
$$x^2 - 2x - 8 = 0$$
$$(x + 2)(x - 4) = 0$$
$$\text{Therefore } x = -2 \text{ or } x = 4$$

The roots of the equation are $x = 1$, $x = 4$ and $x = -2$.

3. $x^2 + 3x - \frac{56}{x(x+3)} = 26$

Solution:

The restrictions are the values for x that would result in the denominator being equal to 0, which would make the fraction undefined. Therefore $x \neq 0$ and $x \neq -3$.

We note that the denominator is equivalent to $x^2 + 3x$.

We notice that $x^2 + 3x$ is a repeated expression and we therefore let $k = x^2 + 3x$ so that the equation becomes

$$k - \frac{56}{k} = 26$$

The restrictions are the values for k that would result in the denominator being equal to 0, which would make the fraction undefined. Therefore $k \neq 0$.

Now we can solve for k

$$\begin{aligned} k - \frac{56}{k} &= 26 \\ k^2 - 56 &= 26k \\ k^2 - 26k - 56 &= 0 \\ (k - 28)(k + 2) &= 0 \\ \text{Therefore } k = 28 &\text{ or } k = -2 \end{aligned}$$

We check these two roots against the restrictions for k and confirm that both are valid.

We can now use values obtained for k to solve for the original variable x

For $k = 28$

$$\begin{aligned} x^2 + 3x &= 28 \\ x^2 + 3x - 28 &= 0 \\ (x + 7)(x - 4) &= 0 \\ \text{Therefore } x = -7 &\text{ or } x = 4 \end{aligned}$$

For $k = -2$

$$\begin{aligned} x^2 + 3x &= -2 \\ x^2 + 3x + 2 &= 0 \\ (x + 2)(x + 1) &= 0 \\ \text{Therefore } x = -1 &\text{ or } x = -2 \end{aligned}$$

We check these roots against the restrictions for x and confirm that all four values are valid.

The roots of the equation are $x = -7$, $x = 4$, $x = -1$ and $x = -2$.

4. $x^2 - 18 + x + \frac{72}{x^2 + x} = 0$

Solution:

The restrictions are the values for x that would result in the denominator being equal to 0, which would make the fraction undefined. Therefore $x \neq 0$ and $x \neq -1$.

We notice that $x^2 + x$ is a repeated expression and we therefore let $k = x^2 + x$ so that the equation becomes

$$k - 18 + \frac{72}{k} = 0$$

The restrictions are the values for k that would result in the denominator being equal to 0, which would make the fraction undefined. Therefore $k \neq 0$.

Now we can solve for k

$$\begin{aligned} k - 18 + \frac{72}{k} &= 0 \\ k^2 - 18k + 72 &= 0 \\ (k - 12)(k - 6) &= 0 \\ \text{Therefore } k = 12 &\text{ or } k = 6 \end{aligned}$$

We check these two roots against the restrictions for k and confirm that both are valid.

We can now use values obtained for k to solve for the original variable x

For $k = 12$

$$\begin{aligned} x^2 + x &= 12 \\ x^2 + x - 12 &= 0 \\ (x + 4)(x - 3) &= 0 \\ \text{Therefore } x = -4 &\text{ or } x = 3 \end{aligned}$$

For $k = 6$

$$\begin{aligned} x^2 + x &= 6 \\ x^2 + x - 6 &= 0 \\ (x + 3)(x - 2) &= 0 \\ \text{Therefore } x = -3 &\text{ or } x = 2 \end{aligned}$$

We check these roots against the restrictions for x and confirm that all four values are valid.

The roots of the equation are $x = -4$, $x = 3$, $x = -3$ and $x = 2$.

5. $x^2 - 4x + 10 - 7(4x - x^2) = -2$

Solution:

$$x^2 - 4x + 10 - 28x + 7x^2 = -2$$
$$8x^2 - 32x + 12 = 0$$
$$2x^2 - 8x + 3 = 0$$
$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(2)(3)}}{2(2)}$$
$$= \frac{8 \pm \sqrt{64 - 24}}{4}$$
$$= \frac{8 \pm \sqrt{40}}{4}$$
$$\text{therefore } x = \frac{8 + \sqrt{40}}{4} \text{ or } x = \frac{8 - \sqrt{40}}{4}$$

6. $\dfrac{9}{x^2 + 2x - 12} = x^2 + 2x - 12$

Solution:

To find the restrictions we first note that the roots of the denominator are $\frac{-2\pm\sqrt{52}}{2}$

The restrictions are the values for x that would result in the denominator being equal to 0, which would make the fraction undefined. Therefore $x \neq \frac{-2+\sqrt{52}}{2}$ and $x \neq \frac{-2-\sqrt{52}}{2}$.

We notice that $x^2 + 2x - 12$ is a repeated expression and we therefore let $k = x^2 + 2x - 12$ so that the equation becomes

$$\frac{9}{k} = k$$

The restrictions are the values for k that would result in the denominator being equal to 0, which would make the fraction undefined. Therefore $k \neq 0$.

Now we can solve for k

$$\frac{9}{k} = k$$
$$9 = k^2$$
$$k = \pm 3$$
$$\text{Therefore } k = -3 \text{ or } k = 3$$

We check these two roots against the restrictions for k and confirm that both are valid.

We can now use values obtained for k to solve for the original variable x

For $k = 3$

$$x^2 + 2x - 12 = 3$$
$$x^2 + 2x - 15 = 0$$
$$(x + 5)(x - 3) = 0$$
$$\text{Therefore } x = -5 \text{ or } x = 3$$

For $k = -3$

$$
\begin{aligned}
x^2 + 2x - 12 &= -3 \\
x^2 + 2x - 9 &= 0 \\
x &= \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-9)}}{2(1)} \\
&= \frac{-2 \pm \sqrt{4 + 36}}{2} \\
&= \frac{-2 \pm \sqrt{40}}{2} \\
&= \frac{-2 \pm 2\sqrt{10}}{2} \\
&= -1 \pm \sqrt{10}
\end{aligned}
$$

therefore $x = -1 + \sqrt{10}$ or $x = -1 - \sqrt{10}$

We check these roots against the restrictions for x and confirm that all four values are valid.

The roots of the equation are $x = -5$, $x = 3$, $x = -1 + \sqrt{10}$ and $x = -1 - \sqrt{10}$.

2.5 Finding the equation

Exercise 2 – 5: Finding the equation

1. Determine a quadratic equation for a graph that has roots 3 and -2.

   Solution:

$$x = 3 \text{ or } x = -2$$

$$x - 3 = 0 \text{ or } x + 2 = 0$$

$$(x - 3)(x + 2) = 0$$

$$x^2 - x - 6 = 0$$

2. Find a quadratic equation for a graph that has x-intercepts of $(-4; 0)$ and $(4; 0)$.

   Solution:

   The x-intercepts are the same as the roots of the equation, so:

$$x = -4 \text{ or } x = 4$$

$$x + 4 = 0 \text{ or } x - 4 = 0$$

$$(x+4)(x-4) = 0$$

$$x^2 - 16 = 0$$

3. Determine a quadratic equation of the form $ax^2 + bx + c = 0$, where a, b and c are integers, that has roots $-\frac{1}{2}$ and 3.

   Solution:

$$x = -\frac{1}{2} \text{ or } x = 3$$

$$x + \frac{1}{2} = 0 \text{ or } x - 3 = 0$$

$$(2x+1)(x-3) = 0$$

$$2x^2 - 5x - 3 = 0$$

   We note that this solution gives us integer values as required.

4. Determine the value of k and the other root of the quadratic equation $kx^2 - 7x + 4 = 0$ given that one of the roots is $x = 1$.

   Solution:

   We use the given root to find k:

$$\begin{aligned} k(1)^2 - 7(1) + 4 &= 0 \\ k &= 3 \end{aligned}$$

   So the equation is:

   $3x^2 - 7x + 4 = 0$

   Now we can find the other root:

$$\begin{aligned} 3x^2 - 7x + 4 &= 0 \\ (x-1)(3x-4) &= 0 \\ x = 1 \text{ or } x &= \frac{3}{4} \end{aligned}$$

5. One root of the equation $2x^2 - 3x = p$ is $2\frac{1}{2}$. Find p and the other root.

   Solution:

   We use the given root to find p:

$$\begin{aligned} 2\left(\frac{5}{2}\right)^2 - 3\left(\frac{5}{2}\right) - p &= 0 \\ \frac{25}{2} - \frac{15}{2} &= p \\ p &= 5 \end{aligned}$$

So the equation is:

$2x^2 - 3x - 5 = 0$

Now we can find the other root:

$$\begin{aligned}2x^2 - 3x - 5 &= 0\\(2x-5)(x+1) &= 0\\x &= 2\frac{1}{2} \text{ or } x = -1\end{aligned}$$

Exercise 2 – 6: Mixed exercises

Solve the following quadratic equations by either factorisation, using the quadratic formula or completing the square:

- Always try to factorise first, then use the formula if the trinomial cannot be factorised.
- In a test or examination, only use the method of completing the square when specifically asked.
- Answers can be left in surd or decimal form.

1. $24y^2 + 61y - 8 = 0$

   Solution:

$$\begin{aligned}24y^2 + 61y - 8 &= 0\\(8y-1)(3y+8) &= 0\\y &= \frac{1}{8} \text{ or } y = -\frac{8}{3}\end{aligned}$$

2. $8x^2 + 16x = 42$

   Solution:

$$\begin{aligned}8x^2 + 16x - 42 &= 0\\4x^2 + 8x - 21 &= 0\\(2x-3)(2x+7) &= 0\\x &= \frac{3}{2} \text{ or } x = -\frac{7}{2}\end{aligned}$$

3. $9t^2 = 24t - 12$

Solution:

$$\begin{aligned}
9t^2 - 24t + 12 &= 0\\
3t^2 - 8t + 4 &= 0\\
(3t-2)(t-2) &= 0\\
t &= \frac{2}{3} \text{ or } t = 2
\end{aligned}$$

4. $-5y^2 + 0y + 5 = 0$

Solution:

$$\begin{aligned}
-5y^2 + 5 &= 0\\
-5(y^2 - 1) &= 0\\
y^2 - 1 &= 0\\
(y-1)(y+1) &= 0\\
y &= 1 \text{ or } y = -1
\end{aligned}$$

5. $3m^2 + 12 = 15m$

Solution:

We will solve this equation by completing the square for practice:

$$\begin{aligned}
3m^2 + 12 - 15m &= 0\\
m^2 - 5m + 4 &= 0\\
m^2 - 5m &= -4\\
m^2 - 5m + \frac{25}{4} &= -4 + \frac{25}{4}\\
\left(m - \frac{5}{2}\right)^2 &= \frac{9}{4}\\
m - \frac{5}{2} &= \pm\sqrt{\frac{9}{4}}\\
m &= \frac{5}{2} \pm \frac{3}{2}\\
m = 1 &\text{ or } m = 4
\end{aligned}$$

6. $49y^2 + 0y - 25 = 0$

Solution:

$$\begin{aligned}
49y^2 - 25 &= 0\\
(7y-5)(7y+5) &= 0\\
y &= \frac{5}{7} \text{ or } y = -\frac{5}{7}
\end{aligned}$$

7. $72 = 66w - 12w^2$

Solution:

$$
\begin{aligned}
12w^2 - 66w + 72 &= 0 \\
2w^2 - 11w + 12 &= 0 \\
(2w - 3)(w - 4) &= 0 \\
w &= \frac{3}{2} \text{ or } w = 4
\end{aligned}
$$

8. $-40y^2 + 58y - 12 = 0$

   Solution:

$$
\begin{aligned}
-40y^2 + 58y - 12 &= 0 \\
20y^2 - 29y + 6 &= 0 \\
(5y - 6)(4y - 1) &= 0 \\
y &= \frac{6}{5} \text{ or } y = \frac{1}{4}
\end{aligned}
$$

9. $37n + 72 - 24n^2 = 0$

   Solution:

$$
\begin{aligned}
24n^2 - 37n - 72 &= 0 \\
(3n - 8)(8n + 9) &= 0 \\
n &= \frac{8}{3} \text{ or } n = -\frac{9}{8}
\end{aligned}
$$

10. $6y^2 + 7y - 24 = 0$

    Solution:

$$
\begin{aligned}
6y^2 + 7y - 24 &= 0 \\
(3y + 8)(2y - 3) &= 0 \\
y &= -\frac{8}{3} \text{ or } y = \frac{3}{2}
\end{aligned}
$$

11. $3 = x(2x - 5)$

    Solution:

$$
\begin{aligned}
3 &= 2x^2 - 5x \\
2x^2 - 5x - 3 &= 0 \\
(2x + 1)(x - 3) &= 0 \\
x &= -\frac{1}{2} \text{ or } x = 3
\end{aligned}
$$

12. $-18y^2 - 55y - 25 = 0$

Solution:

$$\begin{aligned}
-18y^2 - 55y - 25 &= 0 \\
18y^2 + 55y + 25 &= 0 \\
y &= \frac{-55 \pm \sqrt{(55)^2 - (4)(18)(25)}}{2(18)} \\
&= \frac{-55 \pm \sqrt{3025 - 1800}}{36} \\
&= \frac{-55 \pm \sqrt{1225}}{36} \\
&= \frac{-55 \pm 35}{36} \\
y = -\frac{90}{36} = -\frac{5}{2} &\text{ or } y = -\frac{20}{36} = -\frac{5}{9}
\end{aligned}$$

13. $-25y^2 + 25y - 4 = 0$

Solution:

$$\begin{aligned}
-25y^2 + 25y - 4 &= 0 \\
25y^2 - 25y + 4 &= 0 \\
y^2 - y &= -\frac{4}{25} \\
y^2 - y + \frac{1}{4} &= -\frac{4}{25} + \frac{1}{4} \\
\left(y - \frac{1}{2}\right)^2 &= \frac{-16 + 25}{100} \\
\left(y - \frac{1}{2}\right)^2 &= \frac{9}{100} \\
y - \frac{1}{2} &= \pm\sqrt{\frac{9}{100}} \\
y - \frac{1}{2} &= \pm\frac{3}{10} \\
y = \frac{1}{2} + \frac{3}{10} = \frac{4}{5} &\text{ or } y = \frac{1}{2} - \frac{3}{10} = \frac{1}{5}
\end{aligned}$$

14. $8(1 - 4g^2) + 24g = 0$

Solution:

$$\begin{aligned}
-32g^2 + 24g + 8 &= 0 \\
32g^2 - 24g - 8 &= 0 \\
4g^2 - 3g - 1 &= 0 \\
(4g + 1)(g - 1) &= 0 \\
g &= -\frac{1}{4} \text{ or } g = 1
\end{aligned}$$

15. $9y^2 - 13y - 10 = 0$

Solution:

$$
\begin{aligned}
9y^2 - 13y - 10 &= 0 \\
y &= \frac{-(-13) \pm \sqrt{(-13)^2 - 4(9)(-10)}}{2(9)} \\
&= \frac{-(-13) \pm \sqrt{169 + 360}}{18} \\
&= \frac{13 \pm \sqrt{529}}{18} \\
&= \frac{13 \pm 23}{18} \\
y = \frac{13 + 23}{18} = 2 &\text{ or } y = \frac{13 - 23}{18} = -\frac{5}{9}
\end{aligned}
$$

16. $(7p - 3)(5p + 1) = 0$

Solution:

$$
\begin{aligned}
(7p - 3)(5p + 1) &= 0 \\
p = \frac{3}{7} &\text{ or } p = -\frac{1}{5}
\end{aligned}
$$

17. $-81y^2 - 99y - 18 = 0$

Solution:

$$
\begin{aligned}
-81y^2 - 99y - 18 &= 0 \\
9y^2 + 11y + 2 &= 0 \\
(9y + 2)(y + 1) &= 0 \\
y = -\frac{2}{9} &\text{ or } y = -1
\end{aligned}
$$

18. $14y^2 - 81y + 81 = 0$

Solution:

$$
\begin{aligned}
14y^2 - 81y + 81 &= 0 \\
y &= \frac{-(-81) \pm \sqrt{(-81)^2 - 4(14)(81)}}{2(14)} \\
&= \frac{81 \pm \sqrt{6561 - 4536}}{28} \\
&= \frac{81 \pm \sqrt{2025}}{28} \\
&= \frac{81 \pm 45}{28} \\
y = \frac{81 + 45}{28} = \frac{9}{2} &\text{ or } y = \frac{81 - 45}{28} = \frac{9}{7}
\end{aligned}
$$

Exercise 2 – 7: From past papers

1. Determine the nature of the roots for each of the following equations:

a) $x^2 + 3x = -2$

b) $x^2 + 9 = 6x$

c) $6y^2 - 6y - 1 = 0$

d) $4t^2 - 19t - 5 = 0$

e) $z^2 = 3$

f) $0 = p^2 + 5p + 8$

g) $x^2 = 36$

h) $4m + m^2 = 1$

i) $11 - 3x + x^2 = 0$

j) $y^2 + \frac{1}{4} = y$

Solution:

a) We write the equation in standard form $ax^2 + bx + c = 0$:

$$x^2 + 3x + 2 = 0$$

Identify the coefficients to substitute into the formula for the discriminant

$$a = 1; \qquad b = 3; \qquad c = 2$$

Write down the formula and substitute values

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (3)^2 - 4(1)(2) \\ &= 9 - 8 \\ &= 1\end{aligned}$$

We know that $1 > 0$ and is a perfect square.
We have calculated that $\Delta > 0$ and is a perfect square, therefore we can conclude that the roots are **real, unequal** and **rational**.

b) We write the equation in standard form $ax^2 + bx + c = 0$:

$$x^2 - 6x + 9 = 0$$

Identify the coefficients to substitute into the formula for the discriminant

$$a = 1; \qquad b = -6; \qquad c = 9$$

Write down the formula and substitute values

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (-6)^2 - 4(1)(9) \\ &= 36 - 36 \\ &= 0\end{aligned}$$

We have calculated that $\Delta = 0$ therefore we can conclude that the roots are **real** and **equal**.

c) We write the equation in standard form $ax^2 + bx + c = 0$:

$$6y^2 - 6y - 1 = 0$$

Identify the coefficients to substitute into the formula for the discriminant

$$a = 6; \qquad b = -6; \qquad c = -1$$

Write down the formula and substitute values

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (-6)^2 - 4(6)(-1) \\ &= 36 + 36 \\ &= 72 \end{aligned}$$

We know that $72 > 0$ and is not a perfect square.

We have calculated that $\Delta > 0$ and is not a perfect square, therefore we can conclude that the roots are **real, unequal** and **irrational**.

d) We write the equation in standard form $ax^2 + bx + c = 0$:

$$4t^2 - 19t - 5 = 0$$

Identify the coefficients to substitute into the formula for the discriminant

$$a = 4; \qquad b = -19; \qquad c = -5$$

Write down the formula and substitute values

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (-19)^2 - 4(4)(-5) \\ &= 361 + 80 \\ &= 441 \end{aligned}$$

We know that $441 > 0$ and is a perfect square.

We have calculated that $\Delta > 0$ and is a perfect square, therefore we can conclude that the roots are **real, unequal** and **rational**.

e) We write the equation in standard form $ax^2 + bx + c = 0$:

$$z^2 - 3 = 0$$

Identify the coefficients to substitute into the formula for the discriminant

$$a = 1; \qquad b = 0; \qquad c = -3$$

Write down the formula and substitute values

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (0)^2 - 4(1)(-3) \\ &= 0 + 12 \\ &= 12 \end{aligned}$$

We know that $12 > 0$ and is not a perfect square.

We have calculated that $\Delta > 0$ and is not a perfect square, therefore we can conclude that the roots are **real, unequal** and **irrational**.

f) We write the equation in standard form $ax^2 + bx + c = 0$:

$$p^2 + 5p + 8 = 0$$

Identify the coefficients to substitute into the formula for the discriminant

$$a = 1; \qquad b = 5; \qquad c = 8$$

Write down the formula and substitute values

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (5)^2 - 4(1)(8) \\ &= 25 - 32 \\ &= -7\end{aligned}$$

We know that $-7 < 0$.

We have calculated that $\Delta < 0$, therefore we can conclude that the roots are **non-real**.

g) We write the equation in standard form $ax^2 + bx + c = 0$:

$$x^2 - 36 = 0$$

Identify the coefficients to substitute into the formula for the discriminant

$$a = 1; \qquad b = 0; \qquad c = -36$$

Write down the formula and substitute values

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (0)^2 - 4(1)(-36) \\ &= 0 + 144 \\ &= 144\end{aligned}$$

We know that $144 > 0$ and is a perfect square.

We have calculated that $\Delta > 0$ and is a perfect square, therefore we can conclude that the roots are **real, unequal** and **rational**.

h) We write the equation in standard form $ax^2 + bx + c = 0$:

$$m^2 + 4m - 1 = 0$$

Identify the coefficients to substitute into the formula for the discriminant

$$a = 1; \qquad b = 4; \qquad c = -1$$

Write down the formula and substitute values

$$
\begin{aligned}
\Delta &= b^2 - 4ac \\
&= (4)^2 - 4(1)(-1) \\
&= 16 + 4 \\
&= 20
\end{aligned}
$$

We know that $20 > 0$ and is not a perfect square.
We have calculated that $\Delta > 0$ and is not a perfect square, therefore we can conclude that the roots are **real, unequal** and **irrational**.

i) We write the equation in standard form $ax^2 + bx + c = 0$:

$$x^2 - 3x + 11 = 0$$

Identify the coefficients to substitute into the formula for the discriminant

$$a = 1; \qquad b = -3; \qquad c = 11$$

Write down the formula and substitute values

$$
\begin{aligned}
\Delta &= b^2 - 4ac \\
&= (-3)^2 - 4(1)(11) \\
&= 9 - 44 \\
&= -35
\end{aligned}
$$

We know that $-35 < 0$.
We have calculated that $\Delta < 0$, therefore we can conclude that the roots are **non-real**.

j) We write the equation in standard form $ax^2 + bx + c = 0$:

$$4y^2 - 4y + 1 = 0$$

Identify the coefficients to substitute into the formula for the discriminant

$$a = 4; \qquad b = -4; \qquad c = 1$$

Write down the formula and substitute values.

$$
\begin{aligned}
\Delta &= b^2 - 4ac \\
&= (-4)^2 - 4(4)(1) \\
&= 16 - 16 \\
&= 0
\end{aligned}
$$

We have calculated that $\Delta = 0$, therefore we can conclude that the roots are **real** and **equal**.

2. Given: $x^2 + bx - 2 + k\left(x^2 + 3x + 2\right) = 0$, $(k \neq -1)$

   a) Show that the discriminant is given by: $\Delta = k^2 + 6bk + b^2 + 8$
   b) If $b = 0$, discuss the nature of the roots of the equation.
   c) If $b = 2$, find the value(s) of k for which the roots are equal.

[IEB, Nov. 2001, HG]

Solution:

a) We write the equation in standard form $ax^2 + bx + c = 0$:

$$(k+1)x^2 + (b+3k)x - 2 + 2k = 0$$

Identify the coefficients to substitute into the formula for the discriminant

$$a = k+1; \qquad b = b+3k; \qquad c = -2+2k$$

Write down the formula and substitute values

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (b+3k)^2 - 4(k+1)(2k-2) \\ &= b^2 + 6bk + 9k^2 - 4(2k^2 - 2) \\ &= b^2 + 6bk + 9k^2 - 8k^2 + 8 \\ &= k^2 + 6bk + b^2 + 8\end{aligned}$$

b) When $b = 0$ the discriminant is:

$$\Delta = k^2 + 8$$

This is positive for all values of k and greater than 0 for all values of k.
For example if $k = 0$ then $\Delta = 8$, if $k = -1$ then $\Delta = 9$ and if $k = 1$ then $\Delta = 9$.
So the roots are real and unequal. We cannot say if the roots are rational or irrational since this depends on the exact value of k.

c) When $b = 2$ the discriminant is:

$$\begin{aligned}\Delta &= k^2 + 6(2)k + (2)^2 + 8 \\ &= k^2 + 12k + 12\end{aligned}$$

We set this equal to 0 since we want to find the values of k that will make the roots equal.

$$\begin{aligned}0 &= k^2 + 12k + 12 \\ k &= \frac{-12 \pm \sqrt{(12)^2 - 4(1)(12)}}{2(1)} \\ &= \frac{-12 \pm \sqrt{144 - 48}}{2} \\ &= \frac{-12 \pm \sqrt{96}}{2} \\ &= \frac{-12 \pm 4\sqrt{6}}{2}\end{aligned}$$

$$k = -6 + 2\sqrt{6} \text{ or } k = -6 - 2\sqrt{6}$$

The roots will be equal if $k = -6 \pm 2\sqrt{6}$.

3. Show that $k^2x^2 + 2 = kx - x^2$ has non-real roots for all real values for k.

   [IEB, Nov. 2002, HG]

   Solution:

$$\begin{aligned} k^2x^2 + x^2 - kx + 2 &= 0 \\ a &= (k^2 + 1) \\ b &= -k \\ c &= 2 \\ \Delta &= b^2 - 4ac \\ &= (-k)^2 - 4(k^2 + 1)(2) \\ &= k^2 - 8k^2 - 8 \\ &= -7k^2 - 8 \\ &= -(7k^2 + 8) \\ \Delta &< 0 \end{aligned}$$

   Therefore the roots are non-real.

4. The equation $x^2 + 12x = 3kx^2 + 2$ has real roots.

   a) Find the greatest value of value k such that $k \in \mathbb{Z}$.

   b) Find one rational value of k for which the above equation has rational roots.

   [IEB, Nov. 2003, HG]

   Solution:

   a)

$$\begin{aligned} x^2 + 12x &= 3kx^2 + 2 \\ 3kx^2 - x^2 - 12x + 2 &= 0 \\ x^2(3k - 1) - 12x + 2 &= 0 \\ a &= 3k - 1 \\ b &= -12 \\ c &= 2 \\ \text{Given } \Delta &\geq 0 \\ \therefore b^2 - 4ac &\geq 0 \\ \therefore (-12)^2 - 4(3k - 1)(2) &\geq 0 \\ 144 - 24k + 8 &\geq 0 \\ 152 - 24k &\geq 0 \\ 152 &\geq 24k \\ \frac{19}{3} &\geq k \\ \therefore k &\leq \frac{19}{3} \\ \therefore k &\leq 6\frac{1}{3} \end{aligned}$$

   Since k needs to be an integer, the greatest value of k is 6.

b) For rational roots we need Δ to be a perfect square.

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= 152 - 24k \\ \text{if } k &= \frac{1}{3} \\ \Delta &= 152 - 24\left(\frac{1}{3}\right) \\ &= 152 - 8 \\ &= 144 \\ &= 12^2 \end{aligned}$$

This is a perfect square. Therefore if $k = \frac{1}{3}$, the roots will be rational.

5. Consider the equation:

$$k = \frac{x^2 - 4}{2x - 5}$$

where $x \neq \frac{5}{2}$.

a) Find a value of k for which the roots are equal.

b) Find an integer k for which the roots of the equation will be rational and unequal.

[IEB, Nov. 2004, HG]

Solution:

a) We first need to write the equation in standard form:

$$\begin{aligned} k(2x - 5) &= x^2 - 4 \\ 2kx - 5k &= x^2 - 4 \\ 0 &= x^2 - 4 - 2kx + 5k \\ 0 &= x^2 - 2kx + 5k - 4 \end{aligned}$$

Next we note that $a = 1$; $b = -2k$; $c = 5k - 4$.
Now we can find the discriminant:

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (-2k)^2 - 4(1)(5k - 4) \\ &= 4k^2 - 20k + 16 \end{aligned}$$

To find the values of k that make the roots equal, we set this equal to 0 and solve for k:

$$\begin{aligned} 0 &= 4k^2 - 20k + 16 \\ &= k^2 - 5k + 4 \\ &= (k - 4)(k - 1) \\ k &= 4 \text{ or } k = 1 \end{aligned}$$

b) The discriminant is:

$$\Delta = 4k^2 - 20k + 16$$

To find a value of k that makes the roots rational and unequal the discriminant must be greater than 0 and a perfect square.
We try setting the discriminant equal to 1:

$$\begin{aligned} \Delta &= 4k^2 - 20k + 16 \\ 1 &= 4k^2 - 20k + 16 \\ 0 &= 4k^2 - 20k + 15 \end{aligned}$$

This does not give an integer value of k so we try 4:

$$\begin{aligned} 4 &= 4k^2 - 20k + 16 \\ 0 &= 4k^2 - 20k + 12 \\ &= k^2 - 5k + 3 \end{aligned}$$

This does not give an integer value of k so we try 9:

$$\begin{aligned} 9 &= 4k^2 - 20k + 16 \\ 0 &= 4k^2 - 20k + 5 \end{aligned}$$

This does not give an integer value of k so we try 16:

$$\begin{aligned} 16 &= 4k^2 - 20k + 16 \\ 0 &= 4k^2 - 20k \\ &= k^2 - 5k \\ k = 0 &\text{ or } k = 5 \end{aligned}$$

So if $k = 0$ or $k = 5$ the roots will be rational and unequal.

6. a) Prove that the roots of the equation $x^2 - (a+b)\,x + ab - p^2 = 0$ are real for all real values of a, b and p.

   b) When will the roots of the equation be equal?

   [IEB, Nov. 2005, HG]

   Solution:

   a) We need to prove that $\Delta \geq 0$.

$$\begin{aligned} \Delta &= (-a-b)^2 - 4(ab - p^2) \\ &= a^2 + 2ab + b^2 - 4ab + 4p^2 \\ &= a^2 - 2ab + b^2 + 4p^2 \\ &= (a-b)^2 + 4p^2 \end{aligned}$$

$\Delta \geq 0$ for all real values of a, b and p. Therefore the roots are real for all real values of a, b and p.

b) The roots are equal when $\Delta = 0$, that is when $a = b$ and $p = 0$.

7. If b and c can take on only the values 1, 2 or 3, determine all pairs $(b;\ c)$ such that $x^2 + bx + c = 0$ has real roots.

   [IEB, Nov. 2005, HG]

   Solution:

   We need to find the values of a, b and c for which $\Delta \geq 0$.

$$\begin{aligned} a &= 1 \\ b &= 1\text{,}2 \text{ or } 3 \\ c &= 1\text{,}2 \text{ or } 3 \end{aligned}$$

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= b^2 - 4(1)c \end{aligned}$$

   Possible pair values of $(b;c)$:
   $(1;1),\ (1;2),\ (1;3),\ (2;1),\ (2;2),\ (2;3),\ (3;1),\ (3;2),\ (3;3).$

   Corresponding values of Δ: $(\Delta < 0),\ (\Delta < 0),\ (\Delta < 0),\ (\Delta = 0),$ $(\Delta < 0),\ (\Delta < 0),\ (\Delta > 0),\ (\Delta > 0),\ (\Delta < 0)$

   $\Delta \geq 0$ (and therefore the roots are real) for $(b;c) = (2;1),\ (3;1),\ (3;2)$

2.7 Quadratic inequalities

Exercise 2 – 8: Solving quadratic inequalities

1. Solve the following inequalities and show each answer on a number line:

   a) $x^2 - x < 12$
   b) $3x^2 > -x + 4$
   c) $y^2 < -y - 2$
   d) $(3 - t)(1 + t) > 0$
   e) $s^2 - 4s > -6$
   f) $0 \geq 7x^2 - x + 8$
   g) $x \geq -4x^2$
   h) $2x^2 + x + 6 \leq 0$
   i) $\frac{x}{x-3} < 2,\ x \neq 3$
   j) $\frac{x^2+4}{x-7} \geq 0,\ x \neq 7$
   k) $\frac{x+2}{x} - 1 \geq 0,\ x \neq 0$

   Solution:

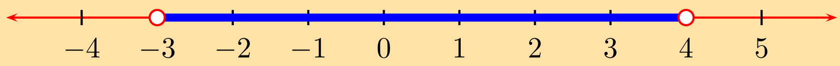
   a)

$$\begin{aligned} x^2 - x - 12 &< 0 \\ (x-4)(x+3) &< 0 \end{aligned}$$

Critical values		$x = -3$		$x = 4$	
$x - 4$	$-$	$-$	$-$	0	$+$
$x + 3$	$-$	0	$+$	$+$	$+$
$f(x) = (x-4)(x+3)$	$+$	0	$-$	0	$+$

From the table we see that $f(x)$ is less than 0 when $-3 < x < 4$

We represent this on a number line:



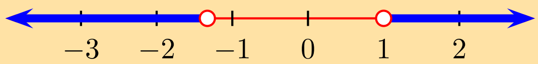
b)

$$3x^2 + x - 4 > 0$$
$$(3x + 4)(x - 1) > 0$$

Critical values		$x = -\frac{4}{3}$		$x = 1$	
$x - 1$	$-$	$-$	$-$	0	$+$
$3x + 4$	$-$	0	$+$	$+$	$+$
$f(x) = (3x + 4)(x - 1)$	$+$	0	$-$	0	$+$

From the table we see that $f(x)$ is greater than 0 when $x < -\frac{4}{3}$ or when $x > 1$

We represent this on a number line:



c)

$$y^2 + y + 2 < 0$$

There are no real solutions.

The graph lies above the x-axis and does not cut the x-axis so the function is never negative. There are no values of y that will solve this inequality.

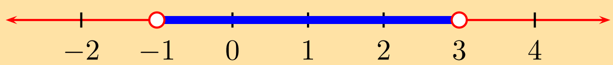
d)

$$(3 - t)(1 + t) > 0$$

Critical values		$t = -1$		$t = 3$	
$3 - t$	$+$	$+$	$+$	0	$-$
$1 + t$	$-$	0	$+$	$+$	$+$
$f(x) = (3 - t)(1 + t)$	$-$	0	$+$	0	$-$

From the table we see that $f(x)$ is greater than 0 when $-1 < t < 3$

We represent this on a number line:



e)

$$s^2 - 4s + 6 > 0$$

Use the quadratic formula to find critical values:

$$\begin{aligned} s &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(6)}}{2(1)} \\ &= \frac{4 \pm \sqrt{16 - 24}}{2} \\ &= \frac{4 \pm \sqrt{-8}}{2} \end{aligned}$$

Therefore there are no real roots and the graph does not cut the x-axis. The graph lies above the x-axis and so this inequality is true for all real values of s.

f) Use the quadratic formula to find critical values:

$$\begin{aligned} x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(7)(8)}}{2(7)} \\ &= \frac{1 \pm \sqrt{1 - 224}}{14} \\ &= \frac{1 \pm \sqrt{-223}}{14} \end{aligned}$$

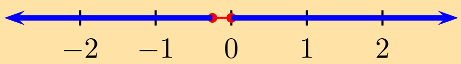
Therefore there are no real roots and the graph does not cut the x-axis. The graph lies above the x-axis and so this inequality is true for all real values of x.

g)

$$\begin{aligned} x &\geq -4x^2 \\ 4x^2 + x &\geq 0 \\ x(4x + 1) &\geq 0 \end{aligned}$$

Critical values		$x = -\frac{1}{4}$		$x = 0$	
x	$-$	$-$	$-$	0	$+$
$4x + 1$	$-$	0	$+$	$+$	$+$
$f(x) = x(4x + 1)$	$+$	0	$-$	0	$+$

From the table we see that $f(x)$ is greater than 0 when $x \leq -\frac{1}{4}$ or $x \geq 0$. We can represent this on a number line:



h)

$$2x^2 + x + 6 \leq 0$$

There are no real roots and the graph does not cut the x-axis. The graph lies above the x-axis and so this inequality is never true.

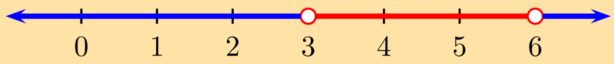
i) We first solve the equation:

$$\begin{aligned} \frac{x}{x-3} &< 2 \\ \frac{x}{x-3} - 2 &< 0 \\ \frac{x - 2(x-3)}{x-3} &< 0 \\ \frac{x - 2x + 6}{x-3} &< 0 \\ \frac{-x + 6}{x-3} &< 0 \\ \frac{-(x-6)}{x-3} &< 0 \\ \frac{x-6}{x-3} &> 0 \\ x &= 6 \end{aligned}$$

Critical values		$x = 3$		$x = 6$	
$x - 3$	$-$	undef	$+$	$+$	$+$
$x - 6$	$-$	$-$	$-$	0	$+$
$f(x) = x - 6$	$+$	undef	$-$	0	$+$

From the table we see that $f(x) > 0$ when $x < 3$ or $x > 6$ with $x \neq 3$.

We can represent this on a number line:



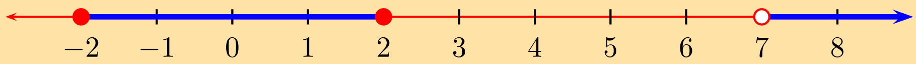
j) We first simplify the fraction:

$$\frac{(x+2)(x-2)}{x-7} \geq 0$$

Critical values		$x = -2$		$x = 2$		$x = 7$	
$x + 2$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x - 2$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x - 7$	$-$	$-$	$-$	$-$	$-$	undef	$+$
$f(x) = \frac{(x+2)(x-2)}{x-7}$	$-$	0	$+$	0	$-$	undef	$+$

From the table we see that $f(x)$ is greater than 0 when $-2 \leq x \leq 2$ and $x > 7$ with $x \neq 7$.

We can represent this on a number line:

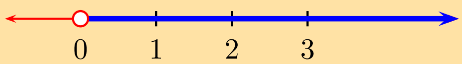


k) We first simplify the equation:

$$\frac{x+2}{x} - 1 \geq 0$$
$$\frac{x+2-x}{x} \geq 0$$
$$\frac{2}{x} \geq 0$$
$$\text{Therefore } x > 0$$

The solution is $x > 0$ with $x \neq 0$.

We can represent this on a number line:



2. Draw a sketch of the following inequalities and solve for x:

a) $2x^2 - 18 > 0$

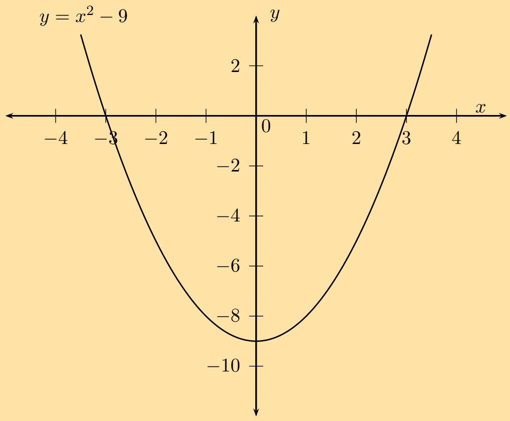
b) $5 - x^2 \leq 0$

c) $x^2 < 0$

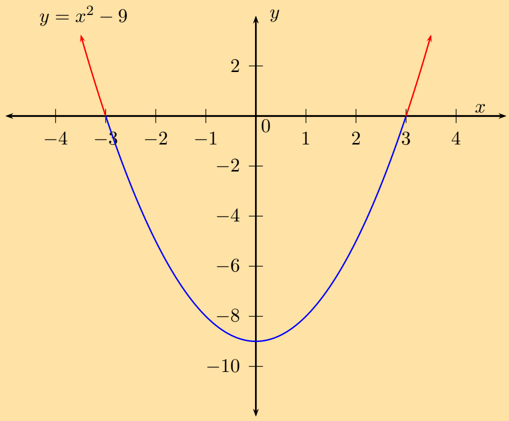
d) $0 \geq 6x^2$

Solution:

a) We draw the parabola:



And now we note that we are looking for the parts of the parabola that are greater than 0. These parts are marked in red below:

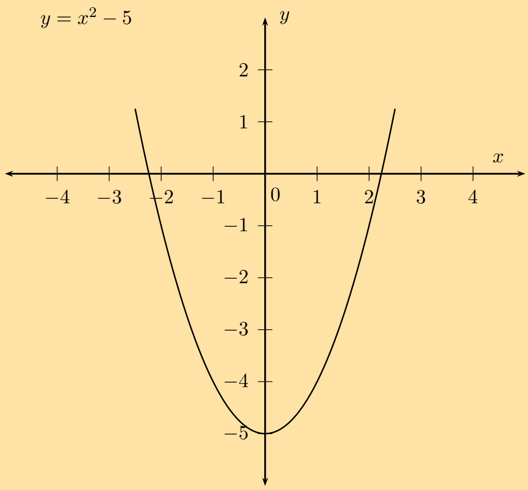


This gives a final answer of $x < -3$ or $x > 3$.

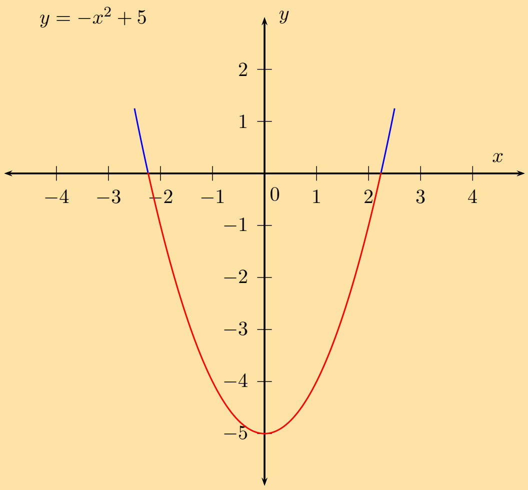
b) We first simplify the equation:

$$\begin{aligned} 5 - x^2 &\leq 0 \\ -(x^2 - 5) &\leq 0 \\ x^2 - 5 &\geq 0 \\ \text{Therefore } x &> 0 \end{aligned}$$

We draw the parabola:

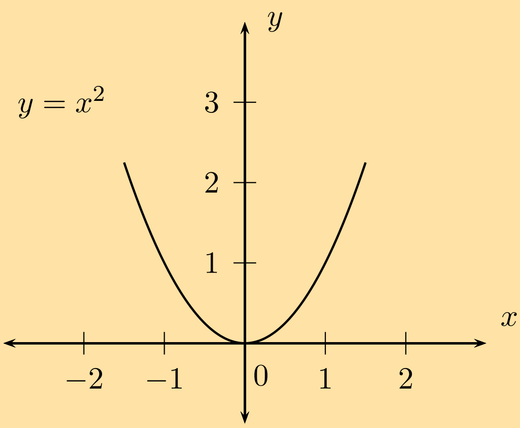


And now we note that we are looking for the parts of the parabola that are less than or equal to 0. These parts are marked in red below:



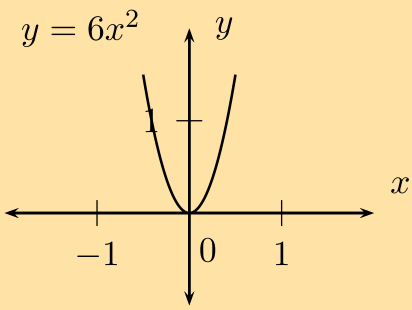
This gives a final answer of $-\sqrt{5} \leq x \leq \sqrt{5}$.

c) We draw the parabola:



We are looking for the parts of the parabola that are less than 0. The parabola does not go below the x-axis and so is therefore never negative. There is no solution to this inequality.

d) We draw the parabola:



We see that the parabola never touches the x-axis and is always positive. Therefore the inequality is true for all real values of x.

Exercise 2 – 9: Solving simultaneous equations

1. Solve the following systems of equations algebraically. Leave your answer in surd form, where appropriate.

   a) $y + x = 5$
   $y - x^2 + 3x - 5 = 0$

   b) $y = 6 - 5x + x^2$
   $y - x + 1 = 0$

   c) $y = \frac{2x + 2}{4}$
   $y - 2x^2 + 3x + 5 = 0$

   d) $a - 2b - 3 = 0$; $a - 3b^2 + 4 = 0$

   e) $x^2 - y + 2 = 3x$
   $4x = 8 + y$

   f) $2y + x^2 + 25 = 7x$
   $3x = 6y + 96$

Solution:

a) We make y the subject of each equation:

$$\begin{aligned} y + x &= 5 \\ y &= 5 - x \end{aligned}$$

$$\begin{aligned} y - x^2 + 3x - 5 &= 0 \\ y &= x^2 - 3x + 5 \end{aligned}$$

Next equate the two equations and solve for x:

$$\begin{aligned} 5 - x &= x^2 - 3x + 5 \\ 0 &= x^2 - 2x \\ 0 &= x(x - 2) \end{aligned}$$

Therefore $x = 0$ or $x = 2$

Now we substitute the values for x back into the first equation to calculate the corresponding y-values

If $x = 0$:

$$\begin{aligned} y &= 5 - 0 \\ \therefore y &= 5 \end{aligned}$$

This gives the point $(0; 5)$.

If $x = 2$:

$$\begin{aligned} y &= 5 - 2 \\ &= 3 \end{aligned}$$

This gives the point $(2; 3)$.

The solution is $x = 0$ and $y = 5$ or $x = 2$ and $y = 3$. These are the coordinate pairs for the points of intersection.

b) We make y the subject of each equation:

$$y = 6 - 5x + x^2$$

$$\begin{aligned} y - x + 1 &= 0 \\ y &= x - 1 \end{aligned}$$

Next equate the two equations and solve for x:

$$\begin{aligned} 6 - 5x + x^2 &= x - 1 \\ x^2 - 6x + 7 &= 0 \\ x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(7)}}{2(1)} \\ x &= \frac{6 \pm \sqrt{36 - 28}}{2} \\ x &= \frac{6 \pm \sqrt{8}}{2} \\ x &= \frac{6 \pm 2\sqrt{2}}{2} \\ x &= 3 \pm \sqrt{2} \end{aligned}$$

Now we substitute the values for x back into the second equation to calculate the corresponding y-values

If $x = 3 - \sqrt{2}$:

$$\begin{aligned} y &= 3 - \sqrt{2} - 1 \\ \therefore y &= 2 - \sqrt{2} \end{aligned}$$

If $x = 3 + \sqrt{2}$:

$$\begin{aligned} y &= 3 + \sqrt{2} - 1 \\ &= 2 + \sqrt{2} \end{aligned}$$

The solution is $x = 3 \pm \sqrt{2}$ and $y = 2 \pm \sqrt{2}$

c) We make y the subject of each equation:

$$y = \frac{2x + 2}{4}$$

$$\begin{aligned} y - 2x^2 + 3x + 5 &= 0 \\ y &= 2x^2 - 3x - 5 \end{aligned}$$

Next equate the two equations and solve for x:

$$\begin{aligned}\frac{2x+2}{4} &= 2x^2 - 3x - 5 \\ 2x + 2 &= 8x^2 - 12x - 20 \\ 0 &= 8x^2 - 14x - 22 \\ 0 &= 4x^2 - 7x - 11 \\ 0 &= (4x - 11)(x + 1) \\ \text{Therefore } x = -1 \text{ or } x &= \frac{11}{4}\end{aligned}$$

Now we substitute the values for x back into the first equation to calculate the corresponding y-values

If $x = -1$:

$$\begin{aligned}y &= \frac{2(-1)+2}{4} \\ y &= 0\end{aligned}$$

This gives the point $(-1; 0)$.

If $x = \frac{1}{4}$:

$$\begin{aligned}y &= \frac{2(\frac{1}{4})+2}{4} \\ y &= \frac{\frac{5}{2}}{4} \\ &= \frac{5}{8}\end{aligned}$$

This gives the point $(\frac{1}{4}; \frac{5}{8})$.

The solution is $x = -1$ and $y = 0$ or $x = \frac{1}{4}$ and $y = \frac{5}{8}$. These are the coordinate pairs for the points of intersection.

d) We make a the subject of each equation:

$$a = 2b + 3$$

$$\begin{aligned}a - 3b^2 + 4 &= 0 \\ a &= 3b^2 - 4\end{aligned}$$

Next equate the two equations and solve for b:

$$\begin{aligned}2b + 3 &= 3b^2 - 4 \\ 0 &= 3b^2 - 2b - 7 \\ 0 &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-7)}}{2(3)} \\ b &= \frac{2 \pm \sqrt{4 + 84}}{6} \\ b &= \frac{2 \pm \sqrt{88}}{6}\end{aligned}$$

Now we substitute the values for b back into the first equation to calculate the corresponding a-values

If $b = \frac{2+\sqrt{88}}{6}$:

$$
\begin{aligned}
a &= 2\left(\frac{2+\sqrt{88}}{6}\right) + 3 \\
&= \frac{2+\sqrt{88}}{3} + 3 \\
&= \frac{11+\sqrt{88}}{3}
\end{aligned}
$$

If $b = \frac{2-\sqrt{88}}{6}$:

$$
\begin{aligned}
a &= 2\left(\frac{2-\sqrt{88}}{6}\right) + 3 \\
&= \frac{2-\sqrt{88}}{3} + 3 \\
&= \frac{11-\sqrt{88}}{3}
\end{aligned}
$$

The solution is $b = \frac{2 \pm \sqrt{88}}{6}$ and $a = \frac{11 \pm \sqrt{88}}{6}$

e) We make y the subject of each equation:

$$
\begin{aligned}
x^2 + y + 2 &= 3x \\
y &= -x^2 + 3x - 2
\end{aligned}
$$

$$
y = 4x - 8
$$

Next equate the two equations and solve for x:

$$
\begin{aligned}
4x - 8 &= -x^2 + 3x - 2 \\
x^2 + x - 6 &= 0 \\
(x+3)(x-2) &= 0 \\
\text{Therefore } x &= -3 \text{ or } x = 2
\end{aligned}
$$

Now we substitute the values for x back into the second equation to calculate the corresponding y-values

If $x = -3$:

$$
\begin{aligned}
y &= 4(-3) - 8 \\
y &= -20
\end{aligned}
$$

This gives the point $(-3; -20)$.

If $x = 2$:

$$
\begin{aligned}
y &= 4(2) - 8 \\
&= 0
\end{aligned}
$$

This gives the point $(2; 0)$.

The solution is $x = -3$ and $y = -20$ or $x = 2$ and $y = 0$. These are the coordinate pairs for the points of intersection.

f) We make y the subject of each equation:

$$
\begin{aligned}
2y + x^2 + 25 &= 7x \\
2y + x^2 + 25 &= 7x - x^2 + 25 \\
y &= \frac{-x^2 + 7x + 25}{2}
\end{aligned}
$$

$$
\begin{aligned}
2y &= x - 32 \\
y &= \frac{x - 32}{2}
\end{aligned}
$$

Next equate the two equations and solve for x:

$$
\begin{aligned}
\frac{x-32}{2} &= \frac{-x^2 + 7x + 25}{2} \\
x - 32 &= -x^2 + 7x + 25 \\
x^2 - 6x - 57 &= 0 \\
x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-57)}}{2(1)} \\
&= \frac{6 \pm \sqrt{36 + 228}}{2} \\
&= \frac{6 \pm \sqrt{264}}{2}
\end{aligned}
$$

Now we substitute the values for x back into the second equation to calculate the corresponding y-values

If $x = \frac{6 + \sqrt{264}}{2}$:

$$
\begin{aligned}
2y &= \frac{6 + \sqrt{264}}{2} - 32 \\
y &= \frac{70 + \sqrt{264}}{4} \\
&= \frac{70 + \sqrt{264}}{4}
\end{aligned}
$$

If $x = \frac{6 - \sqrt{264}}{2}$:

$$
\begin{aligned}
2y &= \frac{6 - \sqrt{264}}{2} - 32 \\
y &= \frac{70 - \sqrt{264}}{4} \\
&= \frac{70 - \sqrt{264}}{4}
\end{aligned}
$$

The solution is $x = \frac{6 \pm \sqrt{264}}{2}$ and $y = \frac{70 \pm \sqrt{264}}{2}$

2. Solve the following systems of equations graphically. Check your solutions by also solving algebraically.

a) $x^2 - 1 - y = 0$
$y + x - 5 = 0$

b) $x + y - 10 = 0$
$x^2 - 2 - y = 0$

c) $xy = 12$
$7 = x + y$

d) $6 - 4x - y = 0$
$12 - 2x^2 - y = 0$

Solution:

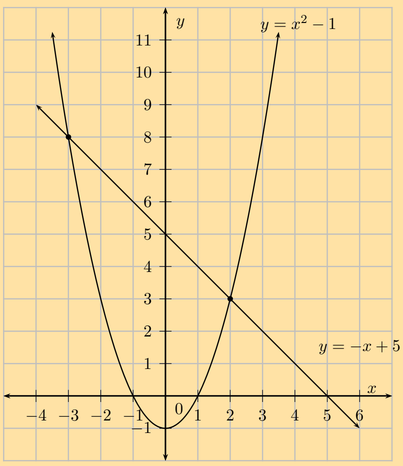
a) Make y the subject of both equations
For the first equation we have:

$$\begin{aligned} x^2 - 1 - y &= 0 \\ y &= x^2 - 1 \end{aligned}$$

and for the second equation:

$$\begin{aligned} y + x - 5 &= 0 \\ y &= -x + 5 \end{aligned}$$

Draw the straight line graph and parabola on the same system of axes:



From the diagram we see that the graphs intersect at $(-3; 8)$ and $(2; 3)$.
We can solve algebraically to check. Doing this we find the same solution.
The solutions to the system of simultaneous equations are $(-3; 8)$ and $(2; 3)$.

b) Make y the subject of both equations
For the first equation we have:

$$\begin{aligned} x + y - 10 &= 0 \\ y &= -x + 10 \end{aligned}$$

and for the second equation:

$$x^2 - 2 - y = 0$$
$$y = x^2 - 2$$

Draw the straight line graph and parabola on the same system of axes:

From the diagram we see that the graphs intersect at $(-4;14)$ and $(3;7)$.

We can solve algebraically to check. Doing this we find the same solution.

The solutions to the system of simultaneous equations are $(-4;14)$ and $(3;7)$.

c) Make y the subject of both equations

For the first equation we have:

$$xy = 12$$
$$y = \frac{12}{x}$$

and for the second equation:

$$7 = x + y$$
$$y = 7 - x$$

Draw the two graphs on the same system of axes:

We also note that this sytem of equations has the followig restrictions: $x \neq 0$ and $y \neq 0$

From the diagram we see that the graphs intersect at $(3;4)$ and $(4;3)$.

We can solve algebraically to check. Doing this we find the same solution.

The solutions to the system of simultaneous equations are $(3;4)$ and $(4;3)$.

d) Make y the subject of both equations

For the first equation we have:

$$6 - 4x - y = 0$$
$$y = -4x + 6$$

and for the second equation:

$$12 - 2x^2 - y = 0$$
$$y = -2x^2 + 12$$

Draw the two graphs on the same system of axes:

From the diagram we see that the graphs intersect at approximately $(4;-10)$ and $(-2;14)$.

We can solve algebraically to check (and to get a more accurate answer). Doing this gives:

$$-4x + 6 = -2x^2 + 12$$
$$2x - 3 = x^2 - 6$$
$$x^2 - 2x - 9 = 0$$
$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-9)}}{2(1)}$$
$$x = \frac{2 \pm \sqrt{(4 + 36}}{2}$$
$$x = \frac{2 \pm \sqrt{40}}{2}$$

Solving for y gives:

$$y = -4\left(\frac{2 \pm \sqrt{40}}{2}\right) + 6$$
$$= -2(2 \pm \sqrt{40}) + 6$$
$$= -4 \pm 2\sqrt{40} + 6$$
$$= 2 \pm 2\sqrt{40}$$

2.9 Word problems

Problem solving strategy

Exercise 2 – 10:

1. Mr. Tsilatsila builds a fence around his rectangular vegetable garden of 8 m^2. If the length is twice the breadth, determine the dimensions of Mr. Tsilatsila's vegetable garden.

   Solution:

   We let the length be l and the breadth be b. The area of the garden is 8 m^2 and is given by $A = l \times b$.

   Since the length is twice the breadth we can express the length in terms of the breadth: $l = 2b$. We now have the following:

$$A = 8 = l \times b$$
$$= 2b(b)$$
$$= 2b^2$$
$$b^2 = 4$$
$$b = \pm 2$$

   Therefore the breadth is 2 m and the length is twice this, 4 m. Note that the breadth cannot be a negative number and so we do not consider this solution.

2. Kevin has played a few games of ten-pin bowling. In the third game, Kevin scored 80 more than in the second game. In the first game Kevin scored 110 less than the third game. His total score for the first two games was 208. If he wants an average score of 146, what must he score on the fourth game?

Solution:

We let the score for the first game be a, the score for the second game be b, the score for the third game be c and the score for the fourth game be d.

Now we note the following:

$$\begin{aligned} c &= 80 + b \\ a &= c - 110 \\ a + b &= 208 \\ \frac{a+b+c+d}{4} &= 146 \end{aligned}$$

We make the c the subject of the first two equations:

$$\begin{aligned} c &= 80 + b \\ c &= a + 110 \end{aligned}$$

And then we use $a = 208 - b$ to solve for b:

$$\begin{aligned} 80 + b &= 208 - b + 110 \\ 2b &= 208 + 110 - 80 \\ 2b &= 238 \\ b &= 119 \end{aligned}$$

Now we can find a:

$$\begin{aligned} a + b &= 208 \\ a + 119 &= 208 \\ a &= 89 \end{aligned}$$

And we can find c:

$$\begin{aligned} c &= 80 + b \\ c &= 80 + 208 \\ c &= 288 \end{aligned}$$

Finally we can find d:

$$\begin{aligned} \frac{a+b+c+d}{4} &= 164 \\ 496 + d &= 656 \\ d &= 187 \end{aligned}$$

Kevin must score 187 in the fourth game.

3. When an object is dropped or thrown downward, the distance, d, that it falls in time, t, is described by the following equation:

   $s = 5t^2 + v_0 t$

   In this equation, v_0 is the initial velocity, in m·s^{-1}. Distance is measured in meters and time is measured in seconds. Use the equation to find how long it takes a tennis ball to reach the ground if it is thrown downward from a hot-air balloon that is 500 m high. The tennis ball is thrown at an initial velocity of 5 m·s^{-1}.

   Solution:

   We are given the distance that the ball falls and the initial velocity so we can solve for t:

$$\begin{aligned} s &= 5t^2 + v_0 t \\ 500 &= 5t^2 + 5t \\ t^2 + t - 100 &= 0 \\ t &= \frac{-(1) \pm \sqrt{(1)^2 - 4(1)(100)}}{2(1)} \\ &= \frac{1 \pm \sqrt{1 + 400}}{2} \\ &= \frac{1 \pm \sqrt{401}}{2} \end{aligned}$$

   Since time cannot be negative the only solution is $t = \frac{1 + \sqrt{401}}{2} \approx 10{,}5$ s.

4. The table below lists the times that Sheila takes to walk the given distances.

time (minutes)	5	10	15	20	25	30
distance (km)	1	2	3	4	5	6

   Plot the points.

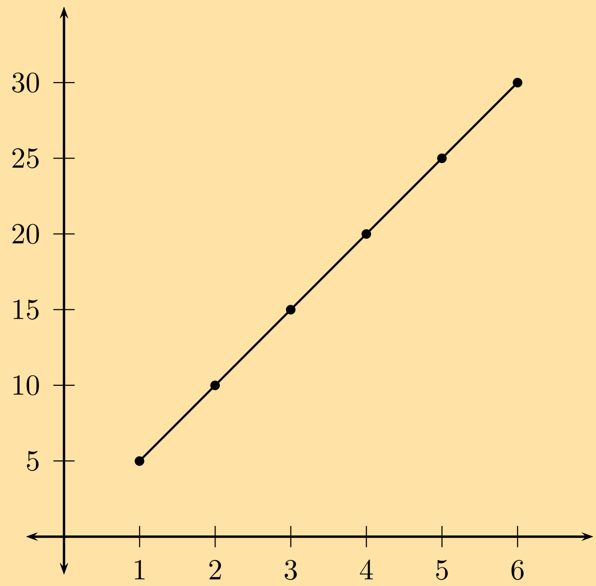
   Find the equation that describes the relationship between time and distance. Then use the equation to answer the following questions:

   a) How long will it take Sheila to walk 21 km?

   b) How far will Sheila walk in 7 minutes?

   If Sheila were to walk half as fast as she is currently walking, what would the graph of her distances and times look like?

   Solution:



The equation is $t = 5d$.

It will take Sheila $t = 5(21) = 105$ minutes to walk 21 km.

Sheila will walk $d = \frac{7}{5} = 1{,}4$ kilometres in 7 minutes.

The gradient of the graph will be twice the gradient of the first graph. The graph will be steeper and lie closer to the y-axis.

5. The power P (in watts) supplied to a circuit by a 12 volt battery is given by the formula $P = 12I - 0{,}5I^2$ where I is the current in amperes.

   a) Since both power and current must be greater than 0, find the limits of the current that can be drawn by the circuit.
   b) Draw a graph of $P = 12I - 0{,}5I^2$ and use your answer to the first question to define the extent of the graph.
   c) What is the maximum current that can be drawn?
   d) From your graph, read off how much power is supplied to the circuit when the current is 10 A. Use the equation to confirm your answer.
   e) At what value of current will the power supplied be a maximum?

Solution:

a) We set the equation equal to 0 to find the limits:

$$\begin{aligned} 0 &= 12I - 0{,}5I^2 \\ 0{,}5I^2 - 12I &= 0 \\ I(0{,}5I - 12) &= 0 \\ I = 0 \text{ or } I &= 24 \end{aligned}$$

b)

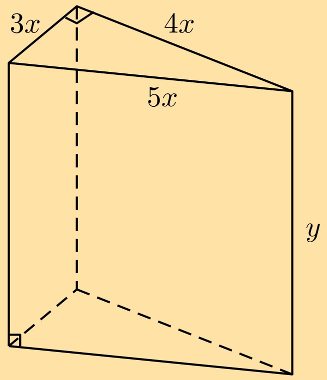
c) The maximum current that can be drawn is 24 A.

d) The power is 70 W.

$$\begin{aligned} P &= 12(10) - \frac{1}{2}(10)^2 \\ &= 120 - 50 \\ &= 70 \end{aligned}$$

e) 12 A. This is the turning point of the parabola.

6. A wooden block is made as shown in the diagram. The ends are right-angled triangles having sides $3x$, $4x$ and $5x$. The length of the block is y. The total surface area of the block is 3600 cm^2.



Show that

$y = \frac{300-x^2}{x}$

Solution:

$$3600 = 3xy + 5xy + 4xy + 2\left(\frac{1}{2}(3x)(4x)\right)$$
$$3600 = 12xy + 12x^2$$
$$3600 - 12x^2 = 12xy$$
$$\frac{3600 - 12x^2}{12x} = y$$
$$\therefore y = \frac{300 - x^2}{x}$$

2.10 Summary

Exercise 2 – 11: End of chapter exercises

1. Solve: $x^2 - x - 1 = 0$. Give your answer correct to two decimal places.

Solution:

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} \\
&= \frac{1 \pm \sqrt{1+4}}{2} \\
&= \frac{1 \pm \sqrt{5}}{2}
\end{aligned}
$$

$x = 1{,}62$ or $x = -0{,}62$

2. Solve: $16\left(x+1\right) = x^2\left(x+1\right)$

   Solution:

$$
\begin{aligned}
16(x+1) &= x^2(x+1) \\
16(x+1) - x^2(x+1) &= 0 \\
(16 - x^2)(x+1) &= 0 \\
x^2 &= 16 \text{ or } x = -1 \\
x &= \pm 4 \text{ or } x = -1
\end{aligned}
$$

3. Solve: $y^2 + 3 + \frac{12}{y^2+3} = 7$

   Solution:

$$
\begin{aligned}
\text{Let } y^2 + 3 &= k \\
\text{Restriction: } y^2 + 3 &\neq 0 \\
\text{Therefore } k &\neq 0 \\
\therefore k + \frac{12}{k} &= 7 \\
k^2 + 12 &= 7k \\
k^2 - 7k + 12 &= 0 \\
(k-3)(k-4) &= 0 \\
k &= 3 \text{ or } k = 4 \\
\therefore y^2 + 3 &= 3 \text{ or } y^2 + 3 = 4 \\
\therefore y^2 &= 0 \text{ or } y^2 = 1 \\
\therefore y &= 0 \text{ or } y = \pm 1
\end{aligned}
$$

4. Solve for x: $2x^4 - 5x^2 - 12 = 0$

   Solution:

$$
\begin{aligned}
2x^4 - 5x^2 - 12 &= 0 \\
(x^2 - 4)(2x^2 + 3) &= 0 \\
x^2 = 4 \text{ or } x^2 &= -\frac{3}{2} \quad \text{(no real solution)} \\
\therefore x &= \pm 2
\end{aligned}
$$

5. Solve for x:

a) $x(x-9)+14=0$
b) $x^2-x=3$ (Show your answer correct to **one** decimal place.)
c) $x+2=\frac{6}{x}$ (Show your answer correct to **two** decimal places.)
d) $\frac{1}{x+1}+\frac{2x}{x-1}=1$

Solution:

a)

$$\begin{aligned} x(x-9)+14&=0\\ x^2-9x+14&=0\\ (x-7)(x-2)&=0\\ x&=7 \text{ or } x=2 \end{aligned}$$

b)

$$\begin{aligned} x^2-x&=3\\ x^2-x-3&=0\\ x&=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-3)}}{2(1)}\\ &=\frac{1\pm\sqrt{1+12}}{2}\\ &=\frac{1\pm\sqrt{13}}{2}\\ x&=2{,}3 \text{ or } x=-1{,}3 \end{aligned}$$

c)

$$\begin{aligned} x+2&=\frac{6}{x}\\ x^2+2x-6&=0\\ x&=\frac{-2\pm\sqrt{2^2-4(1)(-6)}}{2(1)}\\ &=\frac{-2\pm\sqrt{4+24}}{2}\\ &=\frac{-2\pm\sqrt{28}}{2}\\ &=\frac{-2\pm 2\sqrt{7}}{2}\\ &=-1\pm\sqrt{7}\\ x&=1{,}65 \text{ or } x=-3{,}65 \end{aligned}$$

d)

$$\begin{aligned} \frac{1}{x+1}+\frac{2x}{x-1}&=1\\ (x-1)+2x(x+1)&=(x+1)(x-1)\\ x-1+2x^2+2x&=x^2-1\\ x^2+3x&=0\\ x(x+3)&=0\\ x&=0 \text{ or } x=-3 \end{aligned}$$

6. Solve for x in terms of p by completing the square: $x^2 - px - 4 = 0$

Solution:

$$\begin{aligned} x^2 - px - 4 &= 0 \\ x^2 - px + \frac{p^2}{4} &= 4 + \frac{p^2}{4} \\ \left(x - \frac{p}{2}\right)^2 &= \frac{16 + p^2}{4} \\ x - \frac{p}{2} &= \sqrt{\frac{16 + p^2}{4}} \\ x &= \frac{\sqrt{16 + p^2}}{2} - \frac{p}{2} \\ &= \frac{\sqrt{16 + p^2} - 2}{2} \end{aligned}$$

7. The equation $ax^2 + bx + c = 0$ has roots $x = \frac{2}{3}$ and $x = -4$. Find one set of possible values for a, b and c.

Solution:

$$\begin{aligned} (3x - 2)(x + 4) &= 0 \\ 3x^2 + 12x - 2x - 8 &= 0 \\ 3x^2 + 10x - 8 &= 0 \\ \therefore a &= 3 \\ b &= 10 \\ c &= -8 \end{aligned}$$

8. The two roots of the equation $4x^2 + px - 9 = 0$ differ by 5. Calculate the value of p.

Solution:

$$4x^2 + px - 9 = 0$$

Using the quadratic formula:

$$\begin{aligned} x &= \frac{-p \pm \sqrt{p^2 - 4(4)(-9)}}{8} \\ &= \pm\frac{\sqrt{144 + p^2}}{8} \end{aligned}$$

$$\therefore x_1 = \frac{-p+\sqrt{144+p^2}}{8} \text{ and } x_2 = \frac{-p-\sqrt{144+p^2}}{8}$$

$$x_1 - x_2 = 5$$

$$\therefore \frac{-p+\sqrt{144+p^2}}{8} - \frac{-p-\sqrt{144+p^2}}{8} = 5$$

$$-p+\sqrt{144+p^2}+p+\sqrt{144+p^2} = 40$$

$$\sqrt{p^2+144} = 20$$

$$p^2+144 = 400$$

$$p^2 = 256$$

$$\therefore p = \pm 16$$

9. An equation of the form $x^2+bx+c=0$ is written on the board. Saskia and Sven copy it down incorrectly. Saskia has a mistake in the constant term and obtains the solutions -4 and 2. Sven has a mistake in the coefficient of x and obtains the solutions 1 and -15. Determine the correct equation that was on the board.

   Solution:

   Saskia:

   $$(x+4)(x-2) = 0$$

   $$x^2+2x-8 = 0$$

   $$\therefore a = 1 \text{ and } b = 2$$

   Sven:

   $$(x-1)(x+15) = 0$$

   $$x^2+14x-15 = 0$$

   $$\therefore c = -15$$

   Correct equation:

   $$x^2+2x-15 = 0$$

   $$(x+5)(x-3) = 0$$

   $$\therefore \text{correct roots are } x = -5 \text{ and } x = 3$$

10. For which values of b will the expression $\frac{b^2-5b+6}{b+2}$ be:

    a) undefined?

    b) equal to zero?

    Solution:

    a) For the expression to be undefined the denominator must be equal to 0. This means that $b+2=0$ and therefore $b=-2$

    b) We simplify the fraction:

    $$\frac{b^2-5b+6}{b+2} = 0$$

    $$\frac{(b-2)(b-3)}{b+2} = 0$$

Therefore $b = 2$ or $b = 3$ will make the expression equal to 0.

Note that we cannot have $b = -2$ as that will make the denominator 0 and the whole expression will be undefined.

11. Given $\frac{(x^2 - 6)(2x + 1)}{x + 2} = 0$ solve for x if:

    a) x is a real number.
    b) x is a rational number.
    c) x is an irrational number.
    d) x is an integer.

    Solution:

    We first note that the restriction is: $x \neq -2$.

    Next we note that for the fraction to equal 0 the numerator must equal to 0. This gives:

$$
\begin{aligned}
(x^2 - 6)(2x + 1) &= 0 \\
x &= \frac{-1}{2} \text{ or } x = \pm\sqrt{6}
\end{aligned}
$$

    Now we need to decide which of these answers meet the criteria given:

    a) All three solutions are real.
    b) $x = \frac{-1}{2}$
    c) $x = \pm\sqrt{6}$
    d) There are no integer solutions.

12. Given $\frac{(x - 6)^{\frac{1}{2}}}{x^2 + 3}$, for which value(s) of x will the expression be:

    a) equal to zero?
    b) defined?

    Solution:

    a) The expression will be equal to 0 when the numerator is equal to 0. This gives $x = 6$.
    b) The expression is undefined when the denominator is equal to 0. So the expression will be defined for all values of x except where $x = \pm\sqrt{3}$.

13. Solve for a if $\frac{\sqrt{8 - 2a}}{a - 3} \geq 0$.

    Solution:

    We first note that the restriction is: $a \neq 3$.

    Next we note that for the fraction to equal 0 the numerator must equal to 0. This gives:

$$
\begin{aligned}
\sqrt{8 - 2a} &= 0 \\
8 - 2a &= 0 \\
a &= 4
\end{aligned}
$$

    Now we draw up a table of signs to find where the function is positive:

Critical values		$a = 3$		$a = 4$	
$a - 3$	$-$	undef	$+$	$+$	$+$
$a - 4$	$-$	$-$	$-$	0	$+$
	$-$	undef	$-$	0	$+$

From this we see that $a \geq 4$ is the solution.

14. Abdoul stumbled across the following formula to solve the quadratic equation $ax^2 + bx + c = 0$ in a foreign textbook.

$$x = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}$$

a) Use this formula to solve the equation: $2x^2 + x - 3 = 0$.

b) Solve the equation again, using factorisation, to see if the formula works for this equation.

c) Trying to derive this formula to prove that it always works, Abdoul got stuck along the way. His attempt is shown below:

$$ax^2 + bx + c = 0$$

$$a + \frac{b}{x} + \frac{c}{x^2} = 0 \qquad \text{Divided by } x^2 \text{ where } x \neq 0$$

$$\frac{c}{x^2} + \frac{b}{x} + a = 0 \qquad \text{Rearranged}$$

$$\frac{1}{x^2} + \frac{b}{cx} + \frac{a}{c} = 0 \qquad \text{Divided by } c \text{ where } c \neq 0$$

$$\frac{1}{x^2} + \frac{b}{cx} = -\frac{a}{c} \qquad \text{Subtracted } \tfrac{a}{c} \text{ from both sides}$$

$$\therefore \frac{1}{x^2} + \frac{b}{cx} + \ldots \qquad \text{Got stuck}$$

Complete his derivation.

Solution:

a)

$$\begin{aligned} x =& \frac{2c}{-b \pm \sqrt{b^2 - 4ac}} \\ =& \frac{2(-3)}{-(1) \pm \sqrt{1^2 - 4(2)(-3)}} \\ =& \frac{-6}{-1 \pm \sqrt{25}} \\ =& \frac{-6}{-1 \pm 5} \end{aligned}$$

$$x = \frac{-6}{4} = -\frac{3}{2} \text{ or } x = \frac{-6}{-6} = 1$$

b)

$$2x^2 + x - 3 = 0$$
$$(2x + 3)(x - 1) = 0$$
$$x = -\frac{3}{2} \text{ or } x = 1$$

c)

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
a + \frac{b}{x} + \frac{c}{x^2} &= 0 \\
\frac{c}{x^2} + \frac{b}{x} + a &= 0 \\
\frac{1}{x^2} + \frac{b}{cx} + \frac{a}{c} &= 0 \\
\frac{1}{x^2} + \frac{b}{cx} &= -\frac{a}{c} \\
\frac{1}{x^2} + \frac{b}{cx} + \frac{b^2}{4c^2} &= -\frac{a}{c} + \frac{b^2}{4c^2} \\
\left(\frac{1}{x} + \frac{b}{2c}\right)^2 &= \frac{-4ac + b^2}{4c^2} = \frac{b^2 - 4ac}{4c^2} \\
\frac{1}{x} + \frac{b}{2c} &= \pm\sqrt{\frac{b^2 - 4ac}{4c^2}} \\
\frac{1}{x} &= -\frac{b}{2c} \pm \sqrt{\frac{b^2 - 4ac}{4c^2}} \\
\frac{1}{x} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2c} \\
\therefore x &= \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}
\end{aligned}$$

15. Solve for x:

a) $\frac{4}{x-3} \leq 1$

b) $\frac{4}{(x-3)^2} < 1$

c) $\frac{2x-2}{x-3} > 3$

d) $\frac{-3}{(x-3)(x+1)} < 0$

e) $(2x-3)^2 < 4$

f) $2x \leq \frac{15-x}{x}$

g) $\frac{x^2+3}{3x-2} \leq 0$

h) $x - 2 \geq \frac{3}{x}$

i) $\frac{x^2+3x-4}{5+x^4} \leq 0$

j) $\frac{x-2}{3-x} \geq 1$

Solution:

a) $x < 3$ or $x \geq 7$

b) $x < 1$ or $x > 5$

c) $3 < x < 7$

d) $x < -1$ or $x > 3$

e) $0{,}5 < x < 2{,}5$

f) $x \leq -3$ or $0 < x \leq \frac{5}{2}$

g) $x < \frac{2}{3}$

h) $-1 \leq x < 0$ or $x \geq 3$

i) $-4 \leq x \leq 1$

j) $2\frac{1}{2} \leq x < 3$

16. Solve the following systems of equations algebraically. Leave your answer in surd form, where appropriate.

a) $y - 2x = 0$
$y - x^2 - 2x + 3 = 0$

b) $a - 3b = 0$
$a - b^2 + 4 = 0$

c) $y - x^2 - 5x = 0$
$10 = y - 2x$

d) $p = 2p^2 + q - 3$
$p - 3q = 1$

e) $a - b^2 = 0$
$a - 3b + 1 = 0$

f) $a - 2b + 1 = 0$
$a - 2b^2 - 12b + 4 = 0$

g) $y + 4x - 19 = 0$
$8y + 5x^2 - 101 = 0$

h) $a + 4b - 18 = 0$
$2a + 5b^2 - 57 = 0$

Solution:

a) We make y the subject of each equation:

$$\begin{aligned} y - 2x &= 0 \\ y &= 2x \end{aligned}$$

$$\begin{aligned} y - x^2 - 2x + 3 &= 0 \\ y &= x^2 + 2x - 3 \end{aligned}$$

Next equate the two equations and solve for x:

$$\begin{aligned} 2x &= x^2 + 2x - 3 \\ 0 &= x^2 - 3 \\ x^2 &= 3 \\ x &= \pm\sqrt{3} \end{aligned}$$

Now we substitute the values for x back into the first equation to calculate the corresponding y-values
If $x = \sqrt{3}$:

$$y = 2\sqrt{3}$$

This gives the point $(\sqrt{3}; 2\sqrt{3})$.
If $x = -\sqrt{3}$:

$$y = -2(\sqrt{3})$$

This gives the point $(-\sqrt{3}; -2\sqrt{3})$.
The solution is $x = \pm\sqrt{3}$ and $y = \pm 2\sqrt{3}$. These are the coordinate pairs for the points of intersection.

b) We make a the subject of each equation:

$$\begin{aligned} a - 3b &= 0 \\ a &= 3b \end{aligned}$$

$$\begin{aligned} a - b^2 + 4 &= 0 \\ a &= b^2 - 4 \end{aligned}$$

Next equate the two equations and solve for b:

$$
\begin{aligned}
3b &= b^2 - 4 \\
0 &= b^2 - 3b - 4 \\
(b-4)(b+1) &= 0 \\
b &= 4 \text{ or } b = -1
\end{aligned}
$$

Now we substitute the values for b back into the first equation to calculate the corresponding a-values

If $b = 4$:

$$a = 12$$

If $b = -1$:

$$a = -3$$

The solution is $a = -3$ and $b = -1$ or $a = 12$ and $b = 4$. These are the coordinate pairs for the points of intersection.

c) We make y the subject of each equation:

$$
\begin{aligned}
y - x^2 - 5x &= 0 \\
y &= x^2 + 5x
\end{aligned}
$$

$$
\begin{aligned}
10 &= y - 2x \\
y &= 2x + 10
\end{aligned}
$$

Next equate the two equations and solve for x:

$$
\begin{aligned}
x^2 + 5x &= 2x + 10 \\
x^2 + 3x - 10 &= 0 \\
(x+5)(x-2) &= 0 \\
x &= -5 \text{ or } x = 2
\end{aligned}
$$

Now we substitute the values for x back into the second equation to calculate the corresponding y-values

If $x = -5$:

$$
\begin{aligned}
y &= 2(-5) + 10 \\
&= 0
\end{aligned}
$$

If $x = 2$:

$$
\begin{aligned}
y &= 2(2) + 10 \\
&= 14
\end{aligned}
$$

The solution is $x = -5$ and $y = 0$ or $x = 2$ and $y = 14$. These are the coordinate pairs for the points of intersection.

d) We make q the subject of each equation:

$$\begin{aligned} p &= 2p^2 + q - 3 \\ q &= -2p^2 + p + 3 \end{aligned}$$

$$\begin{aligned} p - 3q &= 1 \\ q &= \frac{p-1}{3} \end{aligned}$$

Next equate the two equations and solve for p:

$$\begin{aligned} -2p^2 + p + 3 &= \frac{p-1}{3} \\ -6p^2 + 3p + 9 &= p - 1 \\ 6p^2 - 4p - 10 &= 0 \\ 3p^2 - 2p - 5 &= 0 \\ (3p-5)(p+1) &= 0 \\ p &= \frac{5}{3} \text{ or } p = -1 \end{aligned}$$

Now we substitute the values for p back into the second equation to calculate the corresponding q-values
If $p = \frac{5}{3}$:

$$\begin{aligned} q &= \frac{\frac{5}{3} - 1}{3} \\ &= \frac{2}{9} \end{aligned}$$

If $p = -1$:

$$\begin{aligned} q &= \frac{(-1) - 1}{3} \\ &= \frac{-2}{3} \end{aligned}$$

The solution is $p = \frac{5}{3}$ and $q = \frac{2}{9}$ or $p = -1$ and $q = \frac{-2}{3}$. These are the coordinate pairs for the points of intersection.

e) We make a the subject of each equation:

$$\begin{aligned} a - b^2 &= 0 \\ a &= b^2 \end{aligned}$$

$$\begin{aligned} a - 3b + 1 &= 0 \\ a &= 3b - 1 \end{aligned}$$

Next equate the two equations and solve for b:

$$
\begin{aligned}
b^2 &= 3b - 1 \\
b^2 - 3b + 1 &= 0 \\
b &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(1)}}{2(1)} \\
&= \frac{3 \pm \sqrt{5}}{2}
\end{aligned}
$$

Now we substitute the values for b back into the second equation to calculate the corresponding a-values

If $b = \frac{3+\sqrt{5}}{2}$:

$$
\begin{aligned}
a &= 3\left(\frac{3+\sqrt{5}}{2}\right) - 1 \\
&= \frac{7 + 3\sqrt{5}}{2}
\end{aligned}
$$

If $b = \frac{3-\sqrt{5}}{2}$:

$$
\begin{aligned}
a &= 3\left(\frac{3-\sqrt{5}}{2}\right) - 1 \\
&= \frac{7 - 3\sqrt{5}}{2}
\end{aligned}
$$

The solution is $b = \frac{3\pm\sqrt{5}}{2}$ and $a = \frac{7\pm3\sqrt{5}}{2}$. These are the coordinate pairs for the points of intersection.

f) We make a the subject of each equation:

$$
\begin{aligned}
a - 2b + 1 &= 0 \\
a &= 2b - 1
\end{aligned}
$$

$$
\begin{aligned}
a - 2b^2 - 12b + 4 &= 0 \\
a &= 2b^2 + 12b - 4
\end{aligned}
$$

Next equate the two equations and solve for b:

$$
\begin{aligned}
2b - 1 &= 2b^2 + 12b - 4 \\
2b^2 + 10b - 5 &= 0 \\
b &= \frac{-(10) \pm \sqrt{(10)^2 - 4(2)(-5)}}{2(2)} \\
&= \frac{-10 \pm \sqrt{140}}{4}
\end{aligned}
$$

Now we substitute the values for b back into the first equation to calculate the corresponding a-values

If $b = \frac{-10+\sqrt{140}}{4}$:

$$\begin{aligned} a &= 2\left(\frac{-10+\sqrt{140}}{4}\right) - 1 \\ &= \frac{-24+2\sqrt{140}}{4} \\ &= \frac{-12+\sqrt{140}}{2} \end{aligned}$$

If $b = \frac{-10-\sqrt{140}}{4}$:

$$\begin{aligned} a &= 2\left(\frac{-10-\sqrt{140}}{4}\right) - 1 \\ &= \frac{-24-2\sqrt{140}}{4} \\ &= \frac{-12-\sqrt{140}}{2} \end{aligned}$$

The solution is $b = \frac{-10\pm\sqrt{140}}{4}$ and $a = \frac{-12\pm\sqrt{140}}{2}$. These are the coordinate pairs for the points of intersection.

g) We make y the subject of each equation:

$$\begin{aligned} y + 4x - 19 &= 0 \\ y &= -4x + 19 \end{aligned}$$

$$\begin{aligned} 8y + 5x^2 - 101 &= 0 \\ 8y &= -5x^2 + 101 \\ y &= \frac{-5x^2+101}{8} \end{aligned}$$

Next equate the two equations and solve for x:

$$\begin{aligned} -4x + 19 &= \frac{-5x^2+101}{8} \\ -32x + 152 &= -5x^2 + 101 \\ 5x^2 - 32x + 51 &= 0 \\ x &= \frac{-(-32) \pm \sqrt{(-32)^2 - 4(5)(51)}}{2(5)} \\ &= \frac{32 \pm \sqrt{1024 - 1020}}{10} \\ &= \frac{32 \pm \sqrt{4}}{10} \\ x &= 3{,}4 \text{ or } x = 3 \end{aligned}$$

Now we substitute the values for x back into the first equation to calculate the corresponding y-values

If $x = 3{,}4$:

$$\begin{aligned} y &= -4(3{,}4) + 19 \\ &= 5{,}4 \end{aligned}$$

If $x = 3$:

$$\begin{aligned} y &= -4(3) + 19 \\ &= 5 \end{aligned}$$

The solution is $x = 3{,}4$ and $y = 5{,}4$ or $x = 3$ and $y = 5$. These are the coordinate pairs for the points of intersection.

h) We make a the subject of each equation:

$$\begin{aligned} a + 4b - 18 &= 0 \\ a &= -4b + 18 \end{aligned}$$

$$\begin{aligned} 2a + 5b^2 - 57 &= 0 \\ 2a &= -5b^2 + 57 \\ a &= \frac{-5b^2 + 57}{2} \end{aligned}$$

Next equate the two equations and solve for a:

$$\begin{aligned} -4b + 18 &= \frac{-5b^2 + 57}{2} \\ -8x + 36 &= -5b^2 + 57 \\ 5b^2 - 8x - 21 &= 0 \\ b &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(5)(-21)}}{2(5)} \\ &= \frac{8 \pm \sqrt{64 + 420}}{10} \\ &= \frac{8 \pm \sqrt{484}}{10} \\ b = 3 &\text{ or } b = -1{,}4 \end{aligned}$$

Now we substitute the values for b back into the first equation to calculate the corresponding a-values
If $b = 3$:

$$\begin{aligned} a &= -4(3) + 18 \\ &= 6 \end{aligned}$$

If $b = -1{,}4$:

$$\begin{aligned} y &= -4(-1{,}4) + 18 \\ &= 23{,}6 \end{aligned}$$

The solution is $b = -1{,}4$ and $a = 23{,}6$ or $b = 3$ and $a = 6$. These are the coordinate pairs for the points of intersection.

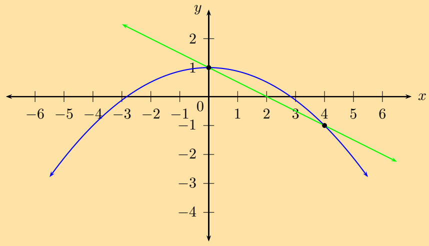
17. Solve the following systems of equations graphically:

a) $2y + x - 2 = 0$
$8y + x^2 - 8 = 0$

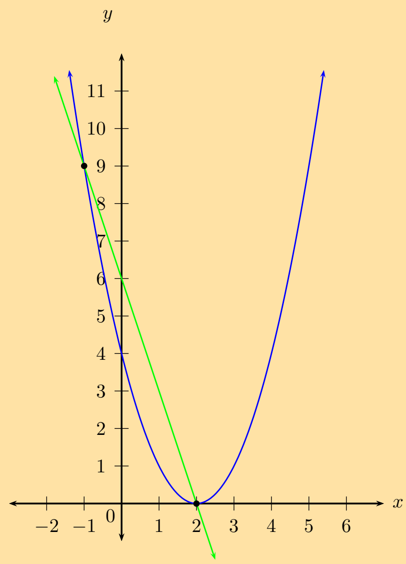
b) $y + 3x - 6 = 0$
$y = x^2 + 4 - 4x$

Solution:

a)



b)



18. A stone is thrown vertically upwards and its height (in metres) above the ground at time t (in seconds) is given by:

    $h(t) = 35 - 5t^2 + 30t$

    Find its initial height above the ground.

    Solution:

    The initial height occurs when $t = 0$. Substituting this in we get:

$$\begin{aligned} h(t) &= 35 - 5t^2 + 30t \\ h(0) &= 35 - 5(0)^2 + 30(0) \\ &= 35 \text{ m} \end{aligned}$$

19. After doing some research, a transport company has determined that the rate at which petrol is consumed by one of its large carriers, travelling at an average speed of x km per hour, is given by:

$$P(x) = \frac{55}{2x} + \frac{x}{200} \quad \text{litres per kilometre}$$

Assume that the petrol costs R 4,00 per litre and the driver earns R 18,00 per hour of travel time. Now deduce that the total cost, C, in Rands, for a 2000 km trip is given by:

$$C(x) = \frac{256\ 000}{x} + 40x$$

Solution:

$$\begin{aligned} C(x) &= 4 \times 2000 \times \left(\frac{55}{2x} + \frac{x}{200}\right) + 18 \times \frac{2000}{x} \\ &= \frac{220\ 000}{x} + 40x + \frac{36\ 000}{x} \\ &= \frac{256\ 000}{x} + 40x \end{aligned}$$

20. Solve the following quadratic equations by either factorisation, completing the square or by using the quadratic formula:

- Always try to factorise first, then use the formula if the trinomial cannot be factorised.
- Solve some of the equations by completing the square.

a) $-4y^2 - 41y - 45 = 0$

b) $16x^2 + 20x = 36$

c) $42p^2 + 104p + 64 = 0$

d) $21y + 3 = 54y^2$

e) $36y^2 + 44y + 8 = 0$

f) $12y^2 - 14 = 22y$

g) $16y^2 + 0y - 81 = 0$

h) $3y^2 + 10y - 48 = 0$

i) $63 - 5y^2 = 26y$

j) $2x^2 - 30 = 2$

k) $2y^2 = 98$

Solution:

a)

$$\begin{aligned} -4y^2 - 41y - 45 &= 0 \\ 4y^2 + 41y + 45 &= 0 \\ (4y + 5)(y + 9) &= 0 \\ y &= -\frac{5}{4} \text{ or } y = -9 \end{aligned}$$

b)

$$\begin{aligned} 16x^2 + 20x - 36 &= 0 \\ 4x^2 + 5x - 9 &= 0 \\ (4x + 9)(x - 1) &= 0 \\ x &= -\frac{9}{4} \text{ or } x = 1 \end{aligned}$$

c)

$$\begin{aligned} 42p^2 + 104p + 64 &= 0 \\ 21p^2 + 52p + 32 &= 0 \\ (7p + 8)(3p + 4) &= 0 \\ p &= -\frac{8}{7} \text{ or } p = -\frac{4}{3} \end{aligned}$$

d)

$$\begin{aligned} -54y^2 + 21y + 3 &= 0 \\ 18y^2 - 7y - 1 &= 0 \\ (9y + 1)(2y - 1) &= 0 \\ y = -\frac{1}{9} &\text{ or } y = \frac{1}{2} \end{aligned}$$

e)

$$\begin{aligned} 36y^2 + 44y + 8 &= 0 \\ 9y^2 + 11y + 2 &= 0 \\ (9y + 2)(y + 1) &= 0 \\ y = -\frac{2}{9} &\text{ or } y = -1 \end{aligned}$$

f)

$$\begin{aligned} 12y^2 - 22y - 14 &= 0 \\ 6y^2 - 11y - 7 &= 0 \\ (3y - 7)(2y + 1) &= 0 \\ y = \frac{7}{3} &\text{ or } y = -\frac{1}{2} \end{aligned}$$

g)

$$\begin{aligned} 16y^2 + 0y - 81 &= 0 \\ (4y - 9)(4y + 9) &= 0 \\ y = \frac{9}{4} &\text{ or } y = -\frac{9}{4} \end{aligned}$$

h)

$$\begin{aligned} 3y^2 + 10y - 48 &= 0 \\ (3y - 8)(y + 6) &= 0 \\ y = \frac{8}{3} &\text{ or } y = -6 \end{aligned}$$

i)

$$\begin{aligned} -5y^2 - 26y + 63 &= 0 \\ 5y^2 + 26y - 63 &= 0 \\ (5y - 9)(y + 7) &= 0 \\ y = \frac{9}{5} &\text{ or } y = -7 \end{aligned}$$

j)

$$\begin{aligned} 2x^2 - 30 &= 2 \\ 2x^2 - 32 &= 0 \\ x^2 - 16 &= 0 \\ (x - 4)(x + 4) &= 0 \\ x &= \pm 4 \end{aligned}$$

k)

$$\begin{aligned} 2y^2 - 98 &= 0 \\ y^2 - 49 &= 0 \\ (y-7)(y+7) &= 0 \\ y &= \pm 7 \end{aligned}$$

21. One root of the equation $9y^2 + 32 = ky$ is 8. Determine the value of k and the other root.

Solution:

We first write the equation in standard form: $9y^2 - ky + 32 = 0$. Now we can solve for k and the other root.

$$\begin{aligned} 9(8)^2 - 8k + 32 &= 0 \\ 576 - 8k + 32 &= 0 \\ 8k &= 608 \\ k &= 76 \end{aligned}$$

The roots are:

$$\begin{aligned} 9y^2 - 76y + 32 &= 0 \\ (9y-4)(y-8) &= 0 \\ y &= \frac{4}{9} \text{ or } y = 8 \end{aligned}$$

22. a) Solve for x in $x^2 - x = 6$.
    b) Hence, solve for y in $(y^2 - y)^2 - (y^2 - y) - 6 = 0$.

Solution:

a)

$$\begin{aligned} x^2 - x - 6 &= 0 \\ (x-3)(x+2) &= 0 \\ x &= 3 \text{ or } x = -2 \end{aligned}$$

b) We note that $y^2 - y$ is a common part and so we let $x = y^2 - y$. Now we note that this gives the same equation as above ($x^2 - x = 6$). We have solved this and so we can use the solution to solve for y.

$$\begin{aligned} y^2 - y &= 3 \\ y^2 - y - 3 &= 0 \\ y &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-3)}}{2(1)} \\ &= \frac{1 \pm \sqrt{13}}{2} \end{aligned}$$

$$y^2 - y = -2$$
$$y^2 - y + 2 = 0$$
$$y = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(2)}}{2(1)}$$
$$= \frac{1 \pm \sqrt{-7}}{2}$$

The second value of x leads to no real solution for y.

23. Solve for x: $x = \sqrt{8-x} + 2$

Solution:

$$x = \sqrt{8-x} + 2$$
$$x - 2 = \sqrt{8-x}$$
$$(x-2)^2 = 8 - x$$
$$x^2 - 4x + 4 = 8 - x$$
$$x^2 - 3x - 4 = 8 - x$$
$$(x-4)(x+1) = 0$$
$$x = 4 \text{ or } x = -1$$

24. a) Solve for y in $-4y^2 + 8y - 3 = 0$.
    b) Hence, solve for p in $4(p-3)^2 - 8(p-3) + 3 = 0$.

Solution:

a)

$$4y^2 - 8y + 3 = 0$$
$$(2y-3)(2y-1) = 0$$
$$y = \frac{3}{2} \text{ or } y = \frac{1}{2}$$

b) We note that $(p-3)$ is a common part and so we let $y = p - 3$. Now we note that this gives the same equation as above ($4y^2 - 8y - 3 = 0$). We have solved this and so we can use the solution to solve for p.

$$p - 3 = \frac{3}{2}$$
$$p = \frac{3}{2} + 3$$
$$p = \frac{9}{2}$$

$$p - 3 = \frac{1}{2}$$
$$p = \frac{1}{2} + 3$$
$$p = \frac{7}{2}$$

25. Solve for x: $2(x+3)^{\frac{1}{2}} = 9$

Solution:

$$\begin{aligned} 2(x+3)^{\frac{1}{2}} &= 9 \\ (x+3)^{\frac{1}{2}} &= \frac{9}{2} \\ (x+3) &= \frac{81}{4} \\ x &= \frac{81}{4} - 3 \\ &= \frac{69}{4} \end{aligned}$$

26. a) Without solving the equation $x + \frac{1}{x} = 3$, determine the value of $x^2 + \frac{1}{x^2}$.

b) Now solve $x + \frac{1}{x} = 3$ and use the result to assess the answer obtained in the question above.

Solution:

7

27. Solve for y: $5(y-1)^2 - 5 = 19 - (y-1)^2$

Solution:

This has a common part of $(y-1)^2$ so we replace that with k:

$$\begin{aligned} 5k - 5 &= 19 - k \\ 6k &= 24 \\ k &= 4 \end{aligned}$$

Now we can use this result to solve for y:

$$\begin{aligned} (y-1)^2 &= 4 \\ y^2 - 2y + 2 &= 4 \\ y^2 - 2y - 2 &= 0 \\ y &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)} \\ &= \frac{2 \pm \sqrt{12}}{2} \end{aligned}$$

28. Solve for t: $2t(t - \frac{3}{2}) = \frac{3}{2t^2 - 3t} + 2$

Solution:

$t = \frac{1}{2}, t = 1$ or $t = \frac{3 \pm \sqrt{33}}{4}$

CHAPTER 3

Number patterns

- Discuss terminology.
- Emphasize the relationship between linear functions (general term) and linear sequences.
- Do not use the formula for arithmetic sequences.
- Emphasize the relationship between quadratic functions (general term) and quadratic sequences.
- Key activity in mathematical description of a pattern: finding the relationship between the number of the term and the value of the term.

3.1 Revision

Exercise 3 – 1: Linear sequences

1. Write down the next three terms in each of the following sequences: $45; 29; 13; -3; \ldots$

   Solution:

   $-19; -35; -51$

2. The general term is given for each sequence below. Calculate the missing terms.

   a) $-4; -9; -14; \ldots; -24$
   $T_n = 1 - 5n$

   b) $6; \ldots; 24; \ldots; 42$
   $T_n = 9n - 3$

   Solution:

   a)

$$\begin{aligned} T_4 &= 1 - 5(4) \\ &= 1 - 20 \\ &= -19 \end{aligned}$$

   b)

$$\begin{aligned} T_2 &= 9(2) - 3 \\ &= 18 - 3 \\ &= 15 \end{aligned}$$

$$\begin{aligned} T_4 &= 9(4) - 3 \\ &= 36 - 3 \\ &= 33 \end{aligned}$$

3. Find the general formula for the following sequences and then find T_{10}, T_{15} and T_{30}:

a) $13; 16; 19; 22; \ldots$

b) $18; 24; 30; 36; \ldots$

c) $-10; -15; -20; -25; \ldots$

Solution:

a)

$$\begin{aligned} d &= T_2 - T_1 \\ &= 16 - 13 \\ &= 3 \\ \therefore T_n &= 10 + 3n \end{aligned}$$

$$\begin{aligned} T_n &= 10 + 3n \\ \therefore T_{10} &= 10 + 3(10) \\ &= 40 \\ \therefore T_{15} &= 10 + 3(15) \\ &= 55 \\ \therefore T_{30} &= 10 + 3(30) \\ &= 100 \end{aligned}$$

b)

$$\begin{aligned} d &= T_2 - T_1 \\ &= 24 - 18 \\ &= 6 \\ \therefore T_n &= 12 + 6n \end{aligned}$$

$$\begin{aligned} T_n &= 12 + 6n \\ \therefore T_{10} &= 12 + 6(10) \\ &= 72 \\ \therefore T_{15} &= 12 + 6(15) \\ &= 102 \\ \therefore T_{30} &= 12 + 6(30) \\ &= 192 \end{aligned}$$

c)

$$\begin{aligned} d &= T_2 - T_1 \\ &= -15 - (-10) \\ &= -5 \\ \therefore T_n &= -5 - 5n \end{aligned}$$

$$
\begin{aligned}
T_n &= -5 - 5n \\
\therefore T_{10} &= -5 - 5(10) \\
&= -55 \\
\therefore T_{15} &= -5 - 5(15) \\
&= -80 \\
\therefore T_{30} &= -5 - 5(30) \\
&= -155
\end{aligned}
$$

4. The seating in a classroom is arranged so that the first row has 20 desks, the second row has 22 desks, the third row has 24 desks and so on. Calculate how many desks are in the ninth row.

   Solution:

$$
\begin{aligned}
d &= T_2 - T_1 \\
&= 22 - 20 \\
&= 2 \\
\therefore T_n &= 18 + 2n
\end{aligned}
$$

$$
\begin{aligned}
T_n &= 18 + 2n \\
\therefore T_9 &= 18 + 2(9) \\
&= 18 + 18 \\
&= 36
\end{aligned}
$$

5. a) Complete the following:

$$
\begin{aligned}
13 + 31 &= \ldots \\
24 + 42 &= \ldots \\
38 + 83 &= \ldots
\end{aligned}
$$

   b) Look at the numbers on the left-hand side, what do you notice about the unit digit and the tens-digit?

   c) Investigate the pattern by trying other examples of 2-digit numbers.

   d) Make a conjecture about the pattern that you notice.

   e) Prove this conjecture.

   Solution:

   a)

$$
\begin{aligned}
13 + 31 &= 44 \\
24 + 42 &= 66 \\
38 + 83 &= 121
\end{aligned}
$$

   b) The unit digit and tens-digit have swapped position.

c)

$$45 + 54 = 99$$
$$71 + 17 = 88$$

d) The sum of the two numbers will always be 11 times the sum of the two digits.

e) Let the first number be $a + 10b$ and let the second number be $b + 10a$:

$$\begin{aligned} \text{Number } 1 : &= a + 10b \\ \text{Number } 2 : &= b + 10a \\ \text{Number } 1 + 2 : &= a + b + 10a + 10b \\ &= 11a + 11b \\ &= 11(a + b) \end{aligned}$$

3.2 Quadratic sequences

Exercise 3 – 2: Quadratic sequences

1. Determine the second difference between the terms for the following sequences:

a) $5; 20; 45; 80; \ldots$

b) $6; 11; 18; 27; \ldots$

c) $1; 4; 9; 16; \ldots$

d) $3; 0; -5; -12; \ldots$

e) $1; 3; 7; 13; \ldots$

f) $0; -6; -16; -30; \ldots$

g) $-1; 2; 9; 20; \ldots$

h) $1; -3; -9; -17; \ldots$

i) $3a+1; 12a+1; 27a+1; 48a+1 \ldots$

j) $2; 10; 24; 44; \ldots$

k) $t - 2; 4t - 1; 9t; 16t + 1; \ldots$

Solution:

a)

$$\begin{aligned} \text{First differences: } &= 15; 25; 35 \\ \text{Second difference: } &= 10 \end{aligned}$$

b)

$$\begin{aligned} \text{First differences: } &= 5; 7; 9 \\ \text{Second difference: } &= 2 \end{aligned}$$

c)

$$\begin{aligned} \text{First differences: } &= 3; 5; 7 \\ \text{Second difference: } &= 2 \end{aligned}$$

d)

$$\text{First differences: } = -3; -5; -7$$
$$\text{Second difference: } = -2$$

e)

$$\text{First differences: } = 2; 4; 6$$
$$\text{Second difference: } = 2$$

f)

$$\text{First differences: } = -6; -10; -14$$
$$\text{Second difference: } = -4$$

g)

$$\text{First differences: } = 3; 7; 11$$
$$\text{Second difference: } = 4$$

h)

$$\text{First differences: } = -4; -6; -8$$
$$\text{Second difference: } = -2$$

i)

$$\text{First differences: } = 9a; 15a; 21a$$
$$\text{Second difference: } = 6a$$

j)

$$\text{First differences: } = 8; 14; 20$$
$$\text{Second difference: } = 6$$

k)

$$\text{First differences: } = 3t+1; 5t+1; 7t+1$$
$$\text{Second difference: } = 2t$$

2. Complete the sequence by filling in the missing term:

a) $11; 21; 35; \ldots; 75$

b) $20; \ldots; 42; 56; 72$

c) $\ldots; 37; 65; 101$

d) $3; \ldots; -13; -27; -45$

e) $24; 35; 48; \ldots; 80$

f) $\ldots; 11; 26; 47$

Solution:

a)

	$n=1$		$n=2$		$n=3$		$n=4$
T_n	11		21		35		**53**
1st difference		10		14		18	
2nd difference			4		4		

b)

	$n=1$		$n=2$		$n=3$		$n=4$
T_n	20		**30**		42		56
1st difference		10		12		14	
2nd difference			2		2		

c)

	$n=1$		$n=2$		$n=3$		$n=4$
T_n	**17**		37		65		101
1st difference		20		28		36	
2nd difference			8		8		

d)

	$n=1$		$n=2$		$n=3$		$n=4$
T_n	3		**-3**		-13		-27
1st difference		-6		-10		-14	
2nd difference			-4		-4		

e)

	$n=1$		$n=2$		$n=3$		$n=4$
T_n	24		35		48		**63**
1st difference		11		13		15	
2nd difference			2		2		

f)

	$n=1$		$n=2$		$n=3$		$n=4$
T_n	**2**		11		26		47
1st difference		9		15		21	
2nd difference			6		6		

3. Use the general term to generate the first four terms in each sequence:

a) $T_n = n^2 + 3n - 1$
b) $T_n = -n^2 - 5$
c) $T_n = 3n^2 - 2n$
d) $T_n = -2n^2 + n + 1$

Solution:

a) $3; 9; 17; 27$
b) $-6; -9; -14; -21$
c) $1; 8; 21; 40$
d) $0; -5; -14; -27$

Exercise 3 – 3: Quadratic sequences

1. Calculate the common second difference for each of the following quadratic sequences:

a) $3; 6; 10; 15; 21; \ldots$

b) $4; 9; 16; 25; 36; \ldots$

c) $7; 17; 31; 49; 71; \ldots$

d) $2; 10; 26; 50; 82; \ldots$

e) $31; 30; 27; 22; 15; \ldots$

Solution:

a)

$$\begin{aligned}\text{First differences: } &= 3; 4; 5; 6\\ \text{Second difference: } &= 1\end{aligned}$$

b)

$$\begin{aligned}\text{First differences: } &= 5; 7; 9; 11\\ \text{Second difference: } &= 2\end{aligned}$$

c)

$$\begin{aligned}\text{First differences: } &= 10; 14; 18; 22\\ \text{Second difference: } &= 4\end{aligned}$$

d)

$$\begin{aligned}\text{First differences: } &= 8; 16; 24; 32\\ \text{Second difference: } &= 8\end{aligned}$$

e)

$$\begin{aligned}\text{First differences: } &= -1; -3; -5; -7\\ \text{Second difference: } &= -2\end{aligned}$$

2. Find the first five terms of the quadratic sequence defined by: $T_n = 5n^2 + 3n + 4$.

Solution:

$$\begin{aligned}T_n &= 5n^2 + 3n + 4\\ T_1 &= 5(1)^2 + 3(1) + 4\\ &= 12\\ T_2 &= 5(2)^2 + 3(2) + 4\\ &= 30\\ T_3 &= 5(3)^2 + 3(3) + 4\\ &= 58\\ T_4 &= 5(4)^2 + 3(4) + 4\\ &= 96\\ T_5 &= 5(5)^2 + 3(5) + 4\\ &= 144\end{aligned}$$

$12; 30; 58; 96; 144$

3. Given $T_n = 4n^2 + 5n + 10$, find T_9.

Solution:

$$
\begin{aligned}
T_n &= 4n^2 + 5n + 10 \\
T_9 &= 4(9)^2 + 5(9) + 10 \\
&= 379
\end{aligned}
$$

4. Given $T_n = 2n^2$, for which value of n does $T_n = 32$?

Solution:

$$
\begin{aligned}
T_n &= 2n^2 \\
32 &= 2n^2 \\
16 &= n^2 \\
4 &= n
\end{aligned}
$$

5. a) Write down the next two terms of the quadratic sequence: $16; 27; 42; 61; \ldots$

   b) Find the general formula for the quadratic sequence above.

Solution:

a)

	$n = 1$		$n = 2$		$n = 3$		$n = 4$
T_n	16		27		42		61
1st difference		11		15		19	
2nd difference			4		4		

$$
\begin{aligned}
\text{Second difference: } &= 4 \\
\text{First differences: } &= 11; 15; 19; 23; 27 \\
\therefore T_5 &= 61 + 23 \\
&= 84 \\
\therefore T_6 &= 84 + 27 \\
&= 111
\end{aligned}
$$

b)

$$
\begin{aligned}
T_n &= an^2 + bn + c \\
T_1 &= a + b + c \\
T_2 &= 4n^2 + 2n + c \\
T_3 &= 9n^2 + 3n + c
\end{aligned}
$$

$$\therefore a + b + c = 16$$
$$\therefore c = 16 - a - b$$
$$4a + 2b + c = 27$$
$$4a + 2b + (16 - a - b) = 27$$
$$3a + b = 11$$
$$\text{And } 9a + 3b + c = 42$$
$$\therefore 9a + 3b + (16 - a - b) = 42$$
$$8a + 2b = 26$$
$$4a + b = 13$$
$$\therefore b = 13 - 4a$$

$$\therefore 3a + b = 11$$
$$3a + (13 - 4a) = 11$$
$$-a = -2$$
$$\therefore a = 2$$
$$b = 13 - 4(2)$$
$$\therefore b = 5$$
$$\text{And } c = 16 - a - b$$
$$\therefore c = 16 - 2 - 5$$
$$= 9$$
$$\therefore T_n = 2n^2 + 5n + 9$$

3.3 Summary

Exercise 3 – 4: End of chapter exercises

1. Find the first five terms of the quadratic sequence defined by:
   $T_n = n^2 + 2n + 1$

   Solution:

   $-4; 9; 16; 25; 36$

2. Determine whether each of the following sequences is:

   - a linear sequence,
   - a quadratic sequence,
   - or neither.

a) $6; 9; 14; 21; 30; \ldots$

b) $1; 7; 17; 31; 49; \ldots$

c) $8; 17; 32; 53; 80; \ldots$

d) $9; 26; 51; 84; 125; \ldots$

e) $2; 20; 50; 92; 146; \ldots$

f) $5; 19; 41; 71; 109; \ldots$

g) $2; 6; 10; 14; 18; \ldots$

h) $3; 9; 15; 21; 27; \ldots$

i) $1; 2{,}5; 5; 8{,}5; 13; \ldots$

j) $10; 24; 44; 70; 102; \ldots$

k) $2\frac{1}{2}; 6; 10\frac{1}{2}; 16; 22\frac{1}{2}; \ldots$

l) $3p^2; 6p^2; 9p^2; 12p^2; 15p^2; \ldots$

m) $2k; 8k; 18k; 32k; 50k; \ldots$

Solution:

a)

$$\begin{aligned} \text{First differences: } &= 3; 5; 7; 9; \\ \text{Second difference: } &= 2 \end{aligned}$$

Quadratic sequence

b)

$$\begin{aligned} \text{First differences: } &= 6; 10; 14; 18 \\ \text{Second difference: } &= 4 \end{aligned}$$

Quadratic sequence

c)

$$\begin{aligned} \text{First differences: } &= 9; 15; 21; 27 \\ \text{Second difference: } &= 6 \end{aligned}$$

Quadratic sequence

d)

$$\begin{aligned} \text{First differences: } &= 17; 25; 33; 41 \\ \text{Second difference: } &= 8 \end{aligned}$$

Quadratic sequence

e)

$$\begin{aligned} \text{First differences: } &= 18; 30; 42; 54 \\ \text{Second difference: } &= 12 \end{aligned}$$

Quadratic sequence

f)

$$\begin{aligned} \text{First differences: } &= 14; 22; 30; 38 \\ \text{Second difference: } &= 8 \end{aligned}$$

Quadratic sequence

g)

$$\text{First difference: } = 4$$

Linear sequence

h)

$$\text{First difference: } = 6$$

Linear sequence

i)

$$\begin{aligned}\text{First differences: } &= 1{,}5;\ 2{,}5;\ 3{,}5;\ 4{,}5\\ \text{Second difference: } &= 1\end{aligned}$$

Quadratic sequence

j)

$$\begin{aligned}\text{First differences: } &= 14; 20; 26; 32\\ \text{Second difference: } &= 16\end{aligned}$$

Quadratic sequence

k)

$$\begin{aligned}\text{First differences: } &= 3{,}5;\ 4{,}5;\ 5{,}5;\ 6{,}5\\ \text{Second difference: } &= 1\end{aligned}$$

Quadratic sequence

l)

$$\text{First difference: } = 3p^2$$

Linear sequence

m)

$$\begin{aligned}\text{First differences: } &= 6k; 10k; 14k; 18k\\ \text{Second difference: } &= 4k\end{aligned}$$

Quadratic sequence

3. Given the pattern: $16; x; 46; \ldots$, determine the value of x if the pattern is linear.

Solution:

$$\begin{aligned}x - 16 &= 46 - x\\ 2x &= 62\\ \therefore x &= 31\end{aligned}$$

4. Given $T_n = 2n^2$, for which value of n does $T_n = 242$?

Solution:

$$\begin{aligned}2n^2 &= 242\\ n^2 &= 121\\ \therefore n &= 11\end{aligned}$$

5. Given $T_n = 3n^2$, find T_{11}.

Solution:

$$\begin{aligned} T_n &= 3n^2 \\ \therefore T_11 &= 3(11)^2 \\ &= 363 \end{aligned}$$

6. Given $T_n = n^2 + 4$, for which value of n does $T_n = 85$?

Solution:

$$\begin{aligned} n^2 + 4 &= 85 \\ n^2 &= 81 \\ \therefore n &= 9 \end{aligned}$$

7. Given $T_n = 4n^2 + 3n - 1$, find T_5.

Solution:

$$\begin{aligned} T_n &= 4n^2 + 3n - 1 \\ \therefore T_5 &= 4(5)^2 + 3(5) - 1 \\ &= 100 + 15 - 1 \\ &= 114 \end{aligned}$$

8. Given $T_n = \frac{3}{2}n^2$, for which value of n does $T_n = 96$?

Solution:

$$\begin{aligned} \frac{3}{2}n^2 &= 96 \\ n^2 &= 96 \times \frac{2}{3} \\ &= 64 \\ \therefore n &= 8 \end{aligned}$$

9. For each of the following patterns, determine:

- the next term in the pattern,
- and the general term,
- the tenth term in the pattern.

a) $3; 7; 11; 15; \ldots$

b) $17; 12; 7; 2; \ldots$

c) $\frac{1}{2}; 1; 1\frac{1}{2}; 2; \ldots$

d) $a; a + b; a + 2b; a + 3b; \ldots$

e) $1; -1; -3; -5; \ldots$

Solution:

a)

$$\begin{aligned}
d &= 7 - 3 \\
&= 4 \\
T_5 &= 15 + 4 \\
&= 19 \\
T_n &= 4n - 1 \\
T_{10} &= 4(10) - 1 \\
&= 39
\end{aligned}$$

b)

$$\begin{aligned}
d &= 12 - 17 \\
&= -5 \\
T_5 &= 2 - 5 \\
&= -3 \\
T_n &= 22 - 5n \\
T_{10} &= 22 - 5(10) \\
&= -28
\end{aligned}$$

c)

$$\begin{aligned}
d &= 1 - \frac{1}{2} \\
&= \frac{1}{2} \\
T_5 &= 2 + \frac{1}{2} \\
&= 2\frac{1}{2} \\
T_n &= \frac{1}{2}n \\
T_{10} &= \frac{1}{2}(10) \\
&= 5
\end{aligned}$$

d)

$$\begin{aligned}
d &= a + b - a \\
&= b \\
T_5 &= a + 3b + b \\
&= a + 4b \\
T_n &= a - b + bn \\
T_{10} &= a - b + b(10) \\
&= a + 9b
\end{aligned}$$

e)

$$\begin{aligned} d &= -1 - 1 \\ &= -2 \\ T_5 &= -5 - 2 \\ &= -7 \\ T_n &= 3 - 2n \\ T_{10} &= 3 - 2(10) \\ &= -17 \end{aligned}$$

10. For each of the following sequences, find the equation for the general term and then use the equation to find T_{100}.

    a) $4; 7; 12; 19; 28; \ldots$
    b) $2; 8; 14; 20; 26; \ldots$
    c) $7; 13; 23; 37; 55; \ldots$
    d) $5; 14; 29; 50; 77; \ldots$

Solution:

a)

	$n = 1$		$n = 2$		$n = 3$		$n = 4$
T_n	4		7		12		19
1st difference		3		5		7	
2nd difference			2		2		

$$\begin{aligned} T_n &= an^2 + bn + c \\ \text{Second difference: } 2a &= 2 \\ \therefore a &= 1 \\ 3a + b &= 3 \\ b &= 3 - 3(1) \\ \therefore b &= 0 \\ a + b + c &= 4 \\ \therefore c &= 4 - a - b \\ &= 4 - 1 \\ \therefore c &= 3 \end{aligned}$$

$$\therefore T_n = n^2 + 3$$

$$\begin{aligned} T_n &= n^2 + 3 \\ \therefore T_{100} &= (10)^2 + 3 \\ &= 10\ 003 \end{aligned}$$

b)

$$\begin{aligned} d &= 8 - 2 \\ &= 6 \\ \therefore T_n &= 6n - 4 \end{aligned}$$

$$
\begin{aligned}
T_n &= 6n - 4 \\
\therefore T_{100} &= 6(100) - 4 \\
&= 596
\end{aligned}
$$

c)

	$n = 1$		$n = 2$		$n = 3$		$n = 4$
T_n	7		13		23		37
1st difference		6		10		14	
2nd difference			4		4		

$$
\begin{aligned}
T_n &= an^2 + bn + c \\
\text{Second difference: } 2a &= 4 \\
\therefore a &= 2 \\
3a + b &= 6 \\
b &= 6 - 3(2) \\
\therefore b &= 0 \\
a + b + c &= 7 \\
\therefore c &= 7 - a - b \\
&= 7 - 2 \\
\therefore c &= 5 \\
\therefore T_n &= 2n^2 + 5
\end{aligned}
$$

$$
\begin{aligned}
T_n &= 2n^2 + 5 \\
\therefore T_{100} &= 2(10)^2 + 5 \\
&= 20\ 005
\end{aligned}
$$

d)

	$n = 1$		$n = 2$		$n = 3$		$n = 4$
T_n	5		14		29		50
1st difference		9		15		21	
2nd difference			6		6		

$$
\begin{aligned}
T_n &= an^2 + bn + c \\
\text{Second difference: } 2a &= 6 \\
\therefore a &= 3 \\
3a + b &= 9 \\
b &= 9 - 3(3) \\
\therefore b &= 0 \\
a + b + c &= 5 \\
\therefore c &= 5 - a - b \\
&= 5 - 3 \\
\therefore c &= 2 \\
\therefore T_n &= 3n^2 + 2
\end{aligned}
$$

$$T_n = 3n^2 + 2$$
$$\therefore T_{100} = 3(10)^2 + 2$$
$$= 30\ 002$$

11. Given: $T_n = 3n - 1$

   a) Write down the first five terms of the sequence.
   b) What do you notice about the difference between any two consecutive terms?
   c) Will this always be the case for a linear sequence?

Solution:

   a) $2; 5; 8; 11; 14$
   b) Constant difference, $d = 3$
   c) Yes

12. Given the following sequence: $-15; -11; -7; \ldots; 173$

   a) Determine the equation for the general term.
   b) Calculate how many terms there are in the sequence.

Solution:

   a)

$$d = -11 - (-15)$$
$$= 4$$
$$T_n = 4n - 19$$

   b)

$$T_n = 4n - 19$$
$$173 = 4n - 19$$
$$192 = 4n$$
$$\therefore 48 = n$$

13. Given $3; 7; 13; 21; 31; \ldots$

   a) Thabang determines that the general term is $T_n = 4n - 1$. Is he correct? Explain.
   b) Cristina determines that the general term is $T_n = n^2 + n + 1$. Is she correct? Explain.

Solution:

   a)

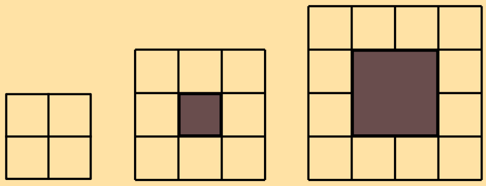
$$\text{First differences:} \quad = 4; 6; 8; 10$$
$$\text{Second difference:} \quad = 2$$

Thabang's general term $T_n = 4n - 1$ describes a linear sequence and this is a quadratic sequence.

b)

$$
\begin{aligned}
T_n &= an^2 + bn + c \\
\text{Second difference: } 2a &= 2 \\
\therefore a &= 1 \\
3a + b &= 4 \\
b &= 4 - 3(1) \\
\therefore b &= 1 \\
a + b + c &= 3 \\
\therefore c &= 3 - a - b \\
&= 3 - 1 - 1 \\
\therefore c &= 1 \\
\therefore T_n &= n^2 + n + 1
\end{aligned}
$$

14. Given the following pattern of blocks:



2 3 4

a) Draw pattern 5.

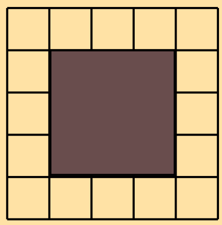
b) Complete the table below:

pattern number (n)	2	3	4	5	10	250	n
number of white blocks (w)	4	8					

c) Is this a linear or a quadratic sequence?

Solution:

a)



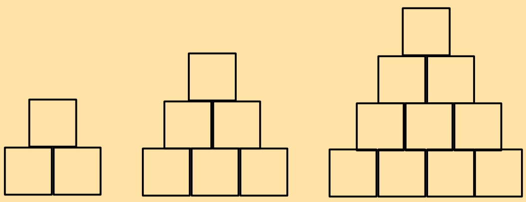
5

b)

pattern number (n)	2	3	4	5	10	250	n
number of white blocks (w)	4	8	12	16	36	996	$4n - 4$

Linear

15. Cubes of volume 1 cm^3 are stacked on top of each other to form a tower:



1 2 3

a) Complete the table for the height of the tower:

tower number (n)	1	2	3	4	10	n
height of tower (h)	2					

b) What type of sequence is this?

c) Now consider the number of cubes in each tower and complete the table below:

tower number (n)	1	2	3	4
number of cubes (c)	3			

d) What type of sequence is this?

e) Determine the general term for this sequence.

f) How many cubes are needed for tower number 21?

g) How high will a tower of 496 cubes be?

Solution:

a)

tower number (n)	1	2	3	4	10	n
height of tower (h)	2	3	4	5	11	$n+1$

b) Linear

c)

tower number (n)	1	2	3	4
number of cubes (c)	3	6	10	15

d) Quadratic

e)

$$\begin{aligned}
T_n &= an^2 + bn + c \\
\text{Second difference: } 2a &= 1 \\
\therefore a &= \frac{1}{2} \\
3a + b &= 3 \\
b &= 3 - 3(\frac{1}{2}) \\
\therefore b &= \frac{3}{2} \\
a + b + c &= 3 \\
\therefore c &= 3 - a - b \\
&= 3 - \frac{1}{2} - \frac{3}{2} \\
\therefore c &= 1 \\
\therefore T_n &= \frac{1}{2}n^2 + \frac{3}{2}n + 1
\end{aligned}$$

f)

$$\begin{aligned}
T_n &= \frac{1}{2}n^2 + \frac{3}{2}n + 1 \\
\therefore T_{21} &= \frac{1}{2}(21)^2 + \frac{3}{2}(21) + 1 \\
&= \frac{441}{2} + \frac{63}{2} + \frac{2}{2} \\
&= 253
\end{aligned}$$

g)

$$\begin{aligned}
T_n &= \frac{1}{2}n^2 + \frac{3}{2}n + 1 \\
496 &= \frac{1}{2}n^2 + \frac{3}{2}n + 1 \\
0 &= \frac{1}{2}n^2 + \frac{3}{2}n - 495 \\
&= n^2 + 3n - 990 \\
&= (n - 30)(n + 33) \\
\therefore n = 30 &\text{ or } n = -33 \\
\therefore n &= 30 \\
\text{And } h &= n + 1 \\
&= 30 + 1 \\
&= 31 \text{ cm}
\end{aligned}$$

16. A quadratic sequence has a second term equal to 1, a third term equal to -6 and a fourth term equal to -14.

a) Determine the second difference for this sequence.

b) Hence, or otherwise, calculate the first term of the pattern.

Solution:

a)

$$\begin{aligned}
T_3 - T_2 &= -6 - (1) \\
&= -7 \\
T_4 - T_3 &= -14 - (-6) \\
&= -8 \\
\therefore \text{Second difference} &= -1
\end{aligned}$$

b)

$$\begin{aligned}
T_1 &= 1 + 6 \\
&= 7
\end{aligned}$$

17. There are 15 schools competing in the U16 girls hockey championship and every team must play two matches — one home match and one away match.

a) Use the given information to complete the table:

no. of schools	no. of matches
1	0
2	
3	
4	
5	

b) Calculate the second difference.

c) Determine a general term for the sequence.

d) How many matches will be played if there are 15 schools competing in the championship?

e) If 600 matches must be played, how many schools are competing in the championship?

Solution:

a)

no. of schools	no. of matches
1	0
2	2
3	6
4	12
5	20

b)

$$\begin{aligned} T_2 - T_1 &= 2 - 0 \\ &= 2 \\ T_3 - T_2 &= 6 - 2 \\ &= 4 \\ T_4 - T_3 &= 12 - 6 \\ &= 6 \\ T_5 - T_4 &= 20 - 12 \\ &= 8 \\ \therefore \text{Second difference} &= 2 \end{aligned}$$

c)

$$\begin{aligned} T_n &= an^2 + bn + c \\ \text{Second difference: } 2a &= 2 \\ \therefore a &= 1 \\ 3a + b &= 2 \\ b &= 2 - 3(1) \\ \therefore b &= -1 \\ a + b + c &= 0 \\ \therefore c &= -a - b \\ &= -1 - (-1) \\ \therefore c &= 0 \\ \therefore T_n &= n^2 - n \end{aligned}$$

d)

$$\begin{aligned} T_n &= n^2 - n \\ T_{15} &= (15)^2 - 15 \\ &= 225 - 15 \\ &= 210 \end{aligned}$$

e)

$$\begin{aligned}
T_n &= n^2 - n \\
600 &= n^2 - n \\
0 &= n^2 - n - 600 \\
&= (n-25)(n+24) \\
\therefore n &= 25 \text{ or } n = -24 \\
\therefore n &= 25
\end{aligned}$$

18. The first term of a quadratic sequence is 4, the third term is 34 and the common second difference is 10. Determine the first six terms in the sequence.

Solution:

$$\begin{aligned}
\text{Let } T_2 &= x \\
\therefore T_2 - T_1 &= x - 4 \\
\text{And } T_3 - T_2 &= 34 - x \\
\text{Second difference } &= (T_3 - T_2) - (T_2 - T_1) \\
&= (34 - x) - (x - 4) \\
\therefore 10 &= 38 - 2x \\
2x &= 28 \\
\therefore x &= 14
\end{aligned}$$

$4; 14; 34; 64; 104; 154$

19. Challenge question:

Given that the general term for a quadratic sequences is $T_n = an^2 + bn + c$, let d be the first difference and D be the second common difference.

a) Show that $a = \frac{D}{2}$.

b) Show that $b = d - \frac{3}{2}D$.

c) Show that $c = T_1 - d + D$.

d) Hence, show that $T_n = \frac{D}{2}n^2 + \left(d - \frac{3}{2}D\right)n + (T_1 - d + D)$.

Solution:

a)

$$\begin{aligned}
T_n &= an^2 + bn + c \\
T_1 &= a(1)^2 + b(1) + c \\
&= a + b + c \\
T_2 &= a(2)^2 + b(2) + c \\
&= 4a + 2b + c \\
T_3 &= a(3)^2 + b(3) + c \\
&= 9a + 6b + c \\
\text{First difference } d &= T_2 - T_1 \\
\therefore d &= (4a + 2b + c) - (a + b + c) \\
&= 3a + b \\
\therefore b &= d - 3a \\
\text{Second difference } D &= (T_3 - T_2) - (T_2 - T_1) \\
\therefore D &= (5a + b) - (3a + b) \\
&= 2a \\
\therefore a &= \frac{D}{2}
\end{aligned}$$

b)

$$\begin{aligned}
a &= \frac{D}{2} \\
b &= d - 3a \\
\therefore b &= d - 3\left(\frac{D}{2}\right) \\
&= d - \frac{3}{2}D
\end{aligned}$$

c)

$$\begin{aligned}
T_1 &= a + b + c \\
\therefore c &= T_1 - a - b \\
&= T_1 - \left(\frac{D}{2}\right) - \left(d - \frac{3}{2}D\right) \\
&= T_1 - \frac{D}{2} - d + \frac{3}{2}D \\
&= T_1 - d + D
\end{aligned}$$

d)

$$\begin{aligned}
T_n &= an^2 + bn + c \\
\therefore T_n &= \frac{D}{2}n^2 + \left(d - \frac{3}{2}D\right)n + (T_1 - d + D)
\end{aligned}$$

CHAPTER 4

Analytical geometry

- Integrate Euclidean Geometry knowledge with Analytical Geometry.
- Emphasize the value and importance of making sketches.
- Emphasize the importance of writing coordinates consistently for the distance formula and gradient.
- Discuss and explain:
  - that parallel lines have equal gradients and equal angles of inclination.
  - that the product of the gradients of perpendicular lines is equal to -1.
  - that horizontal lines have a zero gradient.
  - that vertical lines have an undefined gradient.
- Revise the properties of the different quadrilaterals as this knowledge is needed in many of the exercises.
- Explain that the angle of inclination is never greater than $180°$.

4.1 Revision

Exercise 4 – 1: Revision

1. Determine the length of the line segment between the following points:

   a) $P(-3;5)$ and $Q(-1;-5)$

   b) $R(0{,}75;3)$ and $S(0{,}75;-4)$

   c) $T(2x;y-2)$ and $U(3x+1;y-2)$

   Solution:

   a)

$$\begin{aligned} PQ &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\ &= \sqrt{(-1+3)^2+(-5-5)^2} \\ &= \sqrt{(2)^2+(-10)^2} \\ &= \sqrt{4+100} \\ &= \sqrt{104} \\ &= 2\sqrt{26} \text{ units} \end{aligned}$$

b)

$$\begin{aligned}RS &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\\ &= \sqrt{(0{,}75 - 0{,}75)^2 + (-4 - 3)^2}\\ &= \sqrt{(0)^2 + (-7)^2}\\ &= \sqrt{49}\\ &= 7 \text{ units}\end{aligned}$$

c)

$$\begin{aligned}TU &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\\ &= \sqrt{(3x + 1 - 2x)^2 + (y - 2 - y + 2)^2}\\ &= \sqrt{(x + 1)^2 + (0)^2}\\ &= \sqrt{(x + 1)^2}\\ &= x + 1 \text{ units}\end{aligned}$$

2. Given $Q(4; 1), T(p; 3)$ and length $QT = \sqrt{8}$ units, determine the value of p.

Solution:

$$\begin{aligned}QT &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\\ \sqrt{8} &= \sqrt{(p - 4)^2 + (3 - 1)^2}\\ \left(\sqrt{8}\right)^2 &= (p - 4)^2 + (2)^2\\ 8 &= p^2 - 8p + 16 + 4\\ 0 &= p^2 - 8p + 16 + 4 - 8\\ &= p^2 - 8p + 12\\ &= (p - 6)(p - 2)\\ \therefore p &= 6 \text{ or } p = 2\end{aligned}$$

3. Determine the gradient of the line AB if:

   a) $A(-5; 3)$ and $B(-7; 4)$
   b) $A(3; -2)$ and $B(1; -8)$

Solution:

a)

$$\begin{aligned}m_{AB} &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{4 - 3}{-7 + 5}\\ &= \frac{1}{-2}\\ &= -\frac{1}{2}\end{aligned}$$

b)

$$\begin{aligned} m_{AB} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-8 + 2}{1 - 3} \\ &= \frac{-6}{-2} \\ &= 3 \end{aligned}$$

4. Prove that the line PQ, with $P(0; 3)$ and $Q(5; 5)$, is parallel to the line $5y + 5 = 2x$.

Solution:

$$\begin{aligned} 5y + 5 &= 2x \\ 5y &= 2x - 5 \\ y &= \frac{2}{5}x - 1 \\ \therefore m &= \frac{2}{5} \end{aligned}$$

$$\begin{aligned} m_{PQ} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{5 - 3}{5 - 0} \\ &= \frac{2}{5} \end{aligned}$$

5. Given the points $A(-1; -1)$, $B(2; 5)$, $C(-1; -\frac{5}{2})$ and $D(x; -4)$ and $AB \perp CD$, determine the value of x.

Solution:

$$
\begin{aligned}
m_{AB} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{5+1}{2+1} \\
&= \frac{6}{3} \\
&= 2 \\
m_{CD} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-4+\frac{5}{2}}{x+1} \\
&= \frac{-\frac{3}{2}}{x+1} \\
&= -\frac{3}{2(x+1)}
\end{aligned}
$$

$$
\begin{aligned}
\text{And if } AB &\perp CD \\
\text{then } m_{AB} \times m_{CD} &= -1 \\
-\frac{3}{2(x+1)} \times 2 &= -1 \\
\frac{3}{x+1} &= 1 \\
3 &= x+1 \\
\therefore x &= 2
\end{aligned}
$$

6. Calculate the coordinates of the mid-point $P(x;y)$ of the line segment between the points:
   a) $M(3;5)$ and $N(-1;-1)$
   b) $A(-3;-4)$ and $B(2;3)$

   Solution:

   a)

$$
\begin{aligned}
M(x;y) &= \left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right) \\
&= \left(\frac{3-1}{2};\frac{5-1}{2}\right) \\
&= \left(\frac{2}{2};\frac{4}{2}\right) \\
&= (1;2)
\end{aligned}
$$

   b)

$$
\begin{aligned}
M(x;y) &= \left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right) \\
&= \left(\frac{-3+2}{2};\frac{-4+3}{2}\right) \\
&= \left(\frac{-1}{2};\frac{-1}{2}\right)
\end{aligned}
$$

7. The line joining $A(-2;4)$ and $B(x;y)$ has the mid-point $C(1;3)$. Determine the values of x and y.

Solution:

$$\begin{aligned} M(x;y) &= \left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right) \\ C(1;3) &= \left(\frac{-2+x}{2};\frac{4+y}{2}\right) \\ \therefore 1 &= \frac{-2+x}{2} \\ 2 &= -2+x \\ \therefore 4 &= x \\ \text{And } 3 &= \frac{4+y}{2} \\ 6 &= 4+y \\ \therefore 2 &= y \end{aligned}$$

8. Given quadrilateral $ABCD$ with vertices $A(0;3), B(4;3), C(5;-1)$ and $D(1;-1)$.

    a) Determine the equation of the line AD and the line BC.

    b) Show that $AD \parallel BC$.

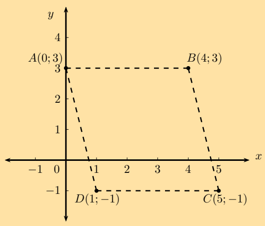
    c) Calculate the lengths of AD and BC.

    d) Determine the equation of the diagonal BD.

    e) What type of quadrilateral is $ABCD$?

Solution:

a)



$$\begin{aligned} m_{AD} &= \frac{y_2-y_1}{x_2-x_1} \\ &= \frac{-1-3}{1-0} \\ &= \frac{-4}{1} \\ &= -4 \\ y &= mx+c \\ y &= -4x+c \\ \text{Subst. } (0;3) \quad 3 &= -4(0)+c \\ \therefore c &= 3 \\ y &= -4x+3 \end{aligned}$$

$$\begin{aligned} m_{BC} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-1-3}{5-4} \\ &= \frac{-4}{1} \\ &= -4 \\ y &= mx + c \\ y &= -4x + c \\ \text{Subst. } (4;3) \quad 3 &= -4(4) + c \\ \therefore c &= 19 \\ y &= -4x + 19 \end{aligned}$$

b)

$$\begin{aligned} m_{AD} &= -4 \\ m_{BC} &= -4 \\ \therefore m_{AD} &= m_{BC} \\ \therefore AD &\parallel BC \end{aligned}$$

c)

$$\begin{aligned} AD &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(1-0)^2 + (-1-3)^2} \\ &= \sqrt{1 + (-4)^2} \\ &= \sqrt{17} \text{ units} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(5-4)^2 + (-1-3)^2} \\ &= \sqrt{1 + (-4)^2} \\ &= \sqrt{17} \text{ units} \end{aligned}$$

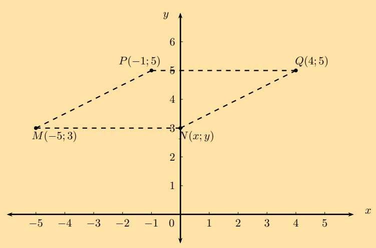
d)

$$\begin{aligned}
m_{BD} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-1-3}{1-4} \\
&= \frac{-4}{-3} \\
&= \frac{4}{3} \\
y &= mx + c \\
y &= \frac{4}{3}x + c \\
\text{Subst. } (4;3) \quad 3 &= \frac{4}{3}(4) + c \\
&= 3 - \frac{16}{3} \\
\therefore c &= -\frac{7}{3} \\
y &= \frac{4}{3}x - \frac{7}{3}
\end{aligned}$$

e) Parallelogram (one opposite side equal and parallel)

9. $MPQN$ is a parallelogram with points $M(-5;3)$, $P(-1;5)$ and $Q(4;5)$. Draw a sketch and determine the coordinates of $N(x;y)$.

Solution:



$MPQN$ is a parallelogram, therefore $PQ = MN$ and $PQ \parallel MN$:

$$
\begin{aligned}
m_{PQ} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{5-5}{4+1} \\
&= 0 \\
\therefore m_{MN} &= 0 \\
\therefore y &= 3 \\
PQ &= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} \\
&= \sqrt{(4+1)^2 + (5-5)^2} \\
&= \sqrt{(5)^2} \\
&= 5 \\
\text{Therefore } x &= -5+5 \\
\therefore x &= 0 \\
\therefore N(x;y) &= (0;3)
\end{aligned}
$$

10. $PQRS$ is a quadrilateral with points $P(-3;1)$, $Q(1;3)$, $R(6;1)$ and $S(2;-1)$ in the Cartesian plane.

a) Determine the lengths of PQ and SR.

b) Determine the mid-point of PR.

c) Show that $PQ \parallel SR$.

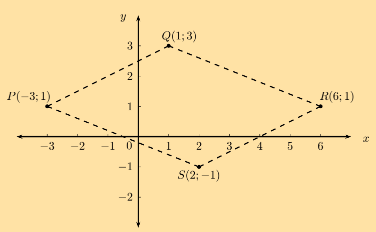
d) Determine the equations of the line PS and the line SR.

e) Is $PS \perp SR$? Explain your answer.

f) What type of quadrilateral is $PQRS$?

Solution:

a)



$$
\begin{aligned}
PQ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(1+3)^2 + (3-1)^2} \\
&= \sqrt{(4)^2 + (2)^2} \\
&= \sqrt{16+4} \\
&= \sqrt{20} \\
SR &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(2-6)^2 + (-1-1)^2} \\
&= \sqrt{(-4)^2 + (-2)^2} \\
&= \sqrt{16+4} \\
&= \sqrt{20}
\end{aligned}
$$

b) Let the mid-point of PR be $M(x;y)$.

$$
\begin{aligned}
M(x;y) &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \\
&= \left(\frac{-3+6}{2}; \frac{1+1}{2}\right) \\
&= \left(\frac{3}{2}; 1\right)
\end{aligned}
$$

c)

$$
\begin{aligned}
m_{PQ} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{3-1}{1+3} \\
&= \frac{2}{4} \\
&= \frac{1}{2} \\
m_{SR} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-1-1}{2-6} \\
&= \frac{-2}{-4} \\
&= \frac{1}{2}
\end{aligned}
$$

Therefore $m_{PQ} = m_{SR}$

$\therefore PQ \parallel SR$

d)

$$
\begin{aligned}
m_{PS} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-1 - 1}{2 + 3} \\
&= \frac{-2}{5} \\
y &= mx + c \\
y &= -\frac{2}{5}x + c \\
\text{Subst. } (-3;1) \quad 1 &= -\frac{2}{5}(-3) + c \\
1 &= \frac{6}{5} + c \\
\therefore -\frac{1}{5} &= c \\
y &= -\frac{2}{5}x - \frac{1}{5}
\end{aligned}
$$

$$
\begin{aligned}
m_{SR} &= \frac{1}{2} \\
y &= mx + c \\
y &= \frac{1}{2}x + c \\
\text{Subst. } (6;1) \quad 1 &= \frac{1}{2}(6) + c \\
1 &= 3 + c \\
\therefore -2 &= c \\
y &= \frac{1}{2}x - 2
\end{aligned}
$$

e) No; $m_{PS} \times m_{SR} \neq -1$.

f) Parallelogram

4.2 Equation of a line

The two-point form of the straight line equation

Exercise 4 – 2: The two-point form of the straight line equation

Determine the equation of the straight line passing through the points:

1. $(3;7)$ and $(-6;1)$

   Solution:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$\frac{y - 7}{x - 3} = \frac{1 - 7}{-6 - 3}$$
$$\frac{y - 7}{x - 3} = \frac{-6}{-9}$$
$$\frac{y - 7}{x - 3} = \frac{2}{3}$$
$$y - 7 = \frac{2}{3}(x - 3)$$
$$y = \frac{2}{3}x - 2 + 7$$
$$\therefore y = \frac{2}{3}x + 5$$

2. $(1; -\frac{11}{4})$ and $(\frac{2}{3}; -\frac{7}{4})$

 Solution:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$\frac{y + \frac{11}{4}}{x - 1} = \frac{-\frac{7}{4} + \frac{11}{4}}{\frac{2}{3} - 1}$$
$$\frac{y + \frac{11}{4}}{x - 1} = \frac{1}{-\frac{1}{3}}$$
$$\frac{y + \frac{11}{4}}{x - 1} = -3$$
$$y + \frac{11}{4} = -3(x - 1)$$
$$y = -3x + 3 - \frac{11}{4}$$
$$\therefore y = -3x + \frac{1}{4}$$

3. $(-2; 1)$ and $(3; 6)$

 Solution:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$\frac{y - 1}{x + 2} = \frac{6 - 1}{3 + 2}$$
$$\frac{y - 1}{x + 2} = \frac{5}{5}$$
$$y - 1 = x + 2$$
$$\therefore y = x + 3$$

4. $(2; 3)$ and $(3; 5)$

 Solution:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$\frac{y - 3}{x - 2} = \frac{5 - 3}{3 - 2}$$
$$\frac{y - 3}{x - 2} = 2$$
$$y - 3 = 2(x - 2)$$
$$y = 2x - 4 + 3$$
$$\therefore y = 2x - 1$$

5. $(1; -5)$ and $(-7; -5)$

   Solution:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$\frac{y + 5}{x - 1} = \frac{-5 + 5}{-7 - 1}$$
$$\frac{y + 5}{x - 1} = 0$$
$$y + 5 = 0$$
$$\therefore y = -5$$

6. $(-4; 0)$ and $(1; \frac{15}{4})$

   Solution:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$\frac{y - 0}{x + 4} = \frac{\frac{15}{4} - 0}{1 + 4}$$
$$\frac{y}{x + 4} = \frac{\frac{15}{4}}{5}$$
$$\frac{y}{x + 4} = \frac{15}{20}$$
$$y = \frac{3}{4}(x + 4)$$
$$\therefore y = \frac{3}{4}x + 3$$

7. $(s; t)$ and $(t; s)$

   Solution:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$\frac{y - t}{x - s} = \frac{s - t}{t - s}$$
$$\frac{y - t}{x - s} = \frac{-(t - s)}{t - s}$$
$$\frac{y - t}{x - s} = -1$$
$$y - t = -(x - s)$$
$$y = -x + s + t$$

8. $(-2;-8)$ and $(1;7)$

Solution:

$$\begin{aligned}
\frac{y-y_1}{x-x_1} &= \frac{y_2-y_1}{x_2-x_1}\\
\frac{y+8}{x+2} &= \frac{7+8}{1+2}\\
\frac{y+8}{x+2} &= \frac{15}{3}\\
\frac{y+8}{x+2} &= 5\\
y+8 &= 5(x+2)\\
y+8 &= 5x+10\\
y &= 5x+2
\end{aligned}$$

9. $(2p;q)$ and $(0;-q)$

Solution:

$$\begin{aligned}
\frac{y-y_1}{x-x_1} &= \frac{y_2-y_1}{x_2-x_1}\\
\frac{y-q}{x-2p} &= \frac{-q-q}{0-2p}\\
\frac{y-q}{x-2p} &= \frac{-2q}{-2p}\\
\frac{y-q}{x-2p} &= \frac{q}{p}\\
y-q &= \frac{q}{p}(x-2p)\\
y-q &= \frac{q}{p}x-2q\\
\therefore y &= \frac{q}{p}x-q
\end{aligned}$$

The gradient–point form of the straight line equation

Exercise 4 – 3: Gradient–point form of a straight line equation

Determine the equation of the straight line:

1. passing through the point $(-1;\frac{10}{3})$ and with $m=\frac{2}{3}$.

Solution:

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - \frac{10}{3} &= \frac{2}{3}(x + 1) \\
y - \frac{10}{3} &= \frac{2}{3}x + \frac{2}{3} \\
y &= \frac{2}{3}x + \frac{2}{3} + \frac{10}{3} \\
\therefore y &= \frac{2}{3}x + 4
\end{aligned}
$$

2. with $m = -1$ and passing through the point $(-2; 0)$.

   Solution:

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 0 &= -(x + 2) \\
\therefore y &= -x - 2
\end{aligned}
$$

3. passing through the point $(3; -1)$ and with $m = -\frac{1}{3}$.

   Solution:

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y + 1 &= -\frac{1}{3}(x - 3) \\
y + 1 &= -\frac{1}{3}x + 1 \\
\therefore y &= -\frac{1}{3}x
\end{aligned}
$$

4. parallel to the x-axis and passing through the point $(0; 11)$.

   Solution:

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 11 &= 0(x - 0) \\
\therefore y &= 11
\end{aligned}
$$

5. passing through the point $(1; 5)$ and with $m = -2$.

   Solution:

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 5 &= -2(x - 1) \\
y - 5 &= -2x + 2 \\
\therefore y &= -2x + 7
\end{aligned}
$$

6. perpendicular to the x-axis and passing through the point $(-\frac{3}{2}; 0)$.

Solution:

$x = -\frac{3}{2}$

7. with $m = -0{,}8$ and passing through the point $(10; -7)$.

   Solution:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y + 7 &= -\frac{4}{5}(x - 10) \\ y + 7 &= -\frac{4}{5}x + 8 \\ \therefore y &= -\frac{4}{5}x + 1 \end{aligned}$$

8. with undefined gradient and passing through the point $(4; 0)$.

   Solution:

   $x = 4$

9. with $m = 3a$ and passing through the point $(-2; -6a + b)$.

   Solution:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-6a + b) &= 3a(x + 2) \\ y + 6a - b &= 3ax + 6a \\ \therefore y &= 3ax + b \end{aligned}$$

The gradient–intercept form of a straight line equation

Exercise 4 – 4: The gradient–intercept form of a straight line equation

Determine the equation of the straight line:

1. passing through the point $(\frac{1}{2}; 4)$ and with $m = 2$.

   Solution:

$$\begin{aligned}
y &= mx + c \\
y &= 2x + c \\
4 &= 2\left(\frac{1}{2}\right) + c \\
4 &= 1 + c \\
\therefore c &= 3 \\
\therefore y &= 2x + 3
\end{aligned}$$

2. passing through the points $(\frac{1}{2}; -2)$ and $(2; 4)$.

   Solution:

$$\begin{aligned}
y &= mx + c \\
-2 &= m\left(\frac{1}{2}\right) + c \\
\text{Multiply by } 2: \quad -4 &= m + 2c \ldots (1) \\
4 &= 2m + c \\
\text{Multiply by } 2: \quad 8 &= 4m + 2c \ldots (2) \\
(2) - (1): \quad 8 + 4 &= 4m - m \\
12 &= 3m \\
\therefore 4 &= m \\
\therefore c &= 4 - 2(4) \\
&= -4 \\
\therefore y &= 4x - 4
\end{aligned}$$

3. passing through the points $(2; -3)$ and $(-1; 0)$.

   Solution:

$$\begin{aligned}
y &= mx + c \\
-3 &= 2m + c \ldots (1) \\
0 &= -m + c \ldots (2) \\
(1) - (2): \quad -3 &= 2m + m \\
-3 &= 3m \\
\therefore -1 &= m \\
\therefore c &= -1 \\
\therefore y &= -x - 1
\end{aligned}$$

4. passing through the point $(2; -\frac{6}{7})$ and with $m = -\frac{3}{7}$.

   Solution:

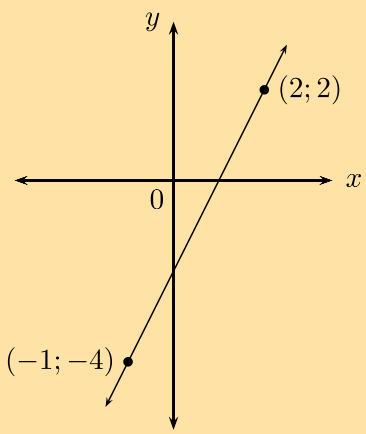
$$
\begin{aligned}
y &= mx + c \\
y &= -\frac{3}{7}x + c \\
-\frac{6}{7} &= -\frac{3}{7}(2) + c \\
-\frac{6}{7} + \frac{6}{7} &= +c \\
\therefore c &= 0 \\
\therefore y &= -\frac{3}{7}x
\end{aligned}
$$

5. which cuts the y-axis at $y = -\frac{1}{5}$ and with $m = \frac{1}{2}$.

   Solution:

$$
\begin{aligned}
y &= mx + c \\
y &= mx - \frac{1}{5} \\
\therefore y &= \frac{1}{2}x - \frac{1}{5}
\end{aligned}
$$

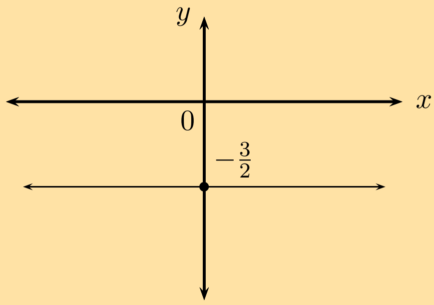
6.



   Solution:

$$
\begin{aligned}
y &= mx + c \\
2 &= 2m + c \ldots (1) \\
-4 &= -m + c \ldots (2) \\
(1) - (2) \quad 2 + 4 &= 2m + m \\
6 &= 3m \\
\therefore m &= 2 \\
\therefore c &= -4 + 2 \\
&= -2 \\
\therefore y &= 2x - 2
\end{aligned}
$$

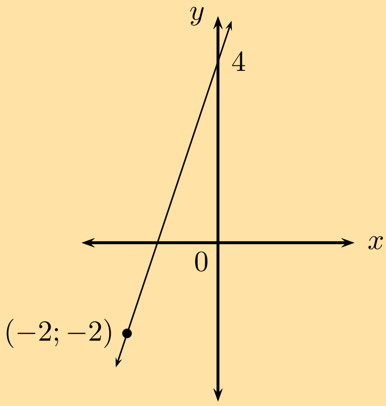
7.



Solution:

$y = -\frac{3}{2}$

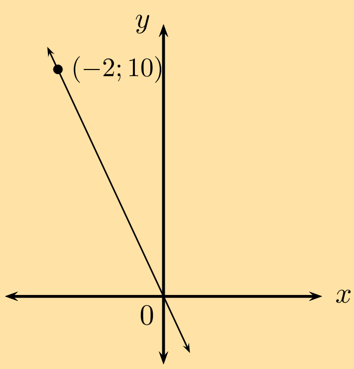
8.



Solution:

$$\begin{aligned} c &= 4 \\ y &= mx + c \\ y &= mx + 4 \\ \text{Subst. } (-2;-2) \quad -2 &= -2m + 4 \\ -2m &= -6 \\ m &= 3 \\ \therefore y &= 3x + 4 \end{aligned}$$

9.



Solution:

$$
\begin{aligned}
c &= 0 \\
y &= mx + c \\
y &= mx + 0 \\
\text{Subst. } (-2;10) \quad 10 &= -2m \\
-2m &= 10 \\
\therefore m &= -5 \\
\therefore y &= -5x
\end{aligned}
$$

4.3 Inclination of a line

Exercise 4 – 5: Angle of inclination

1. Determine the gradient (correct to 1 decimal place) of each of the following straight lines, given that the angle of inclination is equal to:

a) $60°$

b) $135°$

c) $0°$

d) $54°$

e) $90°$

f) $45°$

g) $140°$

h) $180°$

i) $75°$

Solution:

a)

$$
\begin{aligned}
m &= \tan\theta \\
&= \tan 60° \\
\therefore m &= 1{,}7
\end{aligned}
$$

b)

$$
\begin{aligned}
m &= \tan\theta \\
&= \tan 135° \\
\therefore m &= -1
\end{aligned}
$$

c)

$$
\begin{aligned}
m &= \tan\theta \\
&= \tan 0° \\
\therefore m &= 0
\end{aligned}
$$

d)

$$
\begin{aligned}
m &= \tan\theta \\
&= \tan 54° \\
\therefore m &= 1{,}4
\end{aligned}
$$

e)

$$\begin{aligned} m &= \tan\theta \\ &= \tan 90^{\circ} \\ \therefore m &\text{ is undefined} \end{aligned}$$

f)

$$\begin{aligned} m &= \tan\theta \\ &= \tan 45^{\circ} \\ \therefore m &= 1 \end{aligned}$$

g)

$$\begin{aligned} m &= \tan\theta \\ &= \tan 140^{\circ} \\ \therefore m &= -0{,}8 \end{aligned}$$

h)

$$\begin{aligned} m &= \tan\theta \\ &= \tan 180^{\circ} \\ \therefore m &= 0 \end{aligned}$$

i)

$$\begin{aligned} m &= \tan\theta \\ &= \tan 75^{\circ} \\ \therefore m &= 3{,}7 \end{aligned}$$

2. Determine the angle of inclination (correct to 1 decimal place) for each of the following:

   a) a line with $m = \frac{3}{4}$
   b) $2y - x = 6$
   c) the line passes through the points $(-4;-1)$ and $(2;5)$
   d) $y = 4$
   e) $x = 3y + \frac{1}{2}$
   f) $x = -0{,}25$
   g) the line passes through the points $(2;5)$ and $(\frac{2}{3};1)$
   h) a line with gradient equal to $0{,}577$

Solution:

a)

$$\begin{aligned} \tan\theta &= m \\ &= \frac{3}{4} \\ \theta &= \tan^{-1}(0{,}75) \\ \therefore \theta &= 36{,}8^{\circ} \end{aligned}$$

b)

$$
\begin{aligned}
2y - x &= 6 \\
2y &= x + 6 \\
y &= \frac{1}{2}x + 3 \\
\tan\theta &= m \\
&= \frac{1}{2} \\
\theta &= \tan^{-1}(0{,}5) \\
\therefore \theta &= 26{,}6^{\circ}
\end{aligned}
$$

c)

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{5 + 1}{2 + 4} \\
&= \frac{6}{6} \\
\therefore m &= 1 \\
\tan\theta &= 1 \\
\theta &= \tan^{-1}(1) \\
\therefore \theta &= 45^{\circ}
\end{aligned}
$$

d) Horizontal line

e)

$$
\begin{aligned}
x &= 3y + \frac{1}{2} \\
x - \frac{1}{2} &= 3y \\
\frac{1}{3}x - \frac{1}{6} &= y \\
\therefore m &= \frac{1}{3} \\
\theta &= \tan^{-1}\left(\frac{1}{3}\right) \\
\therefore \theta &= 18{,}4^{\circ}
\end{aligned}
$$

f) Vertical line

g)

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{1 - 5}{\frac{2}{3} - 2} \\
&= \frac{-4}{-\frac{4}{3}} \\
\therefore m &= 3 \\
\theta &= \tan^{-1}(3) \\
\therefore \theta &= 71{,}6^{\circ}
\end{aligned}
$$

h)

$$m = 0{,}577$$
$$\theta = \tan^{-1}(0{,}577)$$
$$\therefore \theta = 30^{\circ}$$

Exercise 4 – 6: Inclination of a straight line

1. Determine the angle of inclination for each of the following:

   a) a line with $m = \frac{4}{5}$

   b) $x + y + 1 = 0$

   c) a line with $m = 5{,}69$

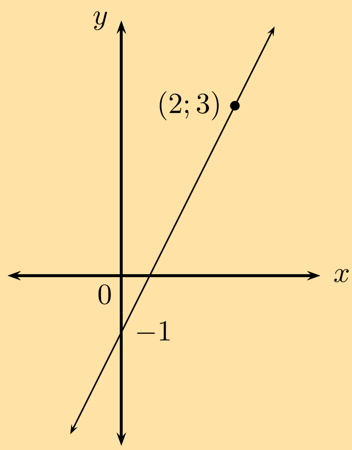
   d) the line that passes through $(1; 1)$ and $(-2; 7)$

   e) $3 - 2y = 9x$

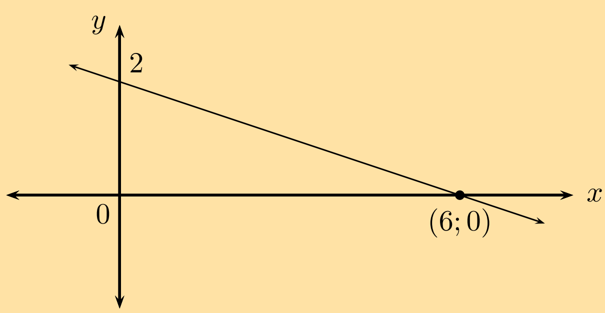
   f) the line that passes through $(-1; -6)$ and $(-\frac{1}{2}; -\frac{11}{2})$

   g) $5 = 10y - 15x$

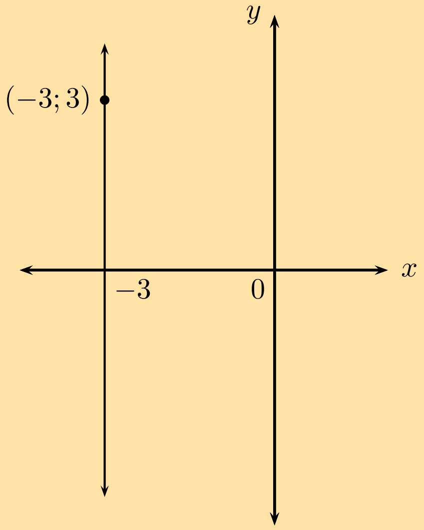
   h)



   i)



   j)



Solution:

a)

$$
\begin{aligned}
m &= \frac{4}{5} \\
\theta &= \tan^{-1}\left(\frac{4}{5}\right) \\
\therefore \theta &= 38{,}7^\circ
\end{aligned}
$$

b)

$$
\begin{aligned}
x + y + 1 &= 0 \\
y &= -x - 1 \\
m &= -1 \\
\theta &= \tan^{-1}(-1) \\
&= -45^\circ \\
\therefore \theta &= 180^\circ - 45^\circ \\
\therefore \theta &= 135^\circ
\end{aligned}
$$

c)

$$
\begin{aligned}
m &= 5{,}69 \\
\theta &= \tan^{-1}(5{,}69) \\
\therefore \theta &= 80^\circ
\end{aligned}
$$

d)

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{7 - 1}{-2 - 1} \\
&= \frac{6}{-3} \\
\therefore m &= -2 \\
\theta &= \tan^{-1}(-2) \\
&= -63{,}4^\circ \\
\therefore \theta &= 180^\circ - 63{,}4^\circ \\
\therefore \theta &= 80^\circ
\end{aligned}
$$

e)

$$\begin{aligned}
3 - 2y &= 9x \\
3 - 9x &= 2y \\
\frac{3}{2} - \frac{9}{2}x &= y \\
\therefore m &= -\frac{9}{2} \\
\theta &= \tan^{-1}\left(-\frac{9}{2}\right) \\
&= -77{,}5^\circ \\
\therefore \theta &= 180^\circ - 77{,}5^\circ \\
\therefore \theta &= 102{,}5^\circ
\end{aligned}$$

f)

$$\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-\frac{11}{2} + 6}{-\frac{1}{2} + 1} \\
&= \frac{\frac{1}{2}}{\frac{1}{2}} \\
\therefore m &= 1 \\
\theta &= \tan^{-1}(1) \\
\therefore \theta &= 45^\circ
\end{aligned}$$

g)

$$\begin{aligned}
5 &= 10y - 15x \\
5 + 15x &= 10y \\
\frac{1}{2} + \frac{3}{2}x &= y \\
\therefore m &= \frac{3}{2} \\
\theta &= \tan^{-1}\left(\frac{3}{2}\right) \\
\therefore \theta &= 56{,}3^\circ
\end{aligned}$$

h)

$$\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{3 + 1}{2 - 0} \\
&= \frac{4}{2} \\
\therefore m &= 2 \\
\theta &= \tan^{-1}(2) \\
\therefore \theta &= 63{,}4^\circ
\end{aligned}$$

i)

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{2 - 0}{0 - 6} \\
&= \frac{2}{-6} \\
\therefore m &= -\frac{1}{3} \\
\theta &= \tan^{-1}\left(-\frac{1}{3}\right) \\
\therefore \theta &= -18{,}4^\circ \\
\therefore \theta &= 180^\circ - 18{,}4^\circ \\
\therefore \theta &= 161{,}6^\circ
\end{aligned}
$$

j) Gradient undefined

2. Determine the acute angle between the line passing through the points $A(-2; \frac{1}{5})$ and $B(0; 1)$ and the line passing through the points $C(1; 0)$ and $D(-2; 6)$.

Solution:

Let the angle of inclination for the line AB be β and let the angle of inclination for the line CD be α. Let the angle between the two lines be θ:

$$
\begin{aligned}
m_{AB} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{1 - \frac{1}{5}}{0 + 2} \\
&= \frac{\frac{4}{5}}{2} \\
\therefore m &= \frac{4}{10} \\
&= \frac{2}{5} \\
\beta &= \tan^{-1}\left(\frac{2}{5}\right) \\
\therefore \beta &= 21{,}8^\circ \\
m_{CD} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{6 - 0}{-2 - 1} \\
&= \frac{6}{-3} \\
\therefore m &= -2 \\
\alpha &= \tan^{-1}(-2) \\
\therefore \alpha &= -63{,}4^\circ \\
\therefore \alpha &= 180^\circ - 63{,}4^\circ \\
\therefore \alpha &= 116{,}6^\circ \\
\text{And } \theta &= \beta + (180^\circ - \alpha) \quad \text{(ext. } \angle\triangle) \\
\therefore \theta &= 21{,}8^\circ + (180^\circ - 116{,}6^\circ) \\
&= 85{,}2^\circ
\end{aligned}
$$

3. Determine the angle between the line $y + x = 3$ and the line $x = y + \frac{1}{2}$.

Solution:

Let the angle of inclination for the line $y + x = 3$ be α and let the angle of inclination for the line $x = y + \frac{1}{2}$ be β. Let the angle between the two lines be θ:

$$
\begin{aligned}
y &= -x + 3 \\
\therefore m &= -1 \\
\alpha &= \tan^{-1}(-1) \\
\therefore \alpha &= -45° \\
\therefore \alpha &= 180° - 45° \\
\therefore \alpha &= 135° \\
x &= y + \frac{1}{2} \\
x - \frac{1}{2} &= y \\
\therefore m &= 1 \\
\beta &= \tan^{-1}(1) \\
\therefore \beta &= 45° \\
\text{And } \theta &= \beta + (180° - \alpha) \quad (\text{ext. } \angle\triangle) \\
\therefore \theta &= 45° + (180° - 135°) \\
&= 90°
\end{aligned}
$$

4. Find the angle between the line $y = 2x$ and the line passing through the points $(-1; \frac{7}{3})$ and $(0; 2)$.

Solution:

Let the angle of inclination for the line $y = 2x$ be β and let the angle of inclination for the other line be α. Let the angle between the two lines be θ:

$$
\begin{aligned}
y &= 2x \\
\therefore m &= 2 \\
\beta &= \tan^{-1}(2) \\
\therefore \beta &= 63{,}4° \\
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{2 - \frac{7}{3}}{0 + 1} \\
&= \frac{-\frac{1}{3}}{1} \\
\therefore m &= -\frac{1}{3} \\
\therefore \alpha &= -18{,}4° \\
\therefore \alpha &= 180° - 18{,}4° \\
\therefore \alpha &= 161{,}6° \\
\text{And } \theta &= \beta + (180° - \alpha) \quad (\text{ext. } \angle\triangle) \\
\therefore \theta &= 63{,}4° + (180° - 161{,}6°) \\
&= 81{,}8°
\end{aligned}
$$

4.4 Parallel lines

Exercise 4 – 7: Parallel lines

1. Determine whether or not the following two lines are parallel:

   a) $y + 2x = 1$ and $-2x + 3 = y$

   b) $\frac{y}{3} + x + 5 = 0$ and $2y + 6x = 1$

   c) $y = 2x - 7$ and the line passing through $(1; -2)$ and $(\frac{1}{2}; -1)$

   d) $y + 1 = x$ and $x + y = 3$

   e) The line passing through points $(-2; -1)$ and $(-4; -3)$ and the line $-y + x - 4 = 0$

   f) $y - 1 = \frac{1}{3}x$ and the line passing through points $(-2; 4)$ and $(1; 5)$

Solution:

a)

$$\begin{aligned} y + 2x &= 1 \\ y &= -2x + 1 \\ \therefore m_1 &= -2 \\ -2x + 3 &= y \\ \therefore m_2 &= -2 \\ \therefore m_1 &= m_2 \end{aligned}$$

$\therefore$Parallel lines.

b)

$$\begin{aligned} \frac{y}{3} + x + 5 &= 0 \\ \frac{y}{3} &= -x - 5 \\ y &= -3x - 15 \\ \therefore m_1 &= -3 \\ 2y + 6x &= 1 \\ 2y &= -6x + 1 \\ y &= -3x + \frac{1}{3} \\ \therefore m_2 &= -3 \\ \therefore m_1 &= m_2 \end{aligned}$$

$\therefore$Parallel lines.

c)

$$\begin{aligned} y &= 2x - 7 \\ \therefore m_1 &= 2 \\ m_2 &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-1 + 2}{\frac{1}{2} - 1} \\ &= \frac{1}{\frac{1}{2}} \\ \therefore m_2 &= 2 \\ \therefore m_1 &= m_2 \end{aligned}$$

∴Parallel lines.

d)

$$\begin{aligned} y + 1 &= x \\ y &= x - 1 \\ \therefore m_1 &= 1 \\ x + y &= 3 \\ y &= -x + 3 \\ \therefore m_2 &= -1 \\ \therefore m_1 &\neq m_2 \end{aligned}$$

∴Not parallel lines.

e)

$$\begin{aligned} -y + x - 4 &= 0 \\ y &= x - 4 \\ \therefore m_1 &= 1 \\ m_2 &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-3 + 1}{-4 + 2} \\ &= \frac{-2}{-2} \\ \therefore m_2 &= 1 \\ \therefore m_1 &= m_2 \end{aligned}$$

∴Parallel lines.

f)

$$\begin{aligned}
y - 1 &= \frac{1}{3}x \\
y &= \frac{1}{3}x + 1 \\
\therefore m_1 &= \frac{1}{3} \\
m_2 &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{5 - 4}{1 + 2} \\
&= \frac{1}{3} \\
\therefore m_1 &= m_2
\end{aligned}$$

$\therefore$Parallel lines.

2. Determine the equation of the straight line that passes through the point $(1;-5)$ and is parallel to the line $y + 2x - 1 = 0$.

   Solution:

$$\begin{aligned}
y + 2x - 1 &= 0 \\
y &= -2x + 1 \\
\therefore m &= -2 \\
y - y_1 &= m(x - x_1) \\
y + 5 &= -2(x - 1) \\
y &= -2x + 2 - 5 \\
\therefore y &= -2x - 3
\end{aligned}$$

3. Determine the equation of the straight line that passes through the point $(-2;-6)$ and is parallel to the line $2y + 1 = 6x$.

   Solution:

$$\begin{aligned}
2y + 1 &= 6x \\
2y &= 6x - 1 \\
y &= 3x - \frac{1}{2} \\
\therefore m &= 3 \\
y - y_1 &= m(x - x_1) \\
y + 6 &= 3(x + 2) \\
y &= 3x + 6 - 6 \\
\therefore y &= 3x
\end{aligned}$$

4. Determine the equation of the straight line that passes through the point $(-2;-2)$ and is parallel to the line with angle of inclination $\theta = 56{,}31^\circ$.

   Solution:

$$\theta = 56{,}31^\circ$$
$$\therefore m = \tan\theta$$
$$= \tan 56{,}31^\circ$$
$$\therefore m = 1{,}5$$
$$y - y_1 = m(x - x_1)$$
$$y + 2 = \frac{3}{2}(x+2)$$
$$y = \frac{3}{2}x + 3 - 2$$
$$\therefore y = \frac{3}{2}x + 1$$

5. Determine the equation of the straight line that passes through the point $(-2; \frac{2}{5})$ and is parallel to the line with angle of inclination $\theta = 145^\circ$.

Solution:

$$\theta = 145^\circ$$
$$\therefore m = \tan\theta$$
$$= \tan 145^\circ$$
$$\therefore m = -0{,}7$$
$$y - y_1 = m(x - x_1)$$
$$y - \frac{2}{5} = -\frac{7}{10}(x+2)$$
$$y = -\frac{7}{10}x - \frac{7}{5} + \frac{2}{5}$$
$$\therefore y = -\frac{7}{10}x - 1$$

4.5 Perpendicular lines

Exercise 4 – 8: Perpendicular lines

1. Calculate whether or not the following two lines are perpendicular:

   a) $y - 1 = 4x$ and $4y + x + 2 = 0$
   b) $10x = 5y - 1$ and $5y - x - 10 = 0$
   c) $x = y - 5$ and the line passing through $(-1; \frac{5}{4})$ and $(3; -\frac{11}{4})$
   d) $y = 2$ and $x = 1$
   e) $\frac{y}{3} = x$ and $3y + x = 9$
   f) $1 - 2x = y$ and the line passing through $(2; -1)$ and $(-1; 5)$
   g) $y = x + 2$ and $2y + 1 = 2x$

Solution:

a)

$$\begin{aligned} y - 1 &= 4x \\ y &= 4x + 1 \\ \therefore m_1 &= 4 \\ 4y + x + 2 &= 0 \\ 4y &= -x - 2 \\ y &= -\frac{1}{4}x - \frac{1}{2} \\ \therefore m_2 &= -\frac{1}{4} \\ \therefore m_1 \times m_2 &= -\frac{1}{4} \times 4 \\ &= -1 \end{aligned}$$

$\therefore$Perpendicular lines.

b)

$$\begin{aligned} 10x &= 5y - 1 \\ 10x + 1 &= 5y \\ 2x + \frac{1}{5} &= y \\ \therefore m_1 &= 2 \\ 5y - x - 10 &= 0 \\ 5y &= -x + 10 \\ y &= -\frac{1}{5}x + 2 \\ \therefore m_2 &= -\frac{1}{5} \\ \therefore m_1 \times m_2 &= -\frac{1}{5} \times 2 \\ &\neq -1 \end{aligned}$$

$\therefore$Not perpendicular lines.

c)

$$\begin{aligned} x &= y - 5 \\ x + 5 &= y \\ \therefore m_1 &= 1 \\ m_2 &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-\frac{11}{4} - \frac{5}{4}}{3 + 1} \\ &= \frac{-4}{4} \\ \therefore m_2 &= -1 \\ \therefore m_1 \times m_2 &= -1 \times 1 \\ &= -1 \end{aligned}$$

$\therefore$Perpendicular lines.

d) Perpendicular: horizontal and vertical lines.

e)

$$\begin{aligned}
\frac{y}{3} &= x \\
y &= 3x \\
\therefore m_1 &= 3 \\
3y + x &= 9 \\
3y &= -x + 9 \\
y &= -\frac{1}{3}x + 3 \\
\therefore m_2 &= -\frac{1}{3} \\
\therefore m_1 \times m_2 &= -\frac{1}{3} \times 3 \\
&= -1
\end{aligned}$$

$\therefore$Perpendicular lines.

f)

$$\begin{aligned}
1 - 2x &= y \\
\therefore m_1 &= -2 \\
m_2 &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{5 + 1}{-1 - 2} \\
&= \frac{6}{-3} \\
\therefore m_2 &= -2 \\
\therefore m_1 \times m_2 &= -2 \times -2 \\
&\neq -1
\end{aligned}$$

$\therefore$Not perpendicular lines.

g)

$$\begin{aligned}
y &= x + 2 \\
\therefore m_1 &= 1 \\
2y + 1 &= 2x \\
2y &= 2x - 1 \\
y &= x - \frac{1}{2} \\
\therefore m_2 &= 1 \\
\therefore m_1 \times m_2 &= 1 \times 1 \\
&\neq -1
\end{aligned}$$

$\therefore$Not perpendicular lines.

2. Determine the equation of the straight line that passes through the point $(-2; -4)$ and is perpendicular to the line $y + 2x = 1$.

Solution:

$$y = -2x + 1$$
$$\therefore m_1 = -2$$
$$\text{For } \perp: m_1 \times m_2 = -1$$
$$-2 \times m_2 = -1$$
$$\therefore m_2 = \frac{1}{2}$$
$$y = mx + c$$
$$y = \frac{1}{2}x + c$$
$$\text{Subst. } (-2; -4): \quad -4 = \frac{1}{2}(-2) + c$$
$$-4 = -1 + c$$
$$\therefore c = -3$$
$$\therefore y = \frac{1}{2}x - 3$$

3. Determine the equation of the straight line that passes through the point $(2; -7)$ and is perpendicular to the line $5y - x = 0$.

Solution:

$$5y - x = 0$$
$$5y = x$$
$$y = \frac{1}{5}x$$
$$\therefore m_1 = \frac{1}{5}$$
$$\text{For } \perp: m_1 \times m_2 = -1$$
$$\frac{1}{5} \times m_2 = -1$$
$$\therefore m_2 = -5$$
$$y = mx + c$$
$$y = -5x + c$$
$$\text{Subst. } (2; 7): \quad -7 = -5(2) + c$$
$$-7 = -10 + c$$
$$\therefore c = 3$$
$$\therefore y = -5x + 3$$

4. Determine the equation of the straight line that passes through the point $(3; -1)$ and is perpendicular to the line with angle of inclination $\theta = 135°$.

Solution:

$$
\begin{aligned}
\theta &= 135^\circ \\
\therefore m &= \tan\theta \\
&= \tan 135^\circ \\
\therefore m &= -1 \\
y &= mx + c \\
y &= -x + c \\
\text{Subst. } (3;-1): \quad -1 &= -(3) + c \\
\therefore c &= 2 \\
\therefore y &= -x + 2
\end{aligned}
$$

5. Determine the equation of the straight line that passes through the point $(-2; \frac{2}{5})$ and is perpendicular to the line $y = \frac{4}{3}$.

   Solution:

   $x = -2$

4.6 Summary

Exercise 4 – 9: End of chapter exercises

1. Determine the equation of the line:

   a) through points $(-1; 3)$ and $(1; 4)$
   b) through points $(7; -3)$ and $(0; 4)$
   c) parallel to $y = \frac{1}{2}x + 3$ and passing through $(-2; 3)$
   d) perpendicular to $y = -\frac{1}{2}x + 3$ and passing through $(-1; 2)$
   e) perpendicular to $3y + x = 6$ and passing through the origin

   Solution:

   a)

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{4-3}{1+1} \\
\therefore m &= \frac{1}{2} \\
y &= mx + c \\
\therefore y &= \frac{1}{2}x + c \\
\text{Subst. } (1;4): \quad 4 &= \frac{1}{2}(1) + c \\
\therefore c &= 3\frac{1}{2} \\
\therefore y &= \frac{1}{2}x + \frac{7}{2}
\end{aligned}
$$

b)

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{4+3}{0-7} \\
&= \frac{7}{-7} \\
\therefore m &= -1 \\
y &= mx + c \\
\therefore y &= -x + c \\
\text{Subst. } (0;4): \quad 4 &= -1(0) + c \\
\therefore c &= 4 \\
\therefore y &= -x + 4
\end{aligned}
$$

c)

$$
\begin{aligned}
\therefore m &= \frac{1}{2} \\
y &= mx + c \\
\therefore y &= \frac{1}{2}x + c \\
\text{Subst. } (-2;3): \quad 3 &= \frac{1}{2}(-2) + c \\
\therefore c &= 4 \\
\therefore y &= \frac{1}{2}x + 4
\end{aligned}
$$

d)

$$
\begin{aligned}
\therefore m &= 2 \\
y &= mx + c \\
\therefore y &= 2x + c \\
\text{Subst. } (-1;2): \quad 2 &= 2(-1) + c \\
\therefore c &= 4 \\
\therefore y &= 2x + 4
\end{aligned}
$$

e)

$$
\begin{aligned}
3y + x &= 6 \\
y &= -\frac{1}{3}x + 2 \\
\therefore m &= 3 \\
y &= mx + c \\
\therefore y &= 3x + c \\
\text{Subst. } (0;0): \quad 0 &= 3(0) + c \\
\therefore c &= 0 \\
\therefore y &= 3x
\end{aligned}
$$

2. Determine the angle of inclination of the following lines:

   a) $y = 2x - 3$

b) $y = \frac{1}{3}x - 7$

c) $4y = 3x + 8$

d) $y = -\frac{2}{3}x + 3$

e) $3y + x - 3 = 0$

Solution:

a)

$$\begin{aligned} \therefore m &= 2 \\ \tan\theta &= m \\ \tan\theta &= 2 \\ \therefore \theta &= \tan^{-1} 2 \\ \theta &= 63{,}4^\circ \end{aligned}$$

b)

$$\begin{aligned} \therefore m &= \frac{1}{3} \\ \tan\theta &= m \\ \tan\theta &= \frac{1}{3} \\ \therefore \theta &= \tan^{-1}\left(\frac{1}{3}\right) \\ \theta &= 18{,}4^\circ \end{aligned}$$

c)

$$\begin{aligned} 4y &= 3x + 8 \\ y &= \frac{3}{4}x + 2 \\ \therefore m &= \frac{3}{4} \\ \tan\theta &= m \\ \tan\theta &= \frac{3}{4} \\ \therefore \theta &= \tan^{-1}\left(\frac{3}{4}\right) \\ \theta &= 36{,}9^\circ \end{aligned}$$

d)

$$\begin{aligned} \therefore m &= -\frac{2}{3} \\ \tan\theta &= m \\ \tan\theta &= -\frac{2}{3} \\ \therefore \theta &= \tan^{-1}\left(-\frac{2}{3}\right) \\ \theta &= -33^\circ + 180^\circ \\ \therefore \theta &= 146{,}3^\circ \end{aligned}$$

e)

$$\begin{aligned}
3y + x - 3 &= 0 \\
y &= -\frac{1}{3} + 1 \\
\therefore m &= -\frac{1}{3} \\
\tan\theta &= m \\
\tan\theta &= -\frac{1}{3} \\
\therefore \theta &= \tan^{-1}\left(-\frac{1}{3}\right) \\
\theta &= -18{,}4^\circ + 180^\circ \\
\therefore \theta &= 161{,}6^\circ
\end{aligned}$$

3. $P(2;3)$, $Q(-4;0)$ and $R(5;-3)$ are the vertices of $\triangle PQR$ in the Cartesian plane. PR intersects the x-axis at S. Determine the following:

   a) the equation of the line PR
   b) the coordinates of point S
   c) the angle of inclination of PR (correct to two decimal places)
   d) the gradient of line PQ
   e) $Q\hat{P}R$
   f) the equation of the line perpendicular to PQ and passing through the origin
   g) the mid-point M of QR
   h) the equation of the line parallel to PR and passing through point M

Solution:

a)

$$\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-3 - 3}{5 - 2} \\
&= \frac{-6}{3} \\
\therefore m &= -2 \\
y &= mx + c \\
\therefore y &= -2x + c \\
\text{Subst. } (2;3): \quad 3 &= -2(2) + c \\
\therefore c &= 7 \\
\therefore y &= -2x + 7
\end{aligned}$$

b)

$$\begin{aligned}
y &= -2x + 7 \\
0 &= -2x + 7 \\
\therefore x &= \frac{7}{2} \\
\therefore S&\left(\frac{7}{2}; 0\right)
\end{aligned}$$

c)

$$\begin{aligned} \therefore m &= -2 \\ \tan\theta &= m \\ \tan\theta &= -2 \\ \therefore \theta &= \tan^{-1}(-2) \\ \theta &= -63{,}4^\circ + 180^\circ \\ \therefore \theta &= 116{,}6^\circ \end{aligned}$$

d)

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3-0}{2+4} \\ &= \frac{3}{6} \\ \therefore m &= \frac{1}{2} \end{aligned}$$

e) $Q\hat{P}R = 90^\circ$

f)

$$\begin{aligned} m_{PQ} &= \frac{1}{2} \\ \therefore m_\perp &= -2 \\ y &= -2x + c \\ c &= 0 \\ \therefore y &= -2x \end{aligned}$$

g)

$$\begin{aligned} M(x;y) &= \left(\frac{x_1+x_2}{2}; \frac{y_1+y_2}{2}\right) \\ &= \left(\frac{-4+5}{2}; \frac{0-3}{2}\right) \\ &= \left(\frac{1}{2}; -\frac{3}{2}\right) \end{aligned}$$

h)

$$\begin{aligned} m &= -2 \\ y &= mx + c \\ y &= -2x + c \\ \text{Subst. } \left(\frac{1}{2}; -\frac{3}{2}\right): \quad -\frac{3}{2} &= -2\left(\frac{1}{2}\right) + c \\ c &= -\frac{1}{2} \\ y &= -2x - \frac{1}{2} \end{aligned}$$

4. Points $A(-3;5)$, $B(-7;-4)$ and $C(2;0)$ are given.

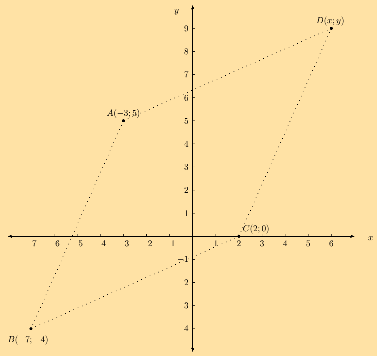
   a) Plot the points on the Cartesian plane.

b) Determine the coordinates of D if $ABCD$ is a parallelogram.

c) Prove that $ABCD$ is a rhombus.

Solution:

a)



b)

$$
\begin{aligned}
m_{BC} &= \frac{-4-0}{-7-2} \\
&= \frac{4}{9} \\
\therefore m_{AD} &= \frac{4}{9}
\end{aligned}
$$

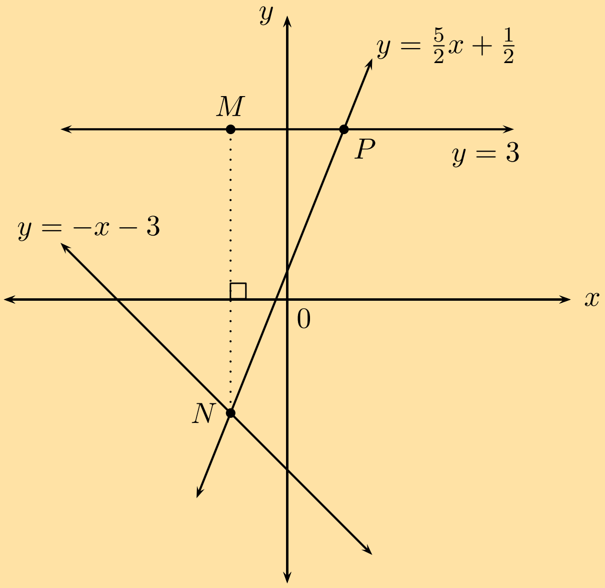
$\therefore$ from $A(-3;5)$: 9 units to the right and 4 units up

c)

$$
\begin{aligned}
AB &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
&= \sqrt{(-7-(-3))^2+(-4-5)^2} \\
&= \sqrt{(-4)^2+(-9)^2} \\
&= \sqrt{16+81} \\
&= \sqrt{97} \\
AD &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
&= \sqrt{(-3-6)^2+(5-9)^2} \\
&= \sqrt{(-9)^2+(-4)^2} \\
&= \sqrt{81+16} \\
&= \sqrt{97} \\
\therefore AB &= AD
\end{aligned}
$$

$\therefore$ parallelogram $ABCD$ is a rhombus (adj. sides equal)

5.



Consider the sketch above, with the following lines shown:

$y = -x - 3$

$y = 3$

$y = \frac{5}{2}x + \frac{1}{2}$

a) Determine the coordinates of the point N.
b) Determine the coordinates of the point P.
c) Determine the equation of the vertical line MN.
d) Determine the length of the vertical line MN.
e) Find $M\hat{N}P$.
f) Determine the equation of the line parallel to NP and passing through the point M.

Solution:

a)

$$\begin{aligned}\frac{5}{2}x + \frac{1}{2} &= -x - 3 \\ \frac{5}{2}x + x &= -3 - \frac{1}{2} \\ \frac{7}{2}x &= -\frac{7}{2} \\ \therefore x &= -1 \\ \therefore y &= -(-1) - 3 \\ &= -2 \\ \therefore N & (-1; -2)\end{aligned}$$

b)

$$\begin{aligned}y &= \frac{5}{2}x + \frac{1}{2} \\ y &= \frac{5}{2}(3) + \frac{1}{2} \\ &= 8 \\ \therefore P & (8; 3)\end{aligned}$$

c) $x = -1$

d)

$$\begin{aligned} MN &= 2 + 3 \\ &= 5 \text{ units} \end{aligned}$$

e)

$$\begin{aligned} \text{Let } M\hat{P}N &= \theta \\ y &= \frac{5}{2}x + \frac{1}{2} \\ \tan\theta &= m \\ \tan\theta &= \frac{5}{2} \\ \therefore \theta &= 68{,}2^\circ \\ \text{In } \triangle MNP \quad M\hat{N}P &= 180^\circ - 90^\circ - 68{,}2^\circ \\ \therefore M\hat{N}P &= 21{,}8^\circ \end{aligned}$$

f)

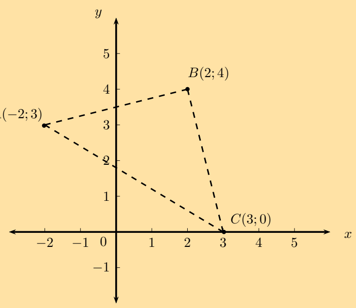
$$\begin{aligned} m &= \frac{5}{2} \\ y &= mx + c \\ y &= \frac{5}{2}x + c \\ \text{Subst. } M(-1;3): \quad 3 &= \frac{5}{2}(-1) + c \\ \therefore c &= 3 + \frac{5}{2} \\ &= \frac{11}{2} \\ \therefore y &= \frac{5}{2}x + \frac{11}{2} \end{aligned}$$

6. The following points are given: $A(-2;3)$, $B(2;4)$, $C(3;0)$.

    a) Plot the points on the Cartesian plane.
    b) Prove that $\triangle ABC$ is a right-angled isosceles triangle.
    c) Determine the equation of the line AB.
    d) Determine the coordinates of D if $ABCD$ is a square.
    e) Determine the coordinates of E, the mid-point of BC.

Solution:

a)



b)

$$
\begin{aligned}
BC &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(3-2)^2 + (0-4)^2} \\
&= \sqrt{(1)^2 + (4)^2} \\
&= \sqrt{1+16} \\
&= \sqrt{17} \\
AB &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(2+2)^2 + (4-3)^2} \\
&= \sqrt{(4)^2 + (1)^2} \\
&= \sqrt{16+1} \\
&= \sqrt{17} \\
\therefore BC &= AB \\
\therefore \triangle ABC & \text{ is isosceles triangle} \\
AC &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(3+2)^2 + (0-3)^2} \\
&= \sqrt{(5)^2 + (-3)^2} \\
&= \sqrt{25+9} \\
&= \sqrt{34} \\
\text{And } BC^2 + AB^2 &= 17 + 17 \\
&= 34 \\
&= AC^2 \\
\therefore \triangle ABC & \text{ is right-angled triangle}
\end{aligned}
$$

c)

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{4-3}{2+2} \\
&= \frac{1}{4} \\
y &= mx + c \\
y &= \frac{1}{4}x + c \\
\text{Subst. } M(-2;3): \quad 3 &= \frac{1}{4}(-2) + c \\
\therefore c &= \frac{7}{2} \\
\therefore y &= \frac{1}{4}x + \frac{7}{2}
\end{aligned}
$$

d) $D(-1;-1)$

e)

$$\begin{aligned} E(x;y) &= \left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right) \\ &= \left(\frac{2+3}{2};\frac{4+0}{2}\right) \\ &= \left(\frac{5}{2};2\right) \end{aligned}$$

7. Given points $S(2;5)$, $T(-3;-4)$ and $V(4;-2)$.

   a) Determine the equation of the line ST.
   b) Determine the size of $T\hat{S}V$.

Solution:

a)

$$\begin{aligned} m &= \frac{y_2-y_1}{x_2-x_1} \\ &= \frac{-4-5}{-3-2} \\ &= \frac{-9}{-6} \\ &= \frac{3}{2} \\ y &= mx+c \\ y &= \frac{3}{2}x+c \\ \text{Subst. } M(2;5): \quad 5 &= \frac{3}{2}(2)+c \\ \therefore c &= 2 \\ \therefore y &= \frac{3}{2}x+2 \end{aligned}$$

b)

$$\begin{aligned} m_{ST} &= \frac{3}{2} \\ \tan\beta &= \frac{3}{2} \\ \therefore \beta &= \tan^{-1}\left(\frac{3}{2}\right) \\ &= 49{,}6^\circ \\ m_{SV} &= \frac{5+2}{2-4} \\ &= \frac{7}{-2} \\ \tan\theta &= -\frac{7}{2} \\ \therefore \theta &= \tan^{-1}\left(-\frac{7}{2}\right) \\ &= -74{,}1^\circ + 180^\circ \\ &= 105{,}9^\circ \\ \therefore T\hat{S}V &= 105{,}9^\circ - 56{,}3^\circ \\ &= 49{,}6^\circ \end{aligned}$$

8. Consider triangle FGH with vertices $F(-1;3)$, $G(2;1)$ and $H(4;4)$.

a) Sketch $\triangle FGH$ on the Cartesian plane.

b) Show that $\triangle FGH$ is an isosceles triangle.

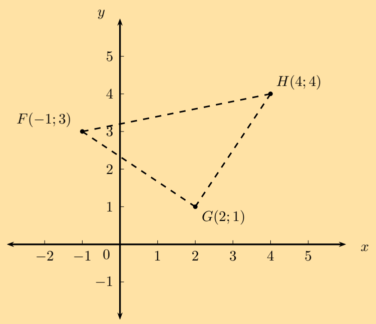
c) Determine the equation of the line PQ, perpendicular bisector of FH.

d) Does G lie on the line PQ?

e) Determine the equation of the line parallel to GH and passing through point F.

Solution:

a)



b)

$$\begin{aligned} FG &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\ &= \sqrt{(-1-2)^2+(3-1)^2} \\ &= \sqrt{(-3)^2+(2)^2} \\ &= \sqrt{9+4} \\ &= \sqrt{13} \\ GH &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\ &= \sqrt{(4-2)^2+(4-1)^2} \\ &= \sqrt{(2)^2+(3)^2} \\ &= \sqrt{4+9} \\ &= \sqrt{13} \\ \therefore FG &= GH \end{aligned}$$

$\therefore \triangle FGH$ is isosceles triangle

c)

$$
\begin{aligned}
Q(x;y) &= \left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right)\\
&= \left(\frac{4-1}{2};\frac{4+3}{2}\right)\\
&= \left(\frac{3}{2};\frac{7}{2}\right)\\
m &= \frac{y_2-y_1}{x_2-x_1}\\
&= \frac{4-3}{4+1}\\
&= \frac{1}{5}\\
\therefore m_\perp &= -5\\
y &= mx+c\\
y &= -5x+c\\
\text{Subst. } Q\left(\frac{3}{2};\frac{7}{2}\right): \quad \frac{7}{2} &= -5\left(\frac{3}{2}\right)+c\\
\therefore c &= 11\\
\therefore y &= -5x+11
\end{aligned}
$$

d)

$$
\begin{aligned}
y &= -5x+11\\
G(2;1) \therefore \text{subst. } x=2: \quad y &= -5\,(2)+11\\
&= -10+11\\
&= 1
\end{aligned}
$$

Yes, G lies on PQ

e)

$$
\begin{aligned}
m_{GH} &= \frac{y_2-y_1}{x_2-x_1}\\
&= \frac{4-1}{4-2}\\
&= \frac{3}{2}\\
y &= mx+c\\
y &= \frac{3}{2}x+c\\
\text{Subst. } F\,(-1;3): \quad 3 &= \frac{3}{2}(-1)+c\\
\therefore c &= \frac{9}{2}\\
\therefore y &= \frac{3}{2}x+\frac{9}{2}
\end{aligned}
$$

9. Given the points $A(-1;5)$, $B(5;-3)$ and $C(0;-6)$. M is the mid-point of AB and N is the mid-point of AC.

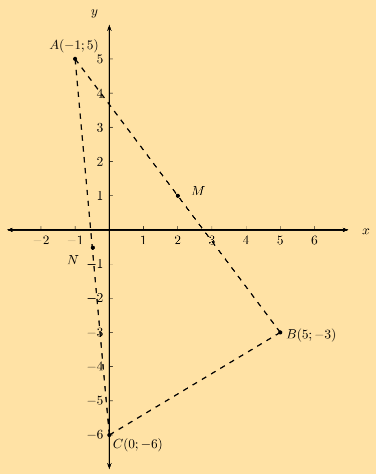
   a) Draw a sketch on the Cartesian plane.

   b) Show that the coordinates of M and N are $(2;1)$ and $(-\frac{1}{2};-\frac{1}{2})$ respectively.

c) Use analytical geometry methods to prove the mid-point theorem. (Prove that $NM \parallel CB$ and $NM = \frac{1}{2}CB$.)

Solution:

a)



b)

$$\begin{aligned} M(x;y) &= \left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right) \\ &= \left(\frac{-1+5}{2};\frac{5-3}{2}\right) \\ &= \left(\frac{4}{2};\frac{2}{2}\right) \\ &= (2;1) \\ N(x;y) &= \left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right) \\ &= \left(\frac{-1+0}{2};\frac{5-6}{2}\right) \\ &= \left(-\frac{1}{2};-\frac{1}{2}\right) \end{aligned}$$

c)

$$
\begin{aligned}
m_{NM} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{1 + \frac{1}{2}}{2 + \frac{1}{2}} \\
&= \frac{\frac{3}{2}}{\frac{5}{2}} \\
&= \frac{3}{5} \\
m_{CB} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-3 + 6}{5 - 0} \\
&= \frac{3}{5} \\
m_{NM} &= m_{CB} \\
\therefore NM &\parallel CB \\
NM &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(2 + \frac{1}{2})^2 + (1 + \frac{1}{2})^2} \\
&= \sqrt{(\frac{5}{2})^2 + (\frac{3}{2})^2} \\
&= \sqrt{\frac{25}{4} + \frac{9}{4}} \\
&= \sqrt{\frac{34}{4}} \\
&= \frac{1}{2}\sqrt{34} \\
CB &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(5 - 0)^2 + (-3 + 6)^2} \\
&= \sqrt{(5)^2 + (3)^2} \\
&= \sqrt{25 + 9} \\
&= \sqrt{34} \\
\therefore NM &= \frac{1}{2}CB
\end{aligned}
$$

CHAPTER 5

Functions

- Learners must be able to determine the equation of a function from a given graph.
- Discuss and explain the characteristics of functions: domain, range, intercepts with the axes, maximum and minimum values, symmetry, etc.
- Emphasize to learners the importance of examining the equation of a function and anticipating the shape of the graph.
- Discuss the different functions and the effects of the parameters in general terms.

5.1 Quadratic functions

Revision

Functions of the form $y = ax^2 + q$

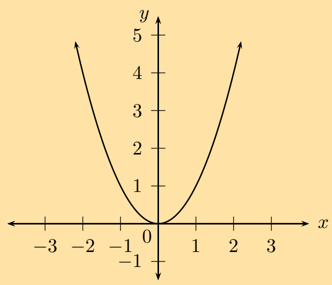
Exercise 5 – 1: Revision

1. On separate axes, accurately draw each of the following functions.
   - Use tables of values if necessary.
   - Use graph paper if available.

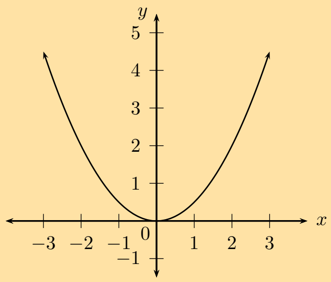
   a) $y_1 = x^2$
   b) $y_2 = \frac{1}{2}x^2$
   c) $y_3 = -x^2 - 1$
   d) $y_4 = -2x^2 + 4$

   Solution:

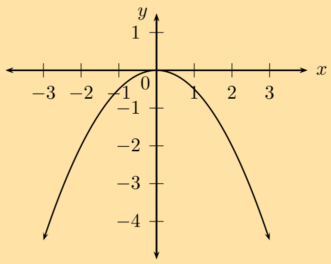
   a)



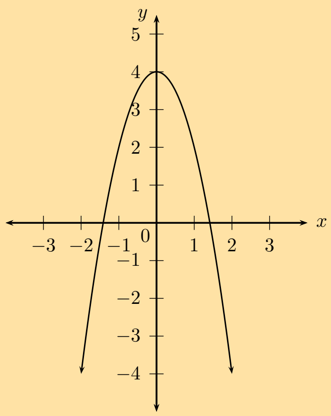
   b)



c)



d)



2. Use your sketches of the functions given above to complete the following table (the first column has been completed as an example):

	y_1	y_2	y_3	y_4
value of q	$q = 0$			
effect of q	$y_{\text{int}} = 0$			
value of a	$a = 1$			
effect of a	standard parabola			
turning point	$(0; 0)$			
axis of symmetry	$x = 0$ (y-axis)			
domain	$\{x : x \in \mathbb{R}\}$			
range	$\{y : y \geq 0\}$			

Solution:

	y_1	y_2	y_3	y_4
value of q	$q = 0$	$q = 0$	$q = -1$	$q = 4$
effect of q	$y_{\text{int}} = 0$	$y_{\text{int}} = 0$	$y_{\text{int}} = -1$, shifts 1 unit down	$y_{\text{int}} = 4$, shifts 4 unit up
value of a	$a = 1$	$a = \frac{1}{2}$	$a = -1$	$a = -2$
effect of a	standard parabola	smile, min. TP	frown, max. TP	frown, max. TP
turning point	$(0; 0)$	$(0; 0)$	$(0; -1)$	$(0; 4)$
axis of symmetry	$x = 0$ (y-axis)	$x = 0$ (y-axis)	$x = 0$ (y-axis)	$x = 0$ (y-axis)
domain	$\{x : x \in \mathbb{R}\}$	$\{x : x \in \mathbb{R}\}$	$\{x : x \in \mathbb{R}\}$	$\{x : x \in \mathbb{R}\}$
range	$\{y : y \geq 0\}$	$\{y : y \geq 0\}$	$\{y : y \leq -1\}$	$\{y : y \leq 4\}$

Functions of the form $y = a(x + p)^2 + q$

Discovering the characteristics

Exercise 5 – 2: Domain and range

Give the domain and range for each of the following functions:

1. $f(x) = (x - 4)^2 - 1$

   Solution:

$$\begin{aligned} \text{Domain: } & \{x : x \in \mathbb{R}\} \\ \text{Range: } & \{y : y \geq -1, y \in \mathbb{R}\} \end{aligned}$$

2. $g(x) = -(x - 5)^2 + 4$

   Solution:

$$\begin{aligned} \text{Domain: } & \{x : x \in \mathbb{R}\} \\ \text{Range: } & \{y : y \leq 4, y \in \mathbb{R}\} \end{aligned}$$

3. $h(x) = x^2 - 6x + 9$

   Solution:

$$\begin{aligned} h(x) &= x^2 - 6x + 9 \\ &= (x - 3)^2 \\ \text{Domain: } & \{x : x \in \mathbb{R}\} \\ \text{Range: } & \{y : y \geq 0, y \in \mathbb{R}\} \end{aligned}$$

4. $j(x) = -2(x+1)^2$

Solution:

$$\begin{aligned}\text{Domain: } & \{x : x \in \mathbb{R}\} \\ \text{Range: } & \{y : y \leq 0, y \in \mathbb{R}\}\end{aligned}$$

5. $k(x) = -x^2 + 2x - 3$

Solution:

$$\begin{aligned}k(x) &= -x^2 + 2x - 3 \\ &= -(x-1)^2 - 2 \\ \text{Domain: } & \{x : x \in \mathbb{R}\} \\ \text{Range: } & \{y : y \leq 2, y \in \mathbb{R}\}\end{aligned}$$

Exercise 5 – 3: Intercepts

Determine the x- and y-intercepts for each of the following functions:

1. $f(x) = (x+4)^2 - 1$

Solution:

$$\begin{aligned}\text{For } x = 0 \quad y &= (0+4)^2 - 1 \\ &= 16 - 1 \\ &\therefore (0; 15) \\ \text{For } y = 0 \quad 0 &= (x+4)^2 - 1 \\ &= x^2 + 8x + 16 - 1 \\ &= x^2 + 8x + 15 \\ &= (x+5)(x+3) \\ x = -5 &\text{ or } x = -3 \\ \therefore (-5; 0) &\text{ and } (-3; 0)\end{aligned}$$

2. $g(x) = 16 - 8x + x^2$

Solution:

$$
\begin{aligned}
\text{For } x = 0 \quad & y = 16 \\
& \therefore (0;16) \\
\text{For } y = 0 \quad & 0 = 16 - 8x + x^2 \\
& = x^2 - 8x + 16 \\
& = (x-4)^2 \\
& \therefore x = 4 \\
& \therefore (4;0)
\end{aligned}
$$

3. $h(x) = -x^2 + 4x - 3$

   Solution:

$$
\begin{aligned}
\text{For } x = 0 \quad & y = -3 \\
& \therefore (0;-3) \\
\text{For } y = 0 \quad & 0 = -x^2 + 4x - 3 \\
& = -(x^2 - 4x + 3) \\
& = -(x-3)(x-1) \\
& x = 3 \text{ or } x = 1 \\
& \therefore (1;0) \text{ and } (3;0)
\end{aligned}
$$

4. $j(x) = 4(x-3)^2 - 1$

   Solution:

$$
\begin{aligned}
\text{For } x = 0 \quad & y = 4(0-3)^2 - 1 \\
& = 36 - 1 \\
& \therefore (0;35) \\
\text{For } y = 0 \quad & 0 = 4(x-3)^2 - 1 \\
& = 4(x^2 - 6x + 9) - 1 \\
& = 4x^2 - 24x + 36 - 1 \\
& = 4x^2 - 36x + 35 \\
& = (2x+5)(2x+7) \\
& x = -\frac{5}{2} \text{ or } x = -\frac{7}{2} \\
& \therefore (-\frac{5}{2};0) \text{ and } (-\frac{7}{2};0)
\end{aligned}
$$

5. $k(x) = 4(x-3)^2 + 1$

   Solution:

$$
\begin{aligned}
\text{For } x = 0 \quad y &= 4(0-3)^2 + 1 \\
&= 36 + 1 \\
&\therefore (0;37) \\
\text{For } y = 0 \quad 0 &= 4(x-3)^2 + 1 \\
&= 4(x^2 - 6x + 9) + 1 \\
&= 4x^2 - 24x + 36 + 1 \\
&= 4x^2 - 36x + 37 \\
\therefore x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{24 \pm \sqrt{(-24)^2 - 4(4)(37)}}{2(4)} \\
&= \frac{576 \pm \sqrt{576 - 592}}{8} \\
&= \frac{576 \pm \sqrt{-16}}{8} \\
&\therefore \text{no real solution}
\end{aligned}
$$

6. $l(x) = 2x^2 - 3x - 4$

Solution:

$$
\begin{aligned}
\text{For } x = 0 \quad y &= -4 \\
&\therefore (0;-4) \\
\text{For } y = 0 \quad 0 &= 2x^2 - 3x - 4 \\
&= 2x^2 - 3x - 4 \\
\therefore x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{3 \pm \sqrt{(-3)^2 - 4(2)(-4)}}{2(2)} \\
&= \frac{3 \pm \sqrt{9 + 32}}{4} \\
&= \frac{3 \pm \sqrt{41}}{4}
\end{aligned}
$$

$$x = \frac{3 - \sqrt{41}}{4} \text{ or } x = \frac{3 + \sqrt{41}}{4}$$

$$\therefore (-0{,}85;0) \text{ and } (2{,}35;0)$$

Exercise 5 – 4: Turning points

Determine the turning point of each of the following:

1. $y = x^2 - 6x + 8$

Solution:

$$\begin{aligned} y &= x^2 - 6x + 8 \\ &= (x-3)^2 - \left(\frac{6}{2}\right)^2 + 8 \\ &= (x-3)^2 - 1 \\ \therefore \text{turning point } &= (3; -1) \end{aligned}$$

2. $y = -x^2 + 4x - 3$

Solution:

$$\begin{aligned} y &= -x^2 + 4x - 3 \\ &= -(x^2 - 4x + 3 \\ &= -\left((x-2)^2 - \left(\frac{4}{2}\right)^2 + 3\right) \\ &= -(x-2)^2 + 1 \\ \therefore \text{turning point } &= (2; 1) \end{aligned}$$

3. $y = \frac{1}{2}(x+2)^2 - 1$

Solution:

$$\begin{aligned} y &= \frac{1}{2}(x+2)^2 - 1 \\ \therefore \text{turning point } &= (-2; -1) \end{aligned}$$

4. $y = 2x^2 + 2x + 1$

Solution:

$$\begin{aligned} y &= 2x^2 + 2x + 1 \\ &= 2\left(x^2 + x + \frac{1}{2}\right) \\ &= 2\left(x + \frac{1}{2}\right)^2 + \frac{1}{2} \\ \therefore \text{turning point } &= (-\frac{1}{2}; \frac{1}{2}) \end{aligned}$$

5. $y = 18 + 6x - 3x^2$

Solution:

$$\begin{aligned} y &= -3x^2 + 6x + 18 \\ &= -3(x^2 - 2x - 6) \\ &= -3\left((x-1)^2 - 7\right) \\ &= -3(x-1)^2 + 21 \\ \therefore \text{turning point } &= (1; 21) \end{aligned}$$

6. $y = -2[(x+1)^2 + 3]$

Solution:

$$\begin{aligned} y &= -2[(x+1)^2 + 3] \\ y &= -2(x+1)^2 - 6 \\ \therefore \text{turning point } &= (-1; -6) \end{aligned}$$

Exercise 5 – 5: Axis of symmetry

1. Determine the axis of symmetry of each of the following:

    a) $y = 2x^2 - 5x - 18$
    b) $y = 3(x-2)^2 + 1$
    c) $y = 4x - x^2$

    Solution:

    a)

    $$\begin{aligned} y &= 2x^2 - 5x - 18 \\ &= 2(x^2 - \frac{5}{2}x - 9) \\ &= 2\left(\left(x - \frac{5}{2}\right)^2 - \frac{25}{16} - 9\right) \\ &= 2\left(\left(x - \frac{5}{2}\right)^2 - \frac{169}{16}\right) \\ &= 2\left(x - \frac{5}{4}\right)^2 - \frac{169}{8} \\ \text{Axis of symmetry: } x &= \frac{5}{4} \end{aligned}$$

    b)

    $$\begin{aligned} y &= 3(x-2)^2 + 1 \\ \text{Axis of symmetry: } x &= 2 \end{aligned}$$

    c)

    $$\begin{aligned} y &= 4x - x^2 \\ &= -(x^2 - 4x) \\ &= -(x-2)^2 + 4 \\ \text{Axis of symmetry: } x &= 2 \end{aligned}$$

2. Write down the equation of a parabola where the y-axis is the axis of symmetry.

    Solution:

    $y = ax^2 + q$

Exercise 5 – 6: Sketching parabolas

1. Sketch graphs of the following functions and determine:
    - intercepts
    - turning point
    - axes of symmetry
    - domain and range

    a) $y = -x^2 + 4x + 5$

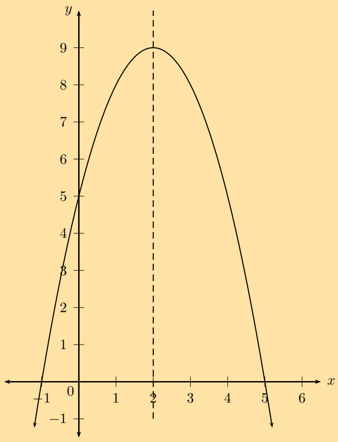
    b) $y = 2(x+1)^2$

    c) $y = 3x^2 - 2(x+2)$

    d) $y = 3(x-2)^2 + 1$

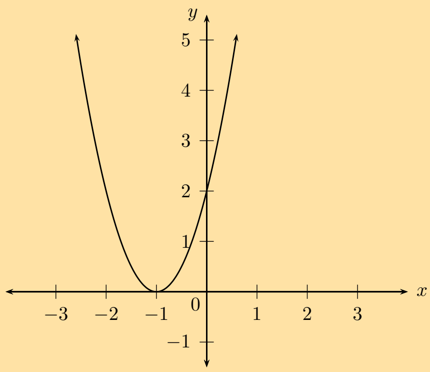
Solution:

a)



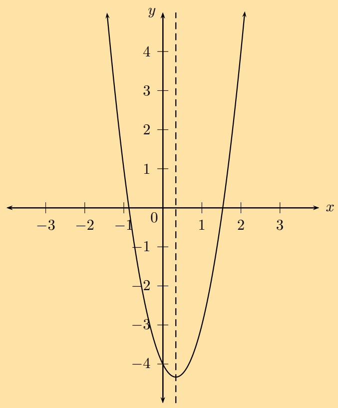
Intercepts: $(-1;0), (5;0), (0;5)$
Turning point: $(2;9)$
Axes of symmetry: $x = 2$
Domain: $\{x : x \in \mathbb{R}\}$
Range: $\{y : y \leq 9, y \in \mathbb{R}\}$

b)



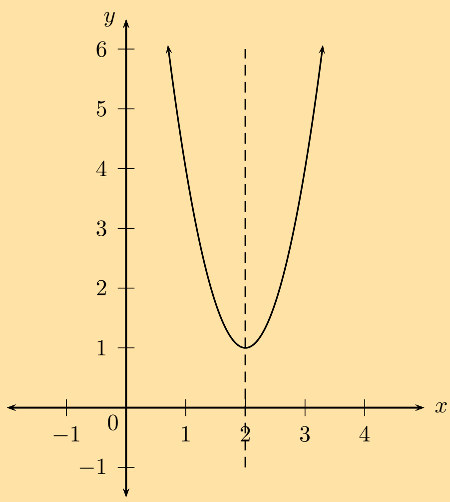
Intercepts: $(-1;0),(0;2)$
Turning point: $(-1;0)$
Axes of symmetry: $x=-1$
Domain: $\{x : x \in \mathbb{R}\}$
Range: $\{y : y \geq 0, y \in \mathbb{R}\}$

c)



Intercepts: $(-0{,}87;0),(1{,}54;0),(0;-4)$
Turning point: $(0{,}33;-4{,}33)$
Axes of symmetry: $x=-0{,}33$
Domain: $\{x : x \in \mathbb{R}\}$
Range: $\{y : y \geq 4{,}33, y \in \mathbb{R}\}$

d)



Intercepts: $(0;13)$
Turning point: $(2;1)$
Axes of symmetry: $x=2$
Domain: $\{x : x \in \mathbb{R}\}$
Range: $\{y : y \geq 1, y \in \mathbb{R}\}$

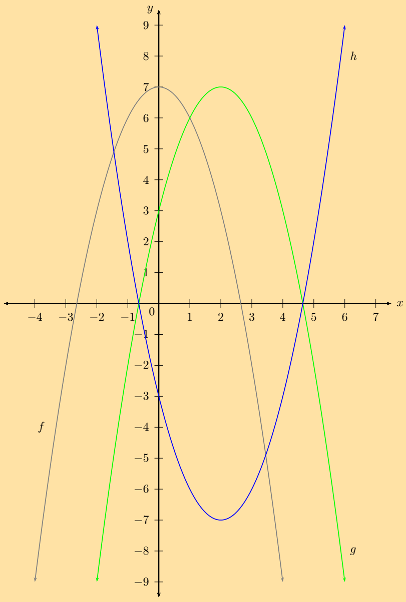
2. Draw the following graphs on the same system of axes:

   $f(x) = -x^2 + 7$

   $g(x) = -(x-2)^2 + 7$

   $h(x) = (x-2)^2 - 7$

   Solution:

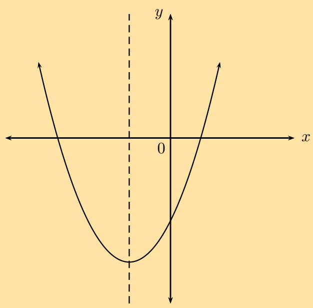


3. Draw a sketch of each of the following graphs:

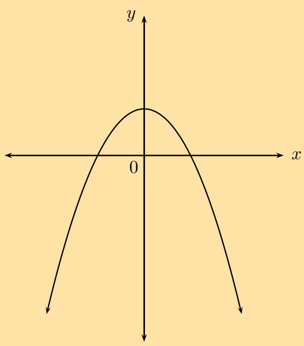
    a) $y = ax^2 + bx + c$ if $a > 0$, $b > 0$, $c < 0$.
    b) $y = ax^2 + bx + c$ if $a < 0$, $b = 0$, $c > 0$.
    c) $y = ax^2 + bx + c$ if $a < 0$, $b < 0$, $b^2 - 4ac < 0$.
    d) $y = (x + p)^2 + q$ if $p < 0$, $q < 0$ and the x-intercepts have different signs.
    e) $y = a(x + p)^2 + q$ if $a < 0$, $p < 0$, $q > 0$ and one root is zero.
    f) $y = a(x + p)^2 + q$ if $a > 0$, $p = 0$, $b^2 - 4ac > 0$.

Solution:

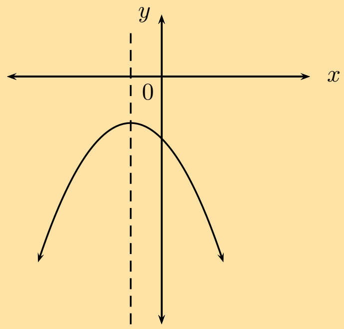
a)



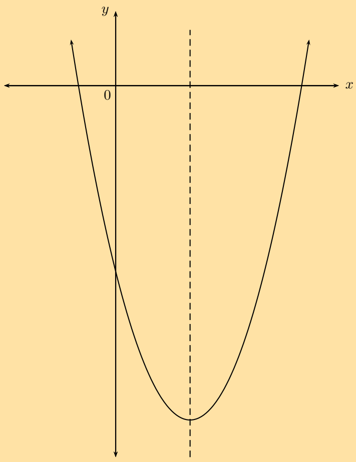
b)



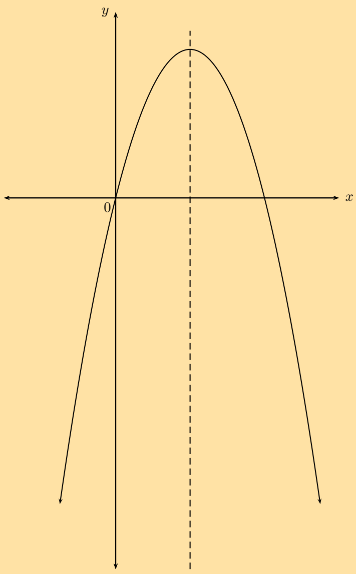
c)



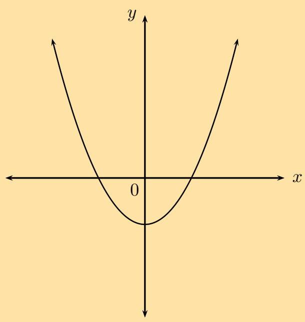
d)



e)



f)



4. Determine the new equation (in the form $y = ax^2 + bx + c$) if:

a) $y = 2x^2 + 4x + 2$ is shifted 3 units to the left.

b) $y = -(x + 1)^2$ is shifted 1 unit up.

c) $y = 3(x - 1)^2 + 2\left(x - \frac{1}{2}\right)$ is shifted 2 units to the right.

Solution:

a)

$$\begin{aligned} y &= 2x^2 + 4x + 2 \\ x &\Rightarrow x + 3 \\ y_{\text{shifted}} &= 2(x + 3)^2 + 4(x + 3) + 2 \\ &= 2(x^2 + 6x + 9) + 4x + 12 + 2 \\ &= 2x^2 + 12x + 18 + 4x + 12 + 2 \\ &= 2x^2 + 16x + 32 \end{aligned}$$

b)

$$\begin{aligned} y &= 2x^2 + 4x + 2 \\ y &\Rightarrow y - 1 \\ y_{\text{shifted}} &= -(x + 1)^2 + 1 \\ &= -(x^2 + 2x + 1) + 1 \\ &= -x^2 - 2x - 1 + 1 \\ &= -x^2 - 2x \end{aligned}$$

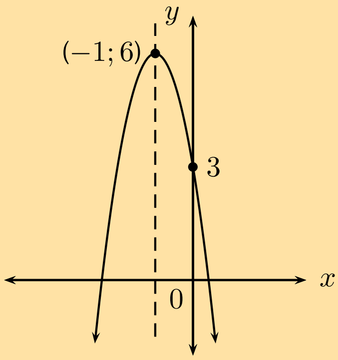
c)

$$\begin{aligned} y &= 3(x - 1)^2 + 2\left(x - \frac{1}{2}\right) \\ x &\Rightarrow x - 2 \\ y_{\text{shifted}} &= 3(x - 2 - 1)^2 + 2\left(x - 2 - \frac{1}{2}\right) \\ &= 3(x - 3)^2 + 2\left(x - \frac{5}{2}\right) \\ &= 3(x^2 - 6x + 9) + 2x - 5 \\ &= 3x^2 - 18x + 27 + 2x - 5 \\ &= 3x^2 - 16x + 22 \end{aligned}$$

Exercise 5 – 7: Finding the equation

Determine the equations of the following graphs. Write your answers in the form $y = a(x + p)^2 + q$.

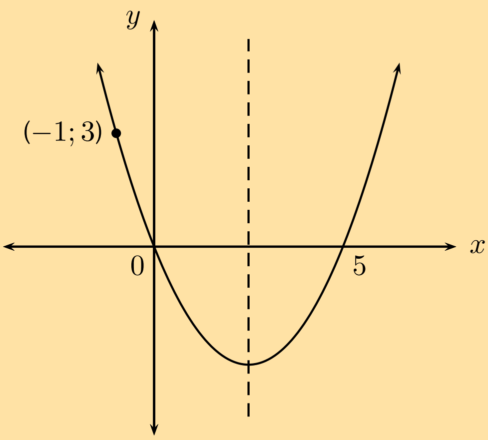
1.



Solution:

$$
\begin{aligned}
y &= a(x+p)^2 + q \\
\text{Subst. } &(-1;6) \\
y &= a(x+1)^2 + 6 \\
&= ax^2 + 2ax + a + 6 \\
y - \text{int: } &= (0;3) \\
3 &= 0 + 0 + a + 6 \\
\therefore 3 &= a + 6 \\
a &= -3 \\
y = -3(x+1)^2 + 6 &\text{ or } y = -3x^2 - 6x + 3
\end{aligned}
$$

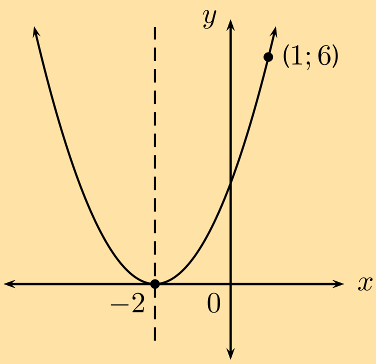
2.



Solution:

$$y = a(x+p)^2 + q$$
$$y = a(x-0)(x-5)$$
$$= ax^2 - 5ax$$
Subst. $(-1;3)$
$$y = ax^2 - 5ax$$
$$3 = a(-1)^2 - 5a(-1)$$
$$3 = a + 5a$$
$$3 = 6a$$
$$\therefore a = \frac{1}{2}$$
$$\therefore y = \frac{1}{2}x^2 - \frac{5}{2}x$$

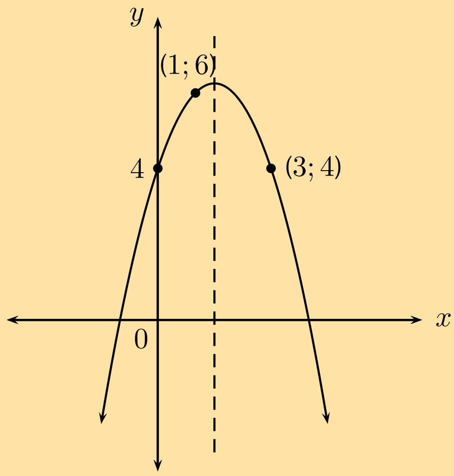
3.



Solution:

$$y = a(x+p)^2 + q$$
Subst. $= (-2;0)$
$$y = a(x+2)^2$$
$$= ax^2 + 4ax + 4a$$
Subst. $(1;6)$
$$6 = a + 4a + 4a$$
$$6 = 9a$$
$$\therefore a = \frac{2}{3}$$
$$\therefore y = \frac{2}{3}(x+2)^2$$

4.



Solution:

$$\begin{aligned}
& y = ax^2 + bx + c \\
& y = ax^2 + bx + 4 \\
\text{Subst. } (1;6):\quad & 6 = a + b + 4 \ldots (1) \\
\text{Eqn. } (1) \times 5:\quad & 30 = 5a + 5b + 20 \ldots (3) \\
\text{Subst. } (1;6):\quad & -6 = 25a + 5b + 4 \ldots (2) \\
(2) - (3)\quad & -36 = 20a - 16 \\
& 20a = -20 \\
& a = -1 \\
& b = 3 \\
& c = 4 \\
& \therefore y = -x^2 + 3x + 4
\end{aligned}$$

5.2 Average gradient

Exercise 5 – 8:

1. a) Determine the average gradient of the curve $f(x) = x(x+3)$ between $x = 5$ and $x = 3$.

   b) Hence, state what you can deduce about the function f between $x = 5$ and $x = 3$.

   Solution:

   a) 11

   b)

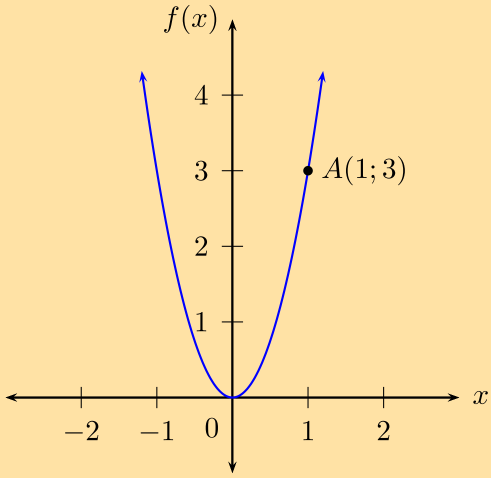
2. $A(1;3)$ is a point on $f(x) = 3x^2$.

   a) Draw a sketch of $f(x)$ and label point A.

b) Determine the gradient of the curve at point A.
c) Determine the equation of the tangent line at A.

Solution:

a)



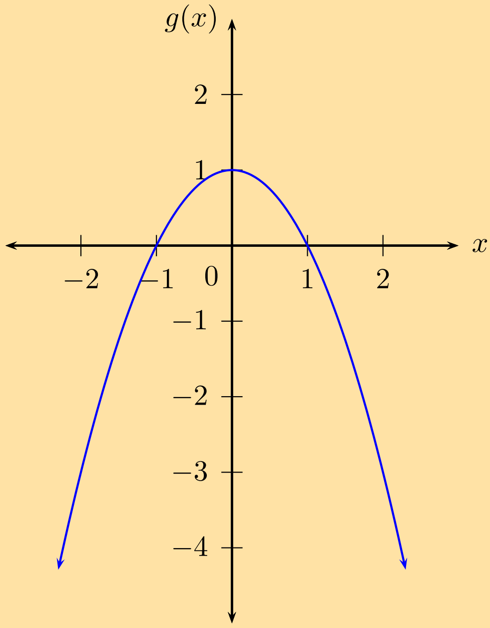
b) 6
c) $y = 6x - 3$

3. Given: $g(x) = -x^2 + 1$.

a) Draw a sketch of $g(x)$.
b) Determine the average gradient of the curve between $x = -2$ and $x = 1$.
c) Determine the gradient of g at $x = 2$.
d) Determine the gradient of g at $x = 0$.

Solution:

a)



b) 1
c) 4
d) 0

5.3 Hyperbolic functions

Revision

Functions of the form $y = \frac{a}{x} + q$

Exercise 5 – 9: Revision

1. Consider the following hyperbolic functions:

   - $y_1 = \frac{1}{x}$
   - $y_2 = -\frac{4}{x}$
   - $y_3 = \frac{4}{x} - 2$
   - $y_4 = -\frac{4}{x} + 1$

   Complete the table to summarise the properties of the hyperbolic function:

	y_1	y_2	y_3	y_4
value of q	$q = 0$			
effect of q	no vertical shift			
value of a	$a = 1$			
effect of a	lies in I and III quad			
asymptotes	y-axis, $x = 0$ x-axis, $y = 0$			
axes of symmetry	$y = x$ $y = -x$			
domain	$\{x : x \in \mathbb{R}, x \neq 0\}$			
range	$\{y : y \in \mathbb{R}, y \neq 0\}$			

Solution:

	y_1	y_2	y_3	y_4
value of q	$q = 0$	$q = 0$	$q = -2$	$q = 1$
effect of q	no vertical shift	no vertical shift	shift 2 units down	shift 1 unit up
value of a	$a = 1$	$a = -1$	$a = 4$	$a = -4$
effect of a	lies in I and III quad	lies in II and IV quad	lies in I and III quad	lies in II and IV quad
asymptotes	y-axis, $x = 0$ x-axis, $y = 0$	y-axis, $x = 0$ x-axis, $y = 0$	y-axis, $x = 0$ $y = -2$	y-axis, $x = 0$ $y = 1$
axes of symmetry	$y = x$ $y = -x$	$y = x$ $y = -x$	$y = x - 2$ $y = -x - 2$	$y = x + 1$ $y = -x + 1$
domain	$\{x : x \in \mathbb{R}, x \neq 0\}$	$\{x : x \in \mathbb{R}, x \neq 0\}$	$\{x : x \in \mathbb{R}, x \neq 0\}$	$\{x : x \in \mathbb{R}, x \neq 0\}$
range	$\{y : y \in \mathbb{R}, y \neq 0\}$	$\{y : y \in \mathbb{R}, y \neq 0\}$	$\{y : y \in \mathbb{R}, y \neq -2\}$	$\{y : y \in \mathbb{R}, y \neq 1\}$

Discovering the characteristics

Exercise 5 – 10: Domain and range

Determine the domain and range for each of the following functions:

1. $y = \frac{1}{x} + 1$

   Solution:

$$\begin{aligned}\text{Domain: } & \{x : x \in \mathbb{R}, x \neq 0\} \\ \text{Range: } & \{y : y \in \mathbb{R}, y \neq 1\}\end{aligned}$$

2. $g(x) = \frac{8}{x-8} + 4$

   Solution:

$$\begin{aligned}\text{Domain: } & \{x : x \in \mathbb{R}, x \neq 8\} \\ \text{Range: } & \{y : y \in \mathbb{R}, y \neq 4\}\end{aligned}$$

3. $y = -\frac{4}{x+1} - 3$

   Solution:

$$\begin{aligned}\text{Domain: } & \{x : x \in \mathbb{R}, x \neq -1\} \\ \text{Range: } & \{y : y \in \mathbb{R}, y \neq -3\}\end{aligned}$$

4. $x = \frac{2}{3-y} + 5$

   Solution:

$$\begin{aligned}x &= \frac{2}{3-y} + 5 \\ x - 5 &= \frac{2}{3-y} \\ (x-5)(3-y) &= 2 \\ 3 - y &= \frac{2}{3x-5} \\ -y &= \frac{2}{x-5} - 3 \\ \therefore y &= -\frac{2}{x-5} + 3 \\ \text{Domain: } & \{x : x \in \mathbb{R}, x \neq 5\} \\ \text{Range: } & \{y : y \in \mathbb{R}, y \neq 3\}\end{aligned}$$

5. $(y-2)(x+2)=3$

Solution:

$$\begin{aligned}(y-2)(x+2)&=3\\ y-2&=\frac{3}{x+2}\\ \therefore y&=\frac{3}{x+2}+2\\ \text{Domain: }&\{x: x\in\mathbb{R}, x\neq -2\}\\ \text{Range: }&\{y: y\in\mathbb{R}, y\neq 2\}\end{aligned}$$

Exercise 5 – 11: Intercepts

Determine the x- and y-intercepts for each of the following functions:

1. $f(x)=\frac{1}{x+4}-2$

Solution:

$$\begin{aligned}f(x)&=\frac{1}{x+4}-2\\ \text{Let } x&=0\\ f(0)&=\frac{1}{4}-2\\ \therefore y_{\text{int}}&=(0;-1\frac{3}{4})\\ \text{Let } y&=0\\ 0&=\frac{1}{x+4}-2\\ 2&=\frac{1}{x+4}\\ 2(x+4)&=1\\ 2x+8&=1\\ 2x&=-7\\ \therefore x&=-\frac{7}{2}\\ \therefore x_{\text{int}}&=\left(-3\frac{1}{2};0\right)\end{aligned}$$

2. $g(x)=-\frac{5}{x}+2$

Solution:

$$
\begin{aligned}
g(x) &= -\frac{5}{x} + 2 \\
\text{Let } x &= 0 \\
\therefore g(x) &\text{is undefined} \\
\therefore \text{no } x &- \text{intercepts} \\
\text{Let } y &= 0 \\
0 &= -\frac{5}{x} + 2 \\
-2 &= -\frac{5}{x} \\
2x &= 5 \\
\therefore x &= \frac{5}{2} \\
\therefore x_{\text{int}} &= \left(\frac{5}{2}; 0\right)
\end{aligned}
$$

3. $j(x) = \frac{2}{x-1} + 3$

Solution:

$$
\begin{aligned}
j(x) &= \frac{2}{x-1} + 3 \\
\text{Let } x &= 0 \\
j(0) &= \frac{2}{-1} + 3 \\
&= 1 \\
\therefore y_{\text{int}} &= (0; 1) \\
\text{Let } y &= 0 \\
0 &= \frac{2}{x-1} + 3 \\
-3 &= \frac{2}{x-1} \\
-3(x-1) &= 2 \\
-3x + 3 &= 2 \\
-3x &= -1 \\
\therefore x &= \frac{1}{3} \\
\therefore x_{\text{int}} &= \left(\frac{1}{3}; 0\right)
\end{aligned}
$$

4. $h(x) = \frac{3}{6-x} + 1$

Solution:

$$h(x) = \frac{3}{6-x} + 1$$
$$\text{Let } x = 0$$
$$h(0) = \frac{3}{6} + 1$$
$$= \frac{3}{2}$$
$$\therefore y_{\text{int}} = \left(0; \frac{3}{2}\right)$$
$$\text{Let } y = 0$$
$$0 = \frac{3}{6-x} + 1$$
$$-1 = \frac{3}{6-x}$$
$$-(6-x) = 3$$
$$-6 + x = 3$$
$$-3x = -1$$
$$\therefore x = \frac{1}{3}$$
$$\therefore x_{\text{int}} = \left(\frac{1}{3}; 0\right)$$

5. $k(x) = \frac{5}{x+2} - \frac{1}{2}$

Solution:

$$k(x) = \frac{5}{x+2} - \frac{1}{2}$$
$$\text{Let } x = 0$$
$$k(0) = \frac{5}{2} - \frac{1}{2}$$
$$= 2$$
$$\therefore y_{\text{int}} = (0; 2)$$
$$\text{Let } y = 0$$
$$0 = \frac{5}{x+2} - \frac{1}{2}$$
$$\frac{1}{2} = \frac{5}{x+2}$$
$$x + 2 = 5(2)$$
$$x = 10 - 2$$
$$\therefore x = 8$$
$$\therefore x_{\text{int}} = (8; 0)$$

Exercise 5 – 12: Asymptotes

Determine the asymptotes for each of the following functions:

1. $y = \frac{1}{x+4} - 2$

   Solution:

   $$\text{Vertical asymptote: } y = -2$$
   $$\text{Horizontal asymptote: } x = -4$$

2. $y = -\frac{5}{x}$

   Solution:

   $$\text{Vertical asymptote: } y = 0$$
   $$\text{Horizontal asymptote: } x = 0$$

3. $y = \frac{3}{2-x} + 1$

   Solution:

   $$\begin{aligned} y &= \frac{3}{2-x} + 1 \\ &= \frac{3}{-(x-2)} + 1 \\ &= -\frac{3}{x-2} + 1 \end{aligned}$$
   $$\text{Vertical asymptote: } y = 1$$
   $$\text{Horizontal asymptote: } x = 2$$

4. $y = \frac{1}{x} - 8$

   Solution:

   $$\text{Vertical asymptote: } y = -8$$
   $$\text{Horizontal asymptote: } x = 0$$

5. $y = -\frac{2}{x-2}$

   Solution:

   $$\text{Vertical asymptote: } y = 0$$
   $$\text{Horizontal asymptote: } x = 2$$

Exercise 5 – 13: Axes of symmetry

1. Complete the following for $f(x)$ and $g(x)$:

    - Sketch the graph.
    - Determine $(-p; q)$.
    - Find the axes of symmetry.

    Compare $f(x)$ and $g(x)$ and also their axes of symmetry. What do you notice?

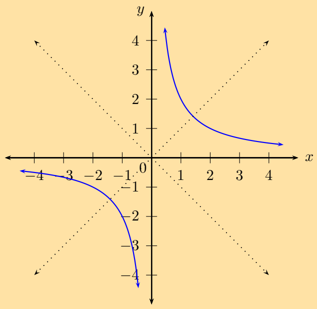
    a) $f(x) = \frac{2}{x}$
    $g(x) = \frac{2}{x} + 1$

    b) $f(x) = -\frac{3}{x}$
    $g(x) = -\frac{3}{x+1}$

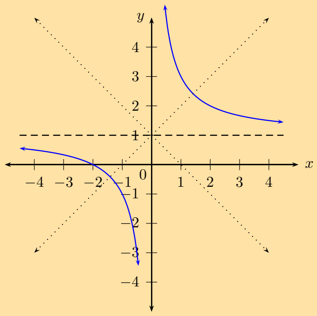
    c) $f(x) = \frac{5}{x}$
    $g(x) = \frac{5}{x-1} - 1$

Solution:

a)

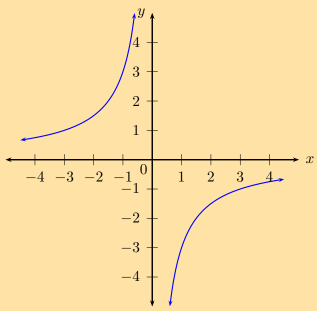


$$(-p; q) = (0; 0)$$
$$y_1 = x$$
$$\therefore y_2 = -x$$

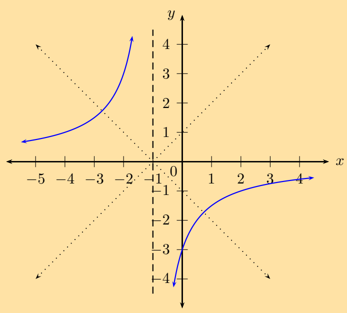


$$
\begin{aligned}
(-p;q) &= (0;1) \\
\text{Let } y_1 &= x + c_1 \\
\text{Subst. } (0;1) \qquad 1 &= 0 + c_1 \\
\therefore c_1 &= 1 \\
\therefore y_1 &= x + 1 \\
\text{Let } y_2 &= -x + c_2 \\
\text{Subst. } (0;1) \qquad 1 &= 0 + c_2 \\
\therefore c_2 &= 1 \\
\therefore y_2 &= -x + 1
\end{aligned}
$$

b)

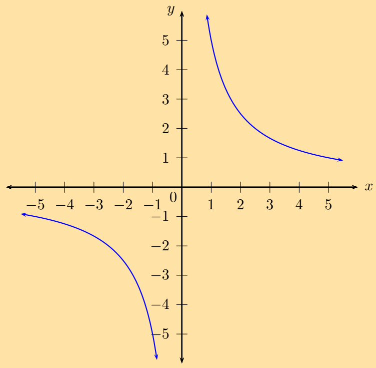


$$
\begin{aligned}
(-p;q) &= (0;0) \\
y_1 &= x \\
\therefore y_2 &= -x
\end{aligned}
$$

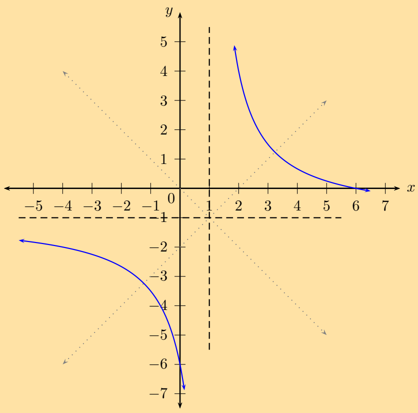


$$
\begin{aligned}
(-p;q) &= (-1;0) \\
\text{Let } y_1 &= x + c_1 \\
\text{Subst. } (-1;0) \qquad 0 &= -1 + c_1 \\
\therefore c_1 &= 1 \\
\therefore y_1 &= x + 1 \\
\text{Let } y_2 &= -x + c_2 \\
\text{Subst. } (-1;0) \qquad 0 &= -(-1) + c_2 \\
\therefore c_2 &= -1 \\
\therefore y_2 &= -x - 1
\end{aligned}
$$

c)



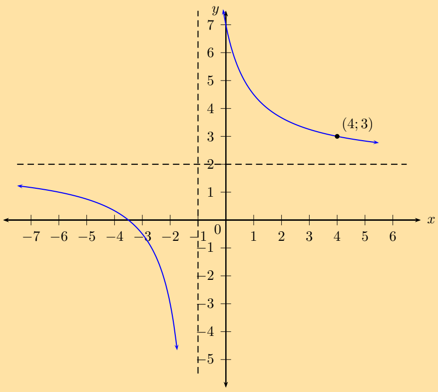
$$(-p;q) = (0;0)$$
$$y_1 = x$$
$$\therefore y_2 = -x$$



$$
\begin{aligned}
(-p;q) &= (1;-1) \\
\text{Let } y_1 &= x + c_1 \\
\text{Subst. } (1;-1) \qquad -1 &= 1 + c_1 \\
\therefore c_1 &= -2 \\
\therefore y_1 &= x - 2 \\
\text{Let } y_2 &= -x + c_2 \\
\text{Subst. } (1;-1) \qquad -1 &= -(1) + c_2 \\
\therefore c_2 &= 0 \\
\therefore y_2 &= -x
\end{aligned}
$$

2. A hyperbola of the form $k(x) = \frac{a}{x+p} + q$ passes through the point $(4;3)$. If the axes of symmetry intersect at $(-1;2)$, determine the equation of $k(x)$.

Solution:



$$\begin{aligned}
\text{Given } k(x) &= \frac{a}{x+p} + q \\
\text{and } (-p;q) &= (-1;-2) \\
\therefore &= \frac{a}{x+1} + 2 \\
\text{Subst. } (4;3) \qquad 3 &= \frac{a}{4+1} + 2 \\
3 - 2 &= \frac{a}{5} \\
\therefore 1(5) &= a \\
\therefore a &= 5 \\
\therefore k(x) &= \frac{5}{x+1} + 2
\end{aligned}$$

Sketching graphs of the form $f(x) = \frac{a}{x+p} + q$

Exercise 5 – 14: Sketching graphs

1. Draw the graphs of the following functions and indicate:

   - asymptotes
   - intercepts, where applicable
   - axes of symmetry
   - domain and range

   a) $y = \frac{1}{x} + 2$

   b) $y = \frac{1}{x+4} - 2$

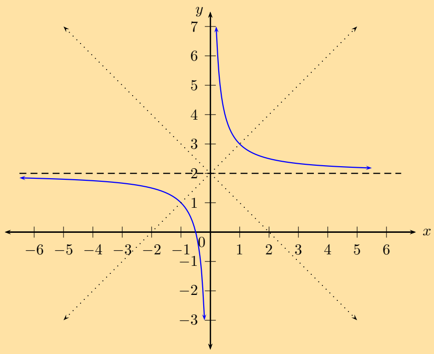
   c) $y = -\frac{1}{x+1} + 3$

   d) $y = -\frac{5}{x-2\frac{1}{2}} - 2$

   e) $y = \frac{8}{x-8} + 4$

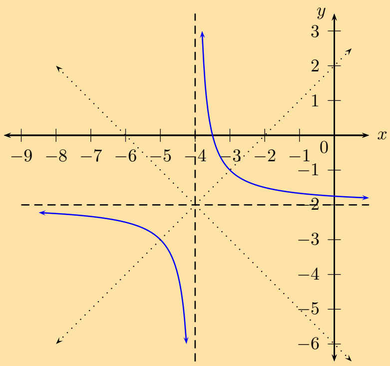
   Solution:

a)



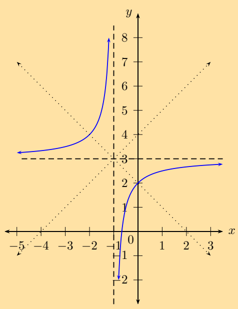
Asymptotes: $x = 0; y = 2$
Intercepts: $\left(-\frac{1}{2}; 0\right)$
Axes of symmetry : $y = x + 2$ and $y = -x + 2$
Domain: $\{x : x \in \mathbb{R}, x \neq 0\}$
Range: $\{y : y \in \mathbb{R}, y \neq 2\}$

b)



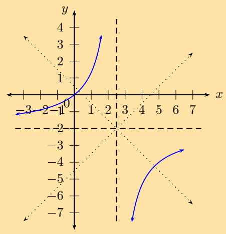
Asymptotes: $x = -4; y = -2$
Intercepts: $\left(-3\frac{1}{2}; 0\right)$ and $\left(0; -1\frac{3}{4}\right)$
Axes of symmetry : $y = x + 2$ and $y = -x - 6$
Domain: $\{x : x \in \mathbb{R}, x \neq -4\}$
Range: $\{y : y \in \mathbb{R}, y \neq -2\}$

c)



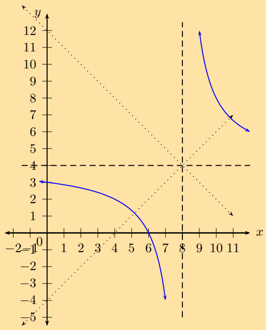
Asymptotes: $x = -1; y = 3$
Intercepts: $\left(-\frac{2}{3}; 0\right)$ and $(0; 2)$
Axes of symmetry : $y = x + 4$ and $y = -x + 2$
Domain: $\{x : x \in \mathbb{R}, x \neq -1\}$
Range: $\{y : y \in \mathbb{R}, y \neq 3\}$

d)



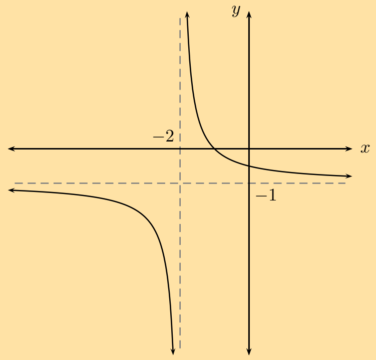
Asymptotes: $x = -2\frac{1}{2}; y = -2$
Intercepts: $(0; 0)$
Axes of symmetry : $y = x - 4\frac{1}{2}$ and $y = -x + \frac{1}{2}$
Domain: $\{x : x \in \mathbb{R}, x \neq -4\}$
Range: $\{y : y \in \mathbb{R}, y \neq -2\}$

e)



Asymptotes: $x = 8; y = 4$
Intercepts: $(6; 0)$ and $(0; 3)$
Axes of symmetry : $y = x - 4$ and $y = -x + 12$
Domain: $\{x : x \in \mathbb{R}, x \neq 8\}$
Range: $\{y : y \in \mathbb{R}, y \neq 4\}$

2. Given the graph of the hyperbola of the form $y = \frac{1}{x+p} + q$, determine the values of p and q.



Solution:

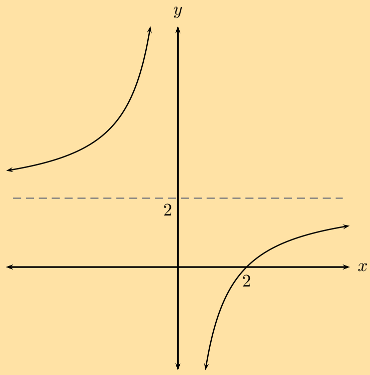
$$y = \frac{1}{x+p} + q$$

From graph $p = 2$

$$q = -1$$

$$\therefore y = \frac{1}{x+2} - 1$$

3. Given a sketch of the function of the form $y = \frac{a}{x+p} + q$, determine the values of a, p and q.



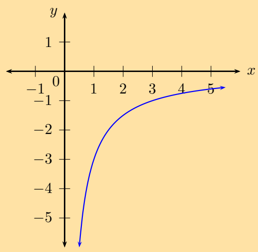
Solution:

$$\begin{aligned} y &= \frac{a}{x+p} + q \\ \text{From graph} \quad p &= 0 \\ q &= 2 \\ \therefore y &= \frac{a}{x} + 2 \\ \text{Subst. } (2;0) \quad 0 &= \frac{a}{2} + 2 \\ -2 &= \frac{a}{2} \\ -2(2) &= a \\ \therefore a &= -4 \\ y &= -\frac{4}{x} + 2 \end{aligned}$$

4. a) Draw the graph of $f(x) = -\frac{3}{x}$, $x > 0$.
   b) Determine the average gradient of the graph between $x = 1$ and $x = 3$.
   c) Is the gradient at $(\frac{1}{2}; -6)$ less than or greater than the average gradient between $x = 1$ and $x = 3$? Illustrate this on your graph.

Solution:

a)



b)

$$\begin{aligned} \text{Average gradient } &= \frac{f(3) - f(1)}{3 - 1} \\ &= \frac{-\frac{3}{3} - \left(-\frac{3}{1}\right)}{3 - 1} \\ &= \frac{-1 + 3}{2} \\ &= \frac{2}{2} \\ &= 1 \end{aligned}$$

Average gradient between $x = 1$ and $x = 3$ is 1.

c)

$$\begin{aligned}
\text{Average gradient} &= \frac{f(a+h) - f(a)}{(a+h) - a} \\
&= \frac{-\frac{3}{a+h} - \left(-\frac{3}{a}\right)}{(a+h) - a} \\
&= \frac{-\frac{3}{a+h} + \frac{3}{a}}{h} \\
&= \frac{\frac{-3(a)+3(a+h)}{a(a+h)}}{h} \\
&= \frac{\frac{-3a+3a+3h}{a^2+ah}}{h} \\
&= \frac{\frac{3h}{a^2+ah}}{h} \\
&= \frac{3h}{a^2+ah} \times \frac{1}{h} \\
&= \frac{3}{a^2+ah} \\
\therefore \text{At } \left(\frac{1}{2}; -6\right) \qquad a &= \frac{1}{2} \\
\therefore \text{And} \qquad h &= 0 \\
\therefore \text{Average gradient} &= \frac{3}{\left(\frac{1}{2}\right)^2 + \frac{1}{2}(0)} \\
&= 3 \times \frac{4}{1} \\
&= 12
\end{aligned}$$

Average gradient at $(\frac{1}{2}; -6)$ is greater than the average gradient between $x = 1$ and $x = 3$.

5.4 Exponential functions

Revision

Functions of the form $y = ab^x + q$

Exercise 5 – 15: Revision

1. On separate axes, accurately draw each of the following functions:

   - Use tables of values if necessary.
   - Use graph paper if available.

   a) $y_1 = 3^x$

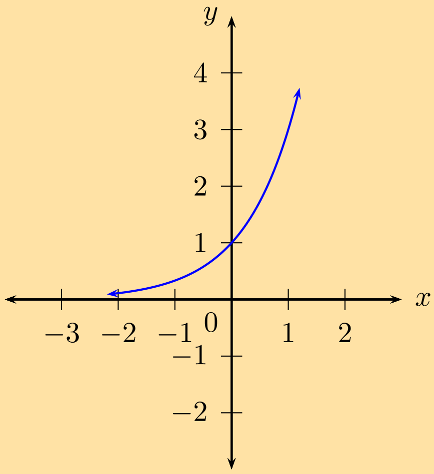
b) $y_2 = -2 \times 3^x$

c) $y_3 = 2 \times 3^x + 1$

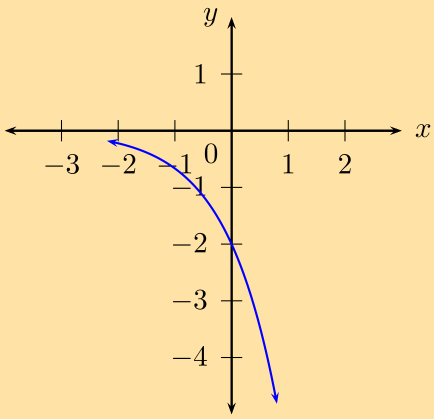
d) $y_4 = 3^x - 2$

Solution:

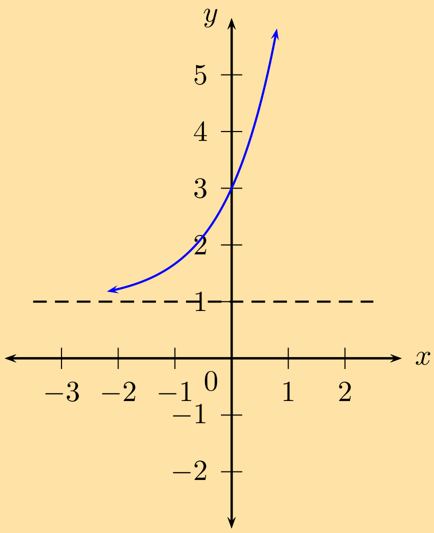
a)



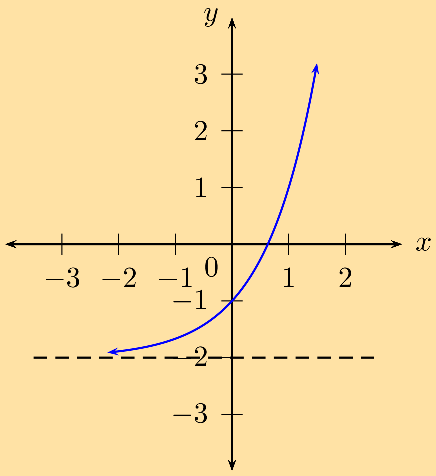
b)



c)



d)



2. Use your sketches of the functions given above to complete the following table (the first column has been completed as an example):

	y_1	y_2	y_3	y_4
value of q	$q = 0$			
effect of q	no vertical shift			
value of a	$a = 1$			
effect of a	increasing			
asymptote	x-axis, $y = 0$			
domain	$\{x : x \in \mathbb{R}\}$			
range	$\{y : y \in \mathbb{R}, y > 0\}$			

Solution:

	y_1	y_2	y_3	y_4
value of q	$q = 0$	$q = 0$	$q = 1$	$q = 2$
effect of q	no vertical shift	no vertical shift	shift 1 unit up	shift 2 units down
value of a	$a = 1$	$a = -2$	$a = 2$	$a = 1$
effect of a	increasing	decreasing	increasing	increasing
asymptote	x-axis, $y = 0$	x-axis, $y = 0$	$y = 1$	$y = -2$
domain	$\{x : x \in \mathbb{R}\}$	$\{x : x \in \mathbb{R}\}$	$\{x : x \in \mathbb{R}\}$	$\{x : x \in \mathbb{R}\}$
range	$\{y : y \in \mathbb{R}, y > 0\}$	$\{y : y \in \mathbb{R}, y < 0\}$	$\{y : y \in \mathbb{R}, y > 1\}$	$\{y : y \in \mathbb{R}, y > -2\}$

Functions of the form $y = ab^{(x+p)} + q$

Discovering the characteristics

Exercise 5 – 16: Domain and range

Give the domain and range for each of the following functions:

1. $y = \left(\frac{3}{2}\right)^{(x+3)}$

Solution:

$$\begin{aligned} \text{Domain: } & \{x : x \in \mathbb{R}\} \\ \text{Range: } & \{y : y > 0, y \in \mathbb{R}\} \end{aligned}$$

2. $f(x) = -5^{(x-2)} + 1$

Solution:

$$\begin{aligned} \text{Domain: } & \{x : x \in \mathbb{R}\} \\ \text{Range: } & \{y : y < 1, y \in \mathbb{R}\} \end{aligned}$$

3. $y + 3 = 2^{(x+1)}$

Solution:

$$\begin{aligned} y + 3 &= 2^{(x+1)} \\ y &= 2^{(x+1)} - 3 \\ \text{Domain: } & \{x : x \in \mathbb{R}\} \\ \text{Range: } & \{y : y > -3, y \in \mathbb{R}\} \end{aligned}$$

4. $y = n + 3^{(x-m)}$

Solution:

$$\begin{aligned} y &= n + 3^{(x-m)} \\ y &= 3^{(x-m)} + n \\ \text{Domain: } & \{x : x \in \mathbb{R}\} \\ \text{Range: } & \{y : y > n, y \in \mathbb{R}\} \end{aligned}$$

5. $\frac{y}{2} = 3^{(x-1)} - 1$

Solution:

$$\begin{aligned} \frac{y}{2} &= 3^{(x-1)} - 1 \\ y &= 2 \times 3^{(x-1)} - 2 \\ \text{Domain: } & \{x : x \in \mathbb{R}\} \\ \text{Range: } & \{y : y > 2, y \in \mathbb{R}\} \end{aligned}$$

Exercise 5 – 17: Intercepts

Determine the x- and y-intercepts for each of the following functions:

1. $f(x) = 2^{(x+1)} - 8$

   Solution:

$$\begin{aligned}
\text{For } x = 0 \quad y &= 2^{(0+1)} - 8 \\
&= 2 - 8 \\
&= -6 \\
&\therefore (0; -6) \\
\text{For } y = 0 \quad 0 &= 2^{(x+1)} - 8 \\
2^3 &= 2^{(x+1)} \\
\therefore 3 &= x + 1 \\
\therefore 2 &= x \\
&\therefore (2; 0)
\end{aligned}$$

2. $y = 2 \times 3^{(x-1)} - 18$

   Solution:

$$\begin{aligned}
\text{For } x = 0 \quad y &= 2 \times 3^{(0-1)} - 18 \\
&= \frac{2}{3} - 18 \\
&= -17\frac{1}{3} \\
&\therefore (0; -17\frac{1}{3}) \\
\text{For } y = 0 \quad 0 &= 2 \times 3^{(x-1)} - 18 \\
18 &= 2 \times 3^{(x-1)} \\
9 &= 3^{(x-1)} \\
3^2 &= 3^{(x-1)} \\
\therefore 2 &= x - 1 \\
\therefore 3 &= x \\
&\therefore (3; 0)
\end{aligned}$$

3. $y + 5^{(x+2)} = 5$

   Solution:

$$y + 5^{(x+2)} = 5$$
$$y = -5^{(x+2)} + 5$$
$$\text{For } x = 0 \quad y = -5^{(0+2)} + 5$$
$$= -25 + 5$$
$$= -20$$
$$\therefore (0; -20)$$
$$\text{For } y = 0 \quad 0 = -5^{(x+2)} + 5$$
$$5^{(x+2)} = 5$$
$$\therefore x + 2 = 1$$
$$\therefore x = -1$$
$$\therefore (-1; 0)$$

4. $y = \frac{1}{2}\left(\frac{3}{2}\right)^{(x+3)} - 0{,}75$

Solution:

$$y = \frac{1}{2}\left(\frac{3}{2}\right)^{(x+3)} - 0{,}75$$
$$y = \frac{1}{2}\left(\frac{3}{2}\right)^{(x+3)} - \frac{3}{4}$$
$$\text{For } x = 0 \quad y = \frac{1}{2}\left(\frac{3}{2}\right)^{(0+3)} - \frac{3}{4}$$
$$= \frac{1}{2}\left(\frac{3}{2}\right)^{3} - \frac{3}{4}$$
$$= \frac{1}{2}\left(\frac{27}{8}\right) - \frac{3}{4}$$
$$= \frac{27}{16} - \frac{3}{4}$$
$$= \frac{15}{16}$$
$$\therefore (0; \frac{15}{16})$$
$$\text{For } y = 0 \quad 0 = \frac{1}{2}\left(\frac{3}{2}\right)^{(x+3)} - \frac{3}{4}$$
$$\frac{3}{4} = \frac{1}{2}\left(\frac{3}{2}\right)^{(x+3)}$$
$$\frac{3}{2} = \left(\frac{3}{2}\right)^{(x+3)}$$
$$\therefore 1 = x + 3$$
$$\therefore -2 = x$$
$$\therefore (-2; 0)$$

Exercise 5 – 18: Asymptote

Give the asymptote for each of the following functions:

1. $y = -5^{(x+1)}$

Solution:

$$\begin{aligned} & y = -5^{(x+1)} \\ \text{Horizontal asymptote:} \quad & y = 0 \end{aligned}$$

2. $y = 3^{(x-2)} + 1$

Solution:

$$\begin{aligned} & y = 3^{(x-2)} + 1 \\ \text{Horizontal asymptote:} \quad & y = 1 \end{aligned}$$

3. $\left(\frac{3y}{2}\right) = 5^{(x+3)} - 1$

Solution:

$$\begin{aligned} \left(\frac{3y}{2}\right) &= 5^{(x+3)} - 1 \\ 3y &= 2 \times 5^{(x+3)} - 2 \\ y &= \frac{2}{3} \times 5^{(x+3)} - \frac{2}{3} \\ \text{Horizontal asymptote:} \quad y &= -\frac{2}{3} \end{aligned}$$

4. $y = 7^{(x+1)} - 2$

Solution:

$$\begin{aligned} & y = 7^{(x+1)} - 2 \\ \text{Horizontal asymptote:} \quad & y = -2 \end{aligned}$$

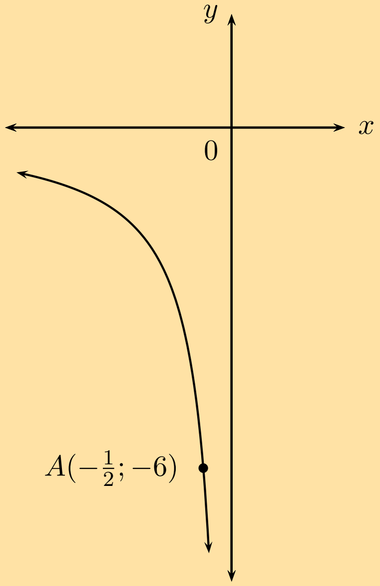
5. $\frac{y}{2} + 1 = 3^{(x+2)}$

Solution:

$$\begin{aligned} \frac{y}{2} + 1 &= 3^{(x+2)} \\ \frac{y}{2} &= 3^{(x+2)} - 1 \\ y &= 2 \times 3^{(x+2)} - 2 \\ \text{Horizontal asymptote:} \quad y &= -2 \end{aligned}$$

Exercise 5 – 19: Mixed exercises

1. Given the graph of the hyperbola of the form $h(x) = \frac{k}{x}$, $x < 0$, which passes though the point $A(-\frac{1}{2}; -6)$.



a) Show that $k = 3$.

b) Write down the equation for the new function formed if $h(x)$:

i. is shifted 3 units vertically upwards

ii. is shifted to the right by 3 units

iii. is reflected about the y-axis

iv. is shifted so that the asymptotes are $x = 0$ and $y = -\frac{1}{4}$

v. is shifted upwards to pass through the point $(-1; 1)$

vi. is shifted to the left by 2 units and 1 unit vertically downwards (for $x < 0$)

Solution:

a)

$$\begin{aligned} y &= \frac{k}{x} \\ \text{Subst. } (-\frac{1}{2}; -6) \qquad -6 &= \frac{k}{-\frac{1}{2}} \\ -6 \times -\frac{1}{2} &= k \\ \therefore k &= 3 \\ \therefore h(x) &= \frac{3}{x} \end{aligned}$$

b) i.

$$h(x) = \frac{3}{x}$$
$$\therefore y \Rightarrow y - 3$$
$$y - 3 = \frac{3}{x}$$
$$y = \frac{3}{x} + 3$$

ii.

$$h(x) = \frac{3}{x}$$
$$\therefore x \Rightarrow x - 3$$
$$y = \frac{3}{x - 3}$$

iii.

$$h(x) = \frac{3}{x}$$
$$\therefore x \Rightarrow -x$$
$$y = \frac{3}{x - 3}$$

iv.

$$h(x) = \frac{3}{x}$$
$$\therefore p = 0 \text{ and } q = -\frac{1}{4}$$
$$y = \frac{3}{x} - \frac{1}{4}$$

v.

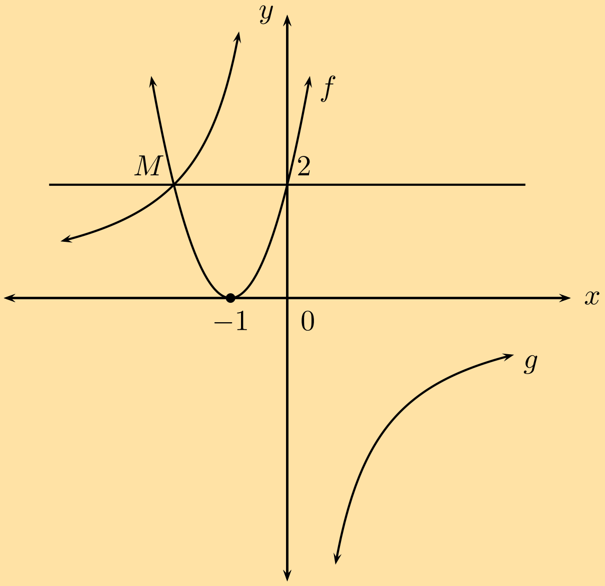
$$h(x) = \frac{3}{x}$$
$$\therefore y \Rightarrow y + m$$
$$y = \frac{3}{x} + m$$
$$\text{Subst.}(-1; 1) \qquad 1 = \frac{3}{-1} + m$$
$$1 + 3 = +m$$
$$\therefore m = 4$$
$$\therefore y = \frac{3}{x} + 4$$

vi.

$$h(x) = \frac{3}{x}$$
$$\therefore x \Rightarrow x + 2$$
$$\therefore y \Rightarrow y + 1$$
$$y + 1 = \frac{3}{x + 2}$$
$$\therefore y = \frac{3}{x + 2} - 1$$

2. Given the graphs of $f(x) = a(x+p)^2$ and $g(x) = \frac{a}{x}$.

The axis of symmetry for $f(x)$ is $x = -1$ and $f(x)$ and $g(x)$ intersect at point M. The line $y = 2$ also passes through M.



Determine:

a) the coordinates of M

b) the equation of $g(x)$

c) the equation of $f(x)$

d) the values for which $f(x) < g(x)$

e) the range of $f(x)$

Solution:

a) $f(x)$ is symmetrical about the line $x = -1$, therefore $M(-2; 2)$.

b)

$$
\begin{aligned}
g(x) &= \frac{a}{x} \\
\text{Subst. } M(-2;2) \qquad 2 &= \frac{a}{-2} \\
\therefore a &= -4 \\
\therefore g(x) &= \frac{-4}{x}
\end{aligned}
$$

c)

$$
\begin{aligned}
f(x) &= a(x+p)^2 + q \\
\text{No vertical shift } \therefore q &= 0 \\
\text{Axis of symmetry } x = -1 \qquad \therefore f(x) &= a(x+1)^2 \\
\text{Subst. } M(-2;2) \qquad 2 &= a(-2+1)^2 \\
2 &= a(-1)^2 \\
\therefore a &= 2 \\
\therefore f(x) &= 2(x+1)^2
\end{aligned}
$$

d) $-2 < x < 0$

e) Range: $\{y : y \in \mathbb{R}, y \geq 0\}$

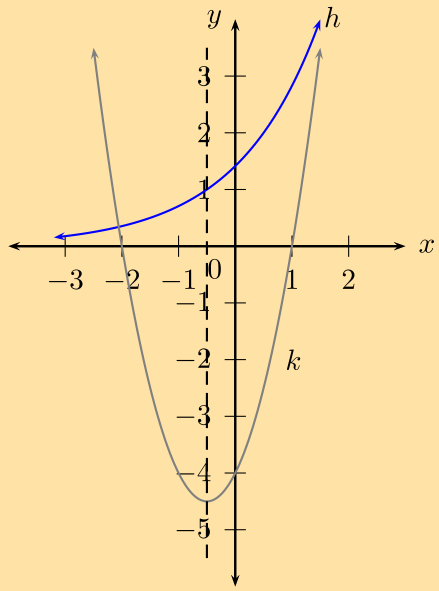
3. On the same system of axes, sketch:

   a) the graphs of $k(x) = 2(x + \frac{1}{2})^2 - 4\frac{1}{2}$ and $h(x) = 2^{(x+\frac{1}{2})}$. Determine all intercepts, turning point(s) and asymptotes.

   b) the reflection of $h(x)$ about the x-axis. Label this function as $j(x)$.

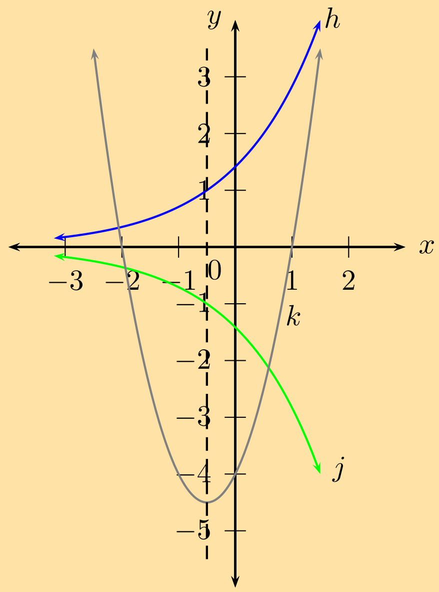
Solution:

a)



For $k(x)$:
Intercepts: $(-2; 0), (1; 0)$ and $(0; -4)$
Turning point: $(-\frac{1}{2}; -4\frac{1}{2})$
Asymptote: none
For $h(x)$:
Intercepts: $(1{,}41; 0)$
Turning point: none
Asymptote: $y = 0$

b)



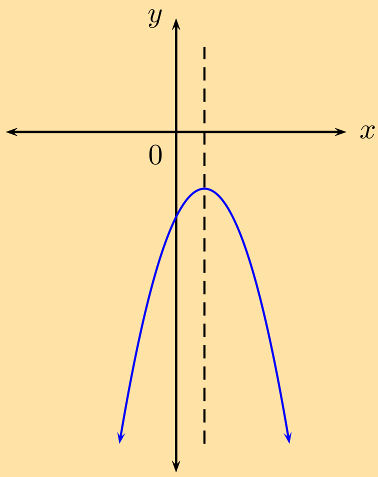
4. Sketch the graph of $y = ax^2 + bx + c$ for:

   a) $a < 0$, $b > 0$, $b^2 < 4ac$

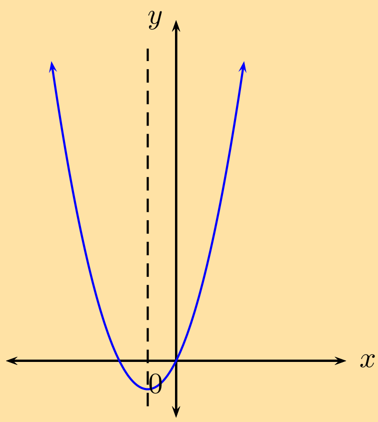
   b) $a > 0$, $b > 0$, one root $= 0$

Solution:

a)



b)



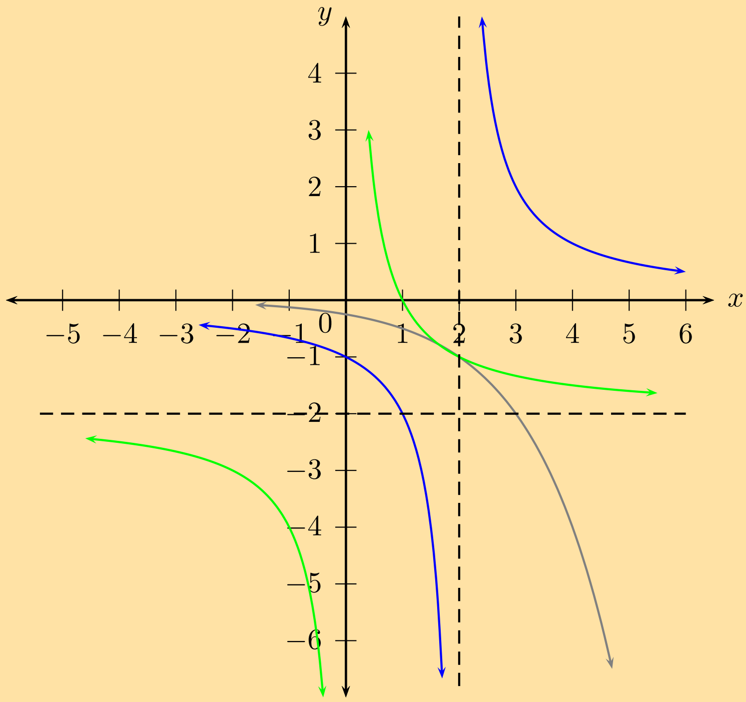
5. On separate systems of axes, sketch the graphs:

$y = \frac{2}{x-2}$

$y = \frac{2}{x} - 2$

$y = -2^{(x-2)}$

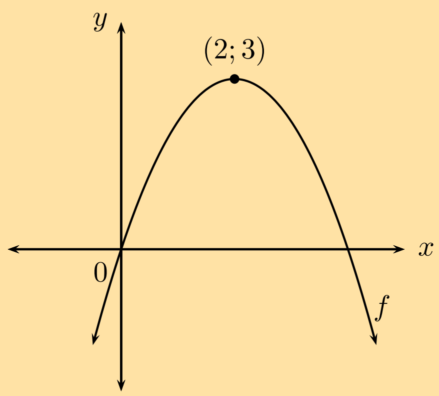
Solution:



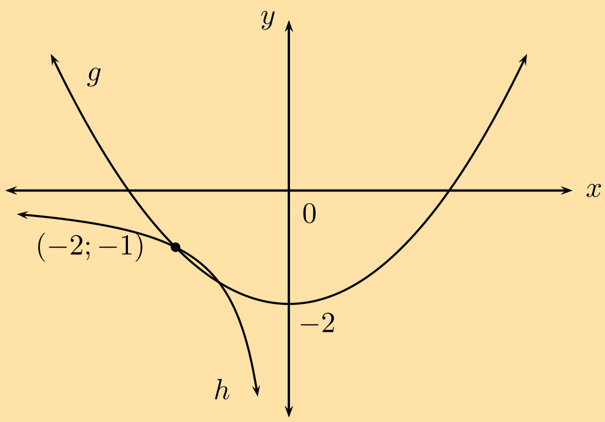
6. For the diagrams shown below, determine:

- the equations of the functions; $f(x) = a(x + p)^2 + q$, $g(x) = ax^2 + q$, $h(x) = \frac{a}{x}, x < 0$ and $k(x) = b^x + q$
- the axes of symmetry of each function
- the domain and range of each function

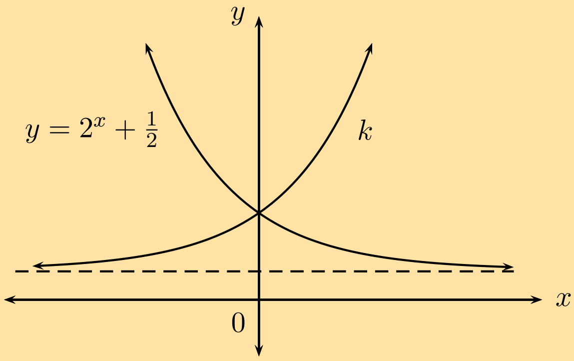
a)



b)



c)



Solution:

a)

$$
\begin{aligned}
f(x) &= a(x+p)^2 + q \\
\text{From turning point: } p = -2 \text{ and } q &= 3 \\
\therefore f(x) &= a(x-2)^2 + 3 \\
\text{Subst. } (0;0) \qquad 0 &= a(0-2)^2 + 3 \\
-3 &= 4a \\
\therefore a &= -\frac{3}{4} \\
f(x) &= -\frac{3}{4}(x-2)^2 + 3 \\
\text{Axes of symmetry: } x &= 2 \\
\text{Domain: } &\{x : x \in \mathbb{R}\} \\
\text{Range: } &\{y : y \in \mathbb{R}, y \leq 3\}
\end{aligned}
$$

b)

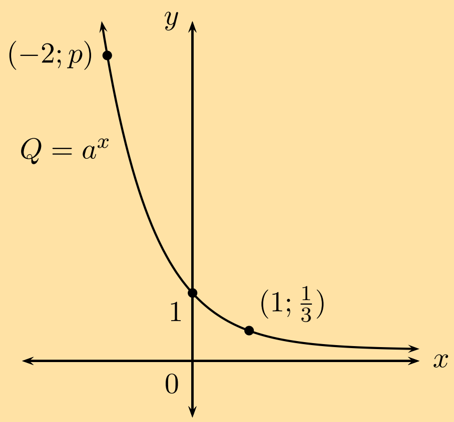
$$
\begin{aligned}
g(x) &= ax^2 + q \\
\text{From turning point: } p = 0 \text{ and } q &= -2 \\
\therefore g(x) &= a(x)^2 - 2 \\
\text{Subst. } (-2;-1) \qquad -1 &= a(-2)^2 - 2 \\
-1 &= 4a - 2 \\
1 &= 4a - 2 \\
\therefore a &= \frac{1}{4} \\
g(x) &= \frac{1}{4}x^2 - 2
\end{aligned}
$$

$$\text{Axes of symmetry: } x = 0$$
$$\text{Domain: } \{x : x \in \mathbb{R}\}$$
$$\text{Range: } \{y : y \in \mathbb{R}, y \geq -2\}$$
$$h(x) = \frac{a}{x+p} + q$$
$$\text{From graph: } p = 0 \text{ and } q = 0$$
$$h(x) = \frac{a}{x}$$
$$\text{Subst. } (-2;-1) \qquad -1 = \frac{a}{-2}$$
$$2 = a$$
$$\therefore h(x) = \frac{2}{x}$$
$$\text{Axes of symmetry: } y = x$$
$$\text{Domain: } \{x : x \in \mathbb{R}, x < 0\}$$
$$\text{Range: } \{y : y \in \mathbb{R}, y < 0\}$$

c)

$$y = 2^x + \frac{1}{2}$$
$$\text{Reflect about } x = 0$$
$$\therefore x \Rightarrow -x$$
$$\therefore k(x) = 2^{-x} + \frac{1}{2}$$
$$= \left(\frac{1}{2}\right)^x + \frac{1}{2}$$
$$\text{Domain: } \{x : x \in \mathbb{R}\}$$
$$\text{Range: } \left\{y : y \in \mathbb{R}, y > \frac{1}{2}\right\}$$

7. Given the graph of the function $Q(x) = a^x$.



a) Show that $a = \frac{1}{3}$.

b) Find the value of p if the point $(-2;p)$ is on Q.

c) Calculate the average gradient of the curve between $x = -2$ and $x = 1$.

d) Determine the equation of the new function formed if Q is shifted 2 units vertically downwards and 2 units to the left.

Solution:

a)

$$\begin{aligned} y &= a^x \\ \text{Subst. } \left(1; \frac{1}{3}\right) \quad \frac{1}{3} &= a^1 \\ \therefore a &= \frac{1}{3} \end{aligned}$$

b)

$$\begin{aligned} Q(x) &= \left(\frac{1}{3}\right)^x \\ \text{Subst. } (-2; p)\ p &= \left(\frac{1}{3}\right)^{-2} \\ p &= 9 \end{aligned}$$

c)

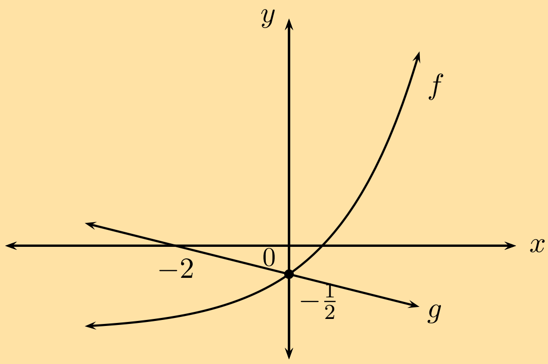
$$\begin{aligned} \text{Average gradient} &= \frac{\left(\frac{1}{3}\right)^{-2} - \left(\frac{1}{3}\right)^1}{-2-(1)} \\ &= \frac{9 - \frac{1}{3}}{-3} \\ &= \frac{8\frac{2}{3}}{-3} \\ &= \frac{-26}{9} \\ &= -2\frac{8}{9} \end{aligned}$$

d)

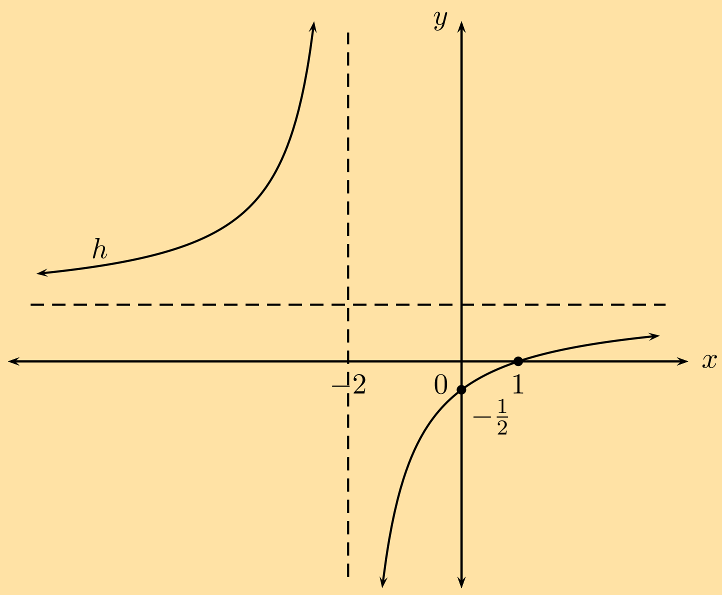
$$\begin{aligned} Q(x) &= \left(\frac{1}{3}\right)^x \\ \therefore x &\Rightarrow x+2 \\ \therefore y &\Rightarrow y+2 \\ y+2 &= \left(\frac{1}{3}\right)^{x+2} \\ y &= \left(\frac{1}{3}\right)^{x+2} - 2 \end{aligned}$$

8. Find the equation for each of the functions shown below:

 a) $f(x) = 2^x + q$
 $g(x) = mx + c$



b) $h(x) = \frac{k}{x+p} + q$



Solution:

a)

$$\begin{aligned}
f(x) &= 2^x + q \\
\text{Subst. } (0; -\frac{1}{2}) \quad -\frac{1}{2} &= 2^0 + q \\
-\frac{1}{2} &= 1 + q \\
\therefore q &= -\frac{3}{2} \\
\therefore f(x) &= 2^x - \frac{3}{2} \\
g(x) &= mx + c \\
\text{Subst. } (0; -\frac{1}{2}) \quad -\frac{1}{2} &= m(0) + c \\
\therefore c &= -\frac{1}{2} \\
g(x) &= mx - \frac{1}{2} \\
\text{Subst. } (-2; 0) \quad 0 &= m(-2) - \frac{1}{2} \\
\frac{1}{2} &= -2m \\
\therefore m &= -\frac{1}{4} \\
\therefore g(x) &= -\frac{1}{4}x - \frac{1}{2}
\end{aligned}$$

b)

$$h(x) = \frac{k}{x+p} + q$$

From graph: $p = -2$

$$h(x) = \frac{k}{x+2} + q$$

Subst. $(0; -\frac{1}{2})$ $\quad -\frac{1}{2} = \frac{k}{2} + q$

$$-1 = k + 2q \ldots (1)$$

Subst. $(1; 0)$ $\quad 0 = \frac{k}{1+2} + q$

$$0 = k + 3q \ldots (2)$$

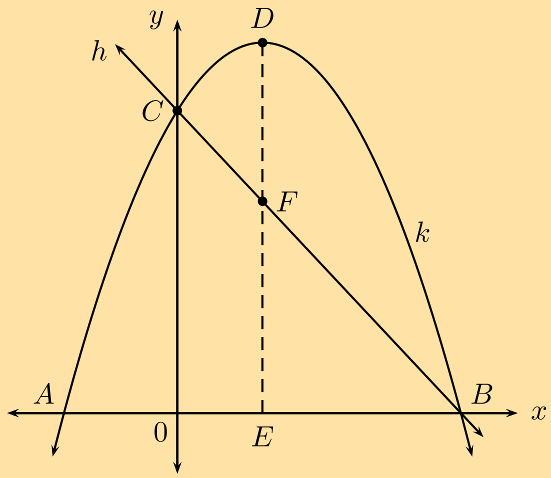
$(2) - (1):$ $\quad 1 = 0 + q$

$$\therefore q = 1$$

and $k = -3$

$$\therefore h(x) = -\frac{3}{x+2} + 1$$

9. Given: the graph of $k(x) = -x^2 + 3x + 10$ with turning point at D. The graph of the straight line $h(x) = mx + c$ passing through points B and C is also shown.



Determine:

a) the lengths AO, OB, OC and DE

b) the equation of DE

c) the equation of $h(x)$

d) the x-values for which $k(x) < 0$

e) the x-values for which $k(x) \geq h(x)$

f) the length of DF

Solution:

a)

$$y = -x^2 + 3x + 10$$
$$\text{Let } y = 0$$
$$0 = -x^2 + 3x + 10$$
$$= x^2 - 3x - 10$$
$$= (x-5)(x+2)$$
$$\therefore x = 5 \text{ or } x = -2$$
$$\therefore AO = 2 \text{ units}$$
$$\therefore BO = 5 \text{ units}$$
$$CO = 10 \text{ units}$$
$$\text{Axes of symmetry: } x = -\frac{5-2}{2}$$
$$= \frac{3}{2}$$
$$\text{Subst. } x = \frac{3}{2}$$
$$y = -\left(\frac{3}{2}\right)^2 + 3\frac{3}{2} + 10$$
$$= 12\frac{1}{4}$$
$$\therefore DE = 12{,}25 \text{ units}$$

b) $DE = 12\frac{1}{4}$

c)

$$h(x) = mx + c$$
$$h(x) = mx + 10$$
$$\text{Subst. } (5;0) \quad 0 = m(5) + 10$$
$$-10 = 5m$$
$$\therefore m = -2$$
$$\therefore h(x) = -2x + 10$$

d) $\{x : x \in \mathbb{R}, x < -2 \text{ and } x > 5\}$

e) $\{x : x \in \mathbb{R}, 0 \leq x \leq 5\}$

f)

$$\text{At } x = \frac{3}{2} k(x) = 12\frac{1}{4}$$
$$\text{At } x = \frac{3}{2} h(x) = -2\left(\frac{3}{2}\right) + 10$$
$$= -3 + 10$$
$$= 7$$
$$\therefore DF = 12\frac{1}{4} - 7$$
$$= 12\frac{1}{4} - 7$$
$$= 5{,}25 \text{ units}$$

5.5 The sine function

Revision

Functions of the form $y = \sin\theta$ for $0° \leq \theta \leq 360°$

Functions of the form $y = a\sin\theta + q$

Exercise 5 – 20: Revision

On separate axes, accurately draw each of the following functions for $0° \leq \theta \leq 360°$.

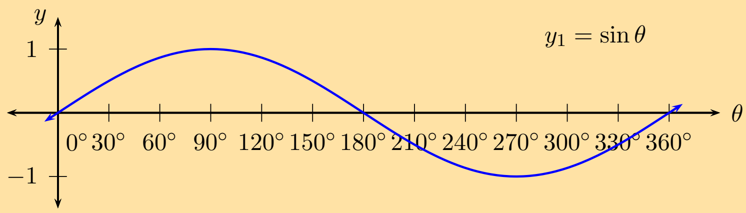
- Use tables of values if necessary.
- Use graph paper if available.

For each function also determine the following:

- Period
- Amplitude
- Domain and range
- x- and y-intercepts
- Maximum and minimum turning points

1. $y_1 = \sin\theta$

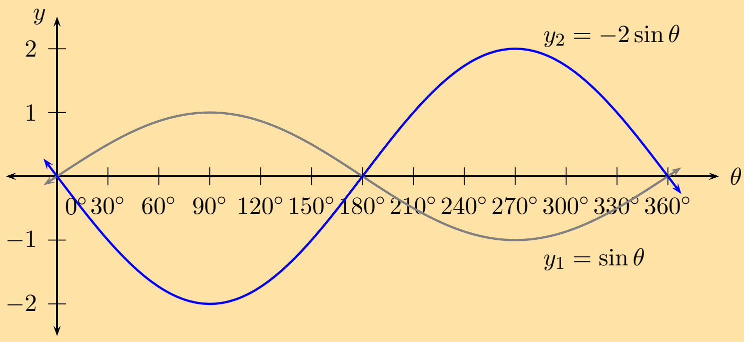
   Solution:



Period: $360°$
Amplitude: 1
Domain: $[0°; 360°]$
Range: $[-1; 1]$
x-intercepts: $(0°; 0); (180°; 0); (360°; 0)$
y-intercepts: $(0°; 0)$
Max. turning point: $(90°; 1)$
Min. turning point: $(270°; -1)$

2. $y_2 = -2\sin\theta$

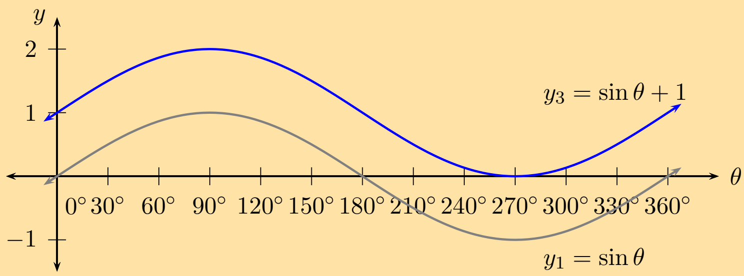
Solution:



Period: $360°$
Amplitude: 2
Domain: $[0°; 360°]$
Range: $[-2; 2]$
x-intercepts: $(0°; 0)$; $(180°; 0)$; $(360°; 0)$
y-intercepts: $(0°; 0)$
Max. turning point: $(270°; 2)$
Min. turning point: $(90°; -2)$

3. $y_3 = \sin\theta + 1$

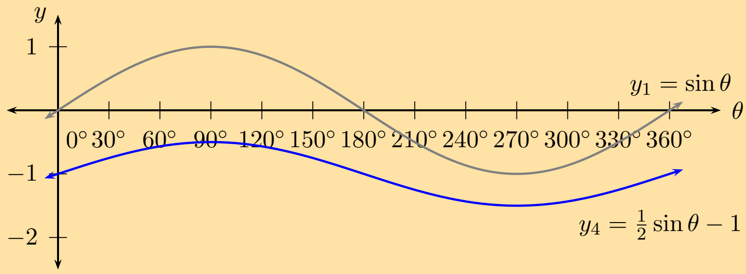
Solution:



Period: $360°$
Amplitude: 1
Domain: $[0°; 360°]$
Range: $[0; 2]$
x-intercepts: $(270°; 0)$
y-intercepts: $(0°; 1)$
Max. turning point: $(90°; 2)$
Min. turning point: $(270°; 0)$

4. $y_4 = \frac{1}{2}\sin\theta - 1$

Solution:



$$
\begin{aligned}
\text{Period: } & 360° \\
\text{Amplitude: } & \frac{1}{2} \\
\text{Domain: } & [0°; 360°] \\
\text{Range: } & [-\frac{1}{2}; -\frac{3}{2}] \\
x\text{-intercepts: } & \text{none} \\
y\text{-intercepts: } & (0°; -\frac{1}{2}) \\
\text{Max. turning point: } & (90°; \frac{1}{2}) \\
\text{Min. turning point: } & (270°; -\frac{3}{2})
\end{aligned}
$$

Functions of the form $y = \sin k\theta$

Discovering the characteristics

Exercise 5 – 21: Sine functions of the form $y = \sin k\theta$

1. Sketch the following functions for $-180° \leq \theta \leq 180°$ and for each graph determine:

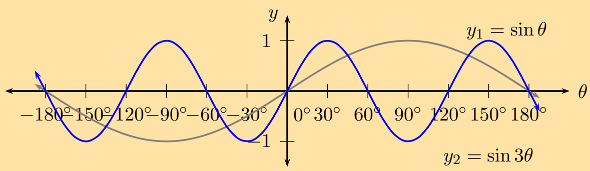
   - Period
   - Amplitude
   - Domain and range
   - x- and y-intercepts
   - Maximum and minimum turning points

   a) $f(\theta) = \sin 3\theta$
   b) $g(\theta) = \sin \frac{\theta}{3}$
   c) $h(\theta) = \sin(-2\theta)$

d) $k(\theta) = \sin \frac{3\theta}{4}$

Solution:

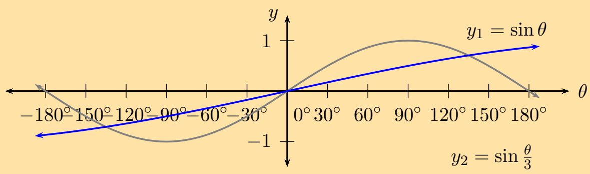
a)



For $f(\theta) = \sin 3\theta$:

Period: $120°$
Amplitude: 1
Domain: $[-180°; 180°]$
Range: $[-1; 1]$
x-intercepts: $(-180°; 0); (-120°; 0); (-60°; 0);$
$(0°; 0); (60°; 0); (120°; 0); (180°; 0)$
y-intercepts: $(0°; 0)$
Max. turning point: $(-90°; 1); (30°; 1); (150°; 1)$
Min. turning point: $(-150°; -1); (-30°; -1); (90°; -1)$

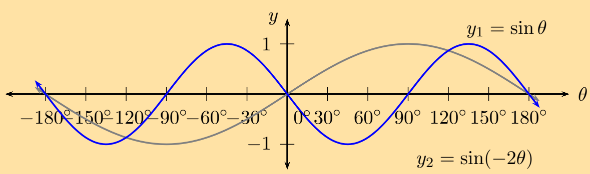
b)



For $g(\theta) = \sin \frac{\theta}{3}$:

Period: $1080°$
Amplitude: 1
Domain: $[-180°; 180°]$
Range: $[-0{,}87; 0{,}87]$
x-intercepts: none
y-intercepts: $(0°; 0)$
Max. turning point: none
Min. turning point: none

c)



For $h(\theta) = \sin(-2\theta)$:

Period: $180°$

Amplitude: 1

Domain: $[-180°; 180°]$

Range: $[-1; 1]$

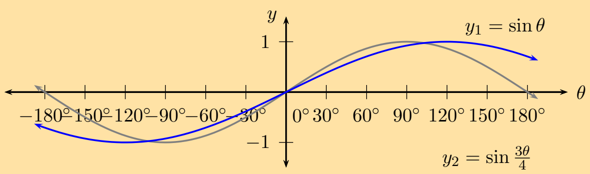
x-intercepts: $(-180°; 0); (-90°; 0); (0°; 0); (90°; 0); (180°; 0)$

y-intercepts: $(0°; 0)$

Max. turning point: $(-45°; 1); (135°; 1)$

Min. turning point: $(-135°; -1); (45°; -1);$

d)



For $k(\theta) = \sin\frac{3\theta}{4}$:

Period: $480°$

Amplitude: 1

Domain: $[-180°; 180°]$

Range: $[-1; 1]$

x-intercepts: $(0°; 0)$

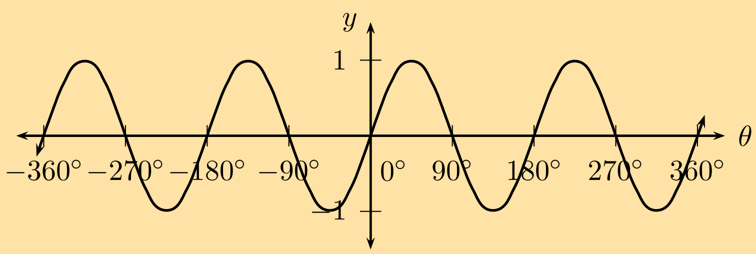
y-intercepts: $(0°; 0)$

Max. turning point: $(120°; 1)$

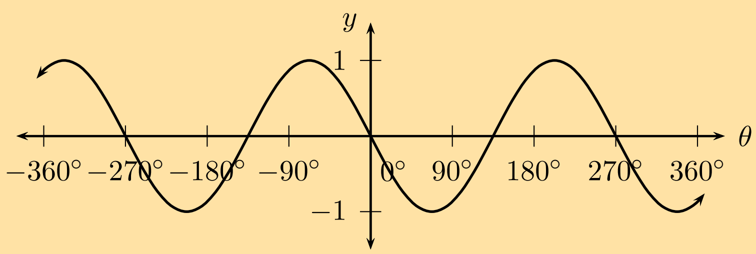
Min. turning point: $(-120°; -1);$

2. For each graph of the form $f(\theta) = \sin k\theta$, determine the value of k:

a)



b)



Solution:

a)

$$\begin{aligned} \text{Period } &= 180° \\ \therefore \frac{360°}{k} &= 180° \\ k &= \frac{360°}{180°} \\ \therefore k &= 2 \end{aligned}$$

b)

$$\begin{aligned} \text{Period } &= 270° \\ \therefore \frac{360°}{k} &= 270° \\ k &= \frac{360°}{270°} \\ \therefore k &= \frac{3}{4} \\ \text{and graph is reflected about the } x-\text{axis } \therefore k &= -\frac{3}{4} \end{aligned}$$

Functions of the form $y = \sin(\theta + p)$

Discovering the characteristics

Exercise 5 – 22: Sine functions of the form $y = \sin(\theta + p)$

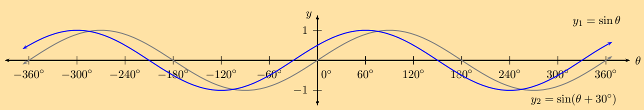
Sketch the following functions for $-360° \leq \theta \leq 360°$.

For each function, determine the following:

- Period
- Amplitude
- Domain and range
- x- and y-intercepts
- Maximum and minimum turning points

1. $f(\theta) = \sin(\theta + 30°)$

   Solution:

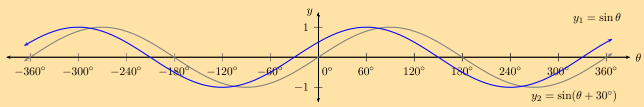


For $f(\theta) = \sin(\theta + 30^\circ)$:

$$\begin{aligned}
\text{Period: } & 360^\circ \\
\text{Amplitude: } & 1 \\
\text{Domain: } & [-360^\circ; 360^\circ] \\
\text{Range: } & [-1; 1] \\
x\text{-intercepts: } & (-210^\circ; 0); (-30^\circ; 0); (150^\circ; 0); (330^\circ; 0) \\
y\text{-intercepts: } & (0^\circ; \frac{1}{2}) \\
\text{Max. turning point: } & (-300^\circ; 1); (60^\circ; 1) \\
\text{Min. turning point: } & (-120^\circ; -1); (240^\circ; -1)
\end{aligned}$$

2. $g(\theta) = \sin(\theta - 45^\circ)$

Solution:

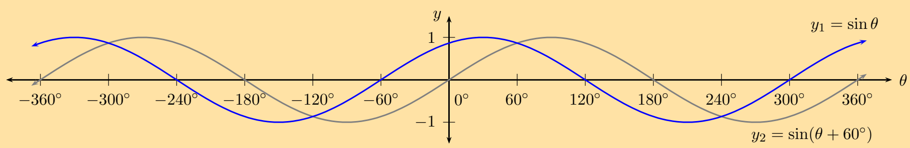


For $g(\theta) = \sin(\theta - 45^\circ)$:

$$\begin{aligned}
\text{Period: } & 360^\circ \\
\text{Amplitude: } & 1 \\
\text{Domain: } & [-360^\circ; 360^\circ] \\
\text{Range: } & [-1; 1] \\
x\text{-intercepts: } & (-315^\circ; 0); (-135^\circ; 0); (45^\circ; 0); (225^\circ; 0) \\
y\text{-intercepts: } & (0^\circ; \frac{1}{\sqrt{2}}) \\
\text{Max. turning point: } & (-300^\circ; 1); (-225^\circ; 1); (135^\circ; 1) \\
\text{Min. turning point: } & (-45^\circ; -1); (315^\circ; -1)
\end{aligned}$$

3. $h(\theta) = \sin(\theta + 60^\circ)$

Solution:



For $h(\theta) = \sin(\theta + 60^\circ)$:

Period: $360°$

Amplitude: 1

Domain: $[-360°; 360°]$

Range: $[-1; 1]$

x-intercepts: $(-240°; 0); (-60°; 0); (120°; 0); (300°; 0)$

y-intercepts: $(0°; \frac{\sqrt{3}}{2})$

Max. turning point: $(-300°; 1); (-330°; 1); (30°; 1)$

Min. turning point: $(-150°; -1); (210°; -1)$

Sketching sine graphs

Exercise 5 – 23: The sine function

1. Sketch the following graphs on separate axes:

   a) $y = 2\sin\frac{\theta}{2}$ for $-360° \leq \theta \leq 360°$

   b) $f(\theta) = \frac{1}{2}\sin(\theta - 45°)$ for $-90° \leq \theta \leq 90°$

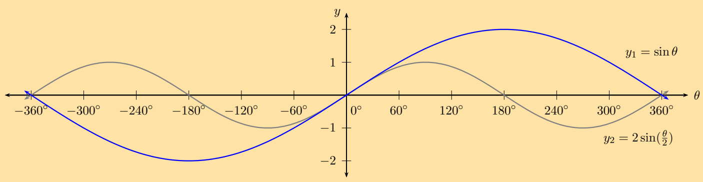
   c) $y = \sin(\theta + 90°) + 1$ for $0° \leq \theta \leq 360°$

   d) $y = \sin(-\frac{3\theta}{2})$ for $-180° \leq \theta \leq 180°$

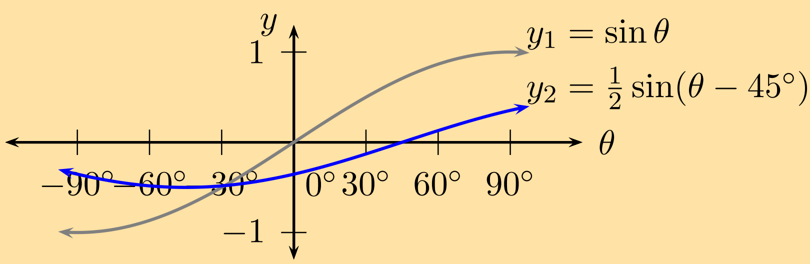
   e) $y = \sin(30° - \theta)$ for $-360° \leq \theta \leq 360°$

Solution:

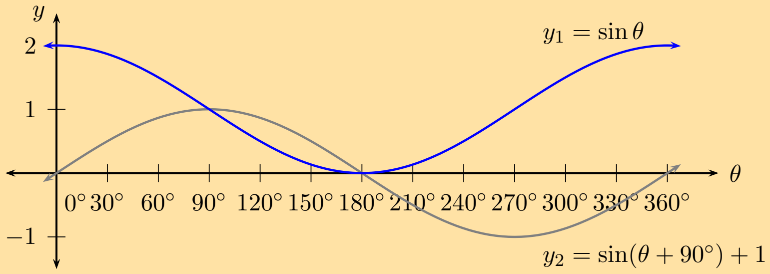
a)



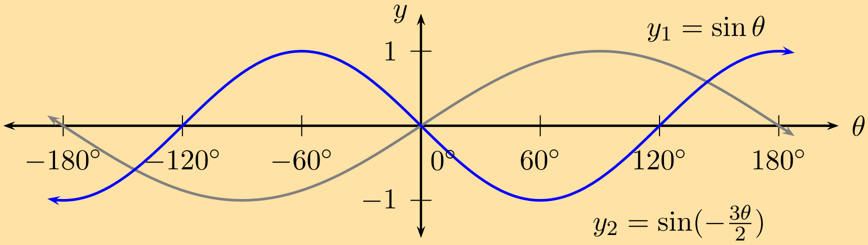
b)



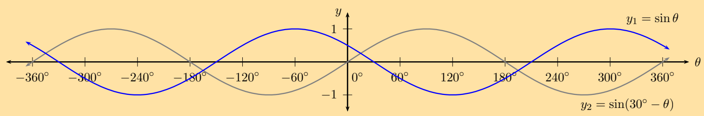
c)



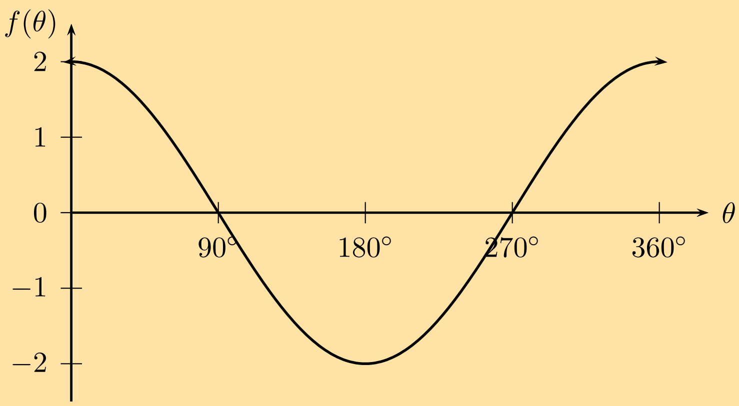
d)



e)



2. Given the graph of the function $y = a\sin(\theta + p)$, determine the values of a and p.



Can you describe this graph in terms of $\cos\theta$?

Solution:

$a = 2$; $p = 90° \therefore y = 2\sin(\theta + 90°)$ and $y = 2\cos\theta$

5.6 The cosine function

Revision

Functions of the form $y = \cos\theta$ for $0° \leq \theta \leq 360°$

Functions of the form $y = a\cos\theta + q$

Exercise 5 – 24: Revision

On separate axes, accurately draw each of the following functions for $0° \leq \theta \leq 360°$:

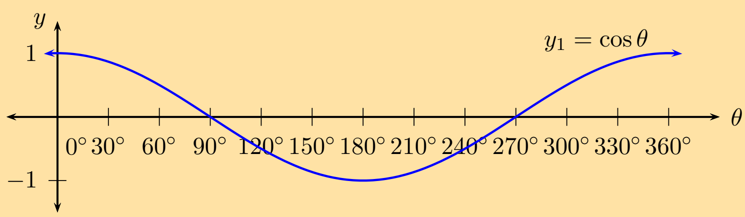
- Use tables of values if necessary.
- Use graph paper if available.

For each function in the previous problem determine the following:

- Period
- Amplitude
- Domain and range
- x- and y-intercepts
- Maximum and minimum turning points

1. $y_1 = \cos\theta$

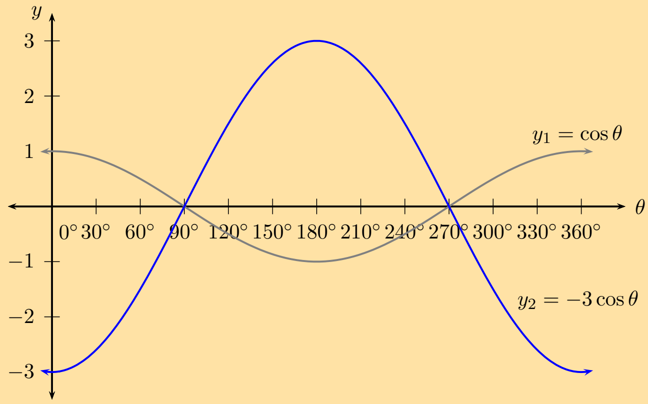
   Solution:



Period: $360°$
Amplitude: 1
Domain: $[0°; 360°]$
Range: $[-1; 1]$
x-intercepts: $(90°; 0); (270°; 0)$
y-intercepts: $(0°; 1)$
Max. turning point: $(0°; 1); (360°; 1)$
Min. turning point: $(180°; -1)$

2. $y_2 = -3\cos\theta$

Solution:



Period: $360°$

Amplitude: 1

Domain: $[0°; 360°]$

Range: $[-3; 3]$

x-intercepts: none

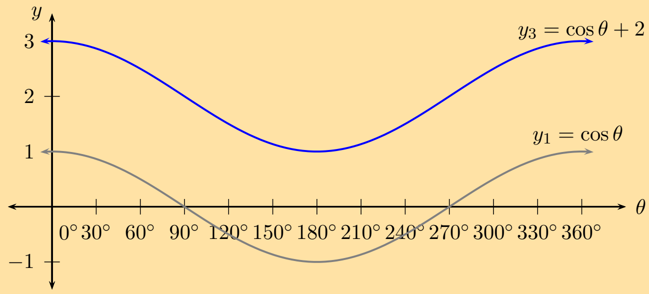
y-intercepts: $(0°; 3)$

Max. turning point: $(0°; 3); (360°; 3)$

Min. turning point: $(180°; -3)$;

3. $y_3 = \cos\theta + 2$

Solution:



Period: $360°$

Amplitude: 1

Domain: $[0°; 360°]$

Range: $[1; 3]$

x-intercepts: none

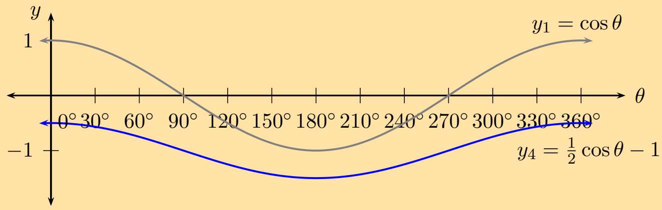
y-intercepts: $(0°; 3)$

Max. turning point: $(0°; 3); (360°; 3)$

Min. turning point: $(180°; 1)$;

4. $y_4 = \frac{1}{2}\cos\theta - 1$

Solution:



Period: $360°$

Amplitude: $\frac{1}{2}$

Domain: $[0°; 360°]$

Range: $[-\frac{1}{2}; -\frac{3}{2}]$

x-intercepts: none

y-intercepts: $(0°; -\frac{1}{2})$

Max. turning point: $(0°; -\frac{1}{2}); (360°; -\frac{1}{2})$

Min. turning point: $(180°; -\frac{3}{2})$;

Functions of the form $y = \cos(k\theta)$

Discovering the characteristics

Exercise 5 – 25: Cosine functions of the form $y = \cos k\theta$

1. Sketch the following functions for $-180° \leq \theta \leq 180°$. For each graph determine:
   - Period
   - Amplitude
   - Domain and range
   - x- and y-intercepts
   - Maximum and minimum turning points

   a) $f(\theta) = \cos 2\theta$

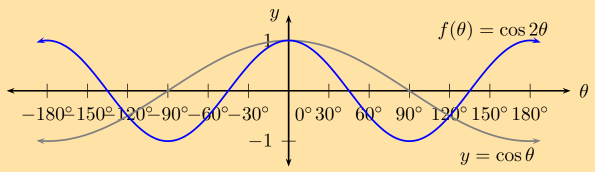
   b) $g(\theta) = \cos \frac{\theta}{3}$

   c) $h(\theta) = \cos(-2\theta)$

   d) $k(\theta) = \cos \frac{3\theta}{4}$

Solution:

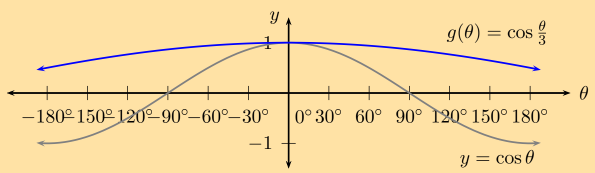
a)



For $f(\theta) = \cos 2\theta$:

Period: 180°
Amplitude: 1
Domain: $[-180^{\circ}; 180^{\circ}]$
Range: $[-1; 1]$
x-intercepts: $(-135^{\circ}; 0); (-45^{\circ}; 0); (45^{\circ}; 0); (135^{\circ}; 0)$
y-intercepts: $(0^{\circ}; 1)$
Max. turning point: $(-180^{\circ}; 1); (0^{\circ}; 1); (180^{\circ}; 1)$
Min. turning point: $(-90^{\circ}; -1); (90^{\circ}; -1)$

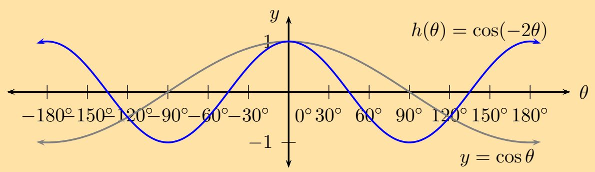
b)



For $g(\theta) = \cos \frac{\theta}{3}$:

Period: 1080°
Amplitude: 1
Domain: $[-180^{\circ}; 180^{\circ}]$
Range: $[\frac{1}{2}; 1]$
x-intercepts: none
Max. turning point: $(0^{\circ}; 1)$
Min. turning point: none

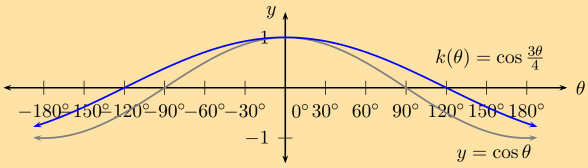
c)



For $h(\theta) = \cos(-2\theta)$:

$$\begin{aligned}
\text{Period: } & 180^\circ \\
\text{Amplitude: } & 1 \\
\text{Domain: } & [-180^\circ; 180^\circ] \\
\text{Range: } & [-1; 1] \\
x\text{-intercepts: } & (-135^\circ; 0); (-45^\circ; 0); (45^\circ; 0); (135^\circ; 0) \\
y\text{-intercepts: } & (0^\circ; 1) \\
\text{Max. turning point: } & (-180^\circ; 1); (0^\circ; 1); (180^\circ; 1) \\
\text{Min. turning point: } & (-90^\circ; -1); (90^\circ; -1)
\end{aligned}$$

d)

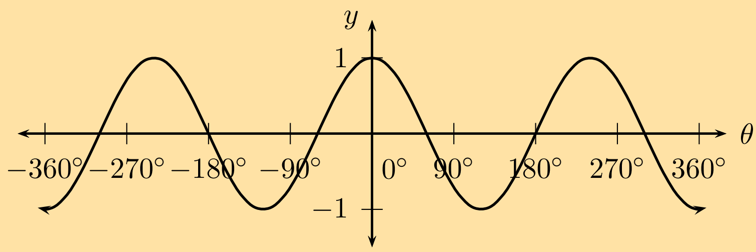


For $k(\theta) = \cos \frac{3\theta}{4}$:

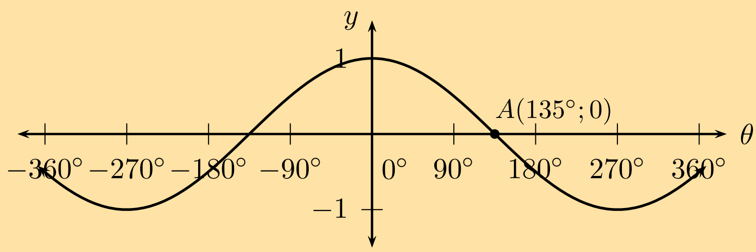
$$\begin{aligned}
\text{Period: } & 480^\circ \\
\text{Amplitude: } & 1 \\
\text{Domain: } & [-180^\circ; 180^\circ] \\
\text{Range: } & [-\frac{1}{\sqrt{2}}; 1] \\
x\text{-intercepts: } & (-120^\circ; 0); (120^\circ; 0) \\
y\text{-intercepts: } & (0^\circ; 1) \\
\text{Max. turning point: } & (0^\circ; 1) \\
\text{Min. turning point: } & \text{none}
\end{aligned}$$

2. For each graph of the form $f(\theta) = \cos k\theta$, determine the value of k:

a)



b)



Solution:

a)

$$\begin{aligned} \text{Period} &= \frac{720^\circ}{3 \text{ complete waves}} \\ &= 240^\circ \\ \therefore \frac{360^\circ}{k} &= 240^\circ \\ \therefore k &= \frac{360^\circ}{240^\circ} \\ &= \frac{3}{2} \end{aligned}$$

b)

$$\begin{aligned} \text{For } y &= \cos\theta \\ 0 &= \cos 90^\circ \\ \text{So for } A(135^\circ; 0) \quad 90^\circ &= k \times 135^\circ \\ \therefore k &= \frac{90^\circ}{135^\circ} \\ &= \frac{2}{3} \end{aligned}$$

Functions of the form $y = \cos(\theta + p)$

Discovering the characteristics

Exercise 5 – 26: Cosine functions of the form $y = \cos(\theta + p)$

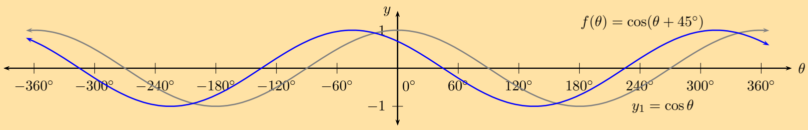
Sketch the following functions for $-360^\circ \leq \theta \leq 360^\circ$.

For each function, determine the following:

- Period
- Amplitude
- Domain and range
- x- and y-intercepts
- Maximum and minimum turning points

1. $f(\theta) = \cos(\theta + 45^\circ)$

   Solution:



Period: $360°$

Amplitude: 1

Domain: $[-360°; 360°]$

Range: $[-1; 1]$

x-intercepts: $(-315°; 0); (-150°; 0); (30°; 0); (210°; 0)$

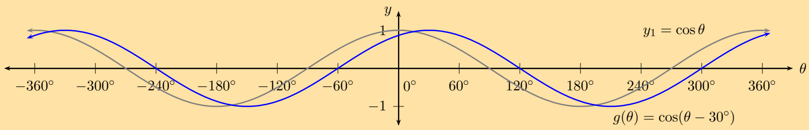
y-intercepts: $(0°; 1)$

Max. turning point: $(-45°; 1); (315°; 1)$

Min. turning point: $(-225°; -1); (135°; -1)$

2. $g(\theta) = \cos(\theta - 30°)$

Solution:



Period: $360°$

Amplitude: 1

Domain: $[-360°; 360°]$

Range: $[-1; 1]$

x-intercepts: $(-240°; 0); (-60°; 0); (120°; 0); (300°; 0)$

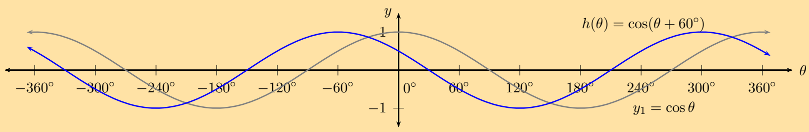
y-intercepts: $(0°; \frac{\sqrt{3}}{2})$

Max. turning point: $(-330°; 1); (30°; 1)$

Min. turning point: $(-150°; -1); (210°; -1)$

3. $h(\theta) = \cos(\theta + 60°)$

Solution:



Period: $360°$

Amplitude: 1

Domain: $[-360°; 360°]$

Range: $[-1; 1]$

x-intercepts: $(-330°; 0); (-150°; 0); (30°; 0); (210°; 0)$

y-intercepts: $(0°; \frac{1}{2})$

Max. turning point: $(-60°; 1); (300°; 1)$

Min. turning point: $(-240°; -1); (120°; -1)$

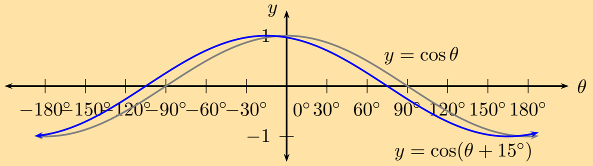
Exercise 5 – 27: The cosine function

1. Sketch the following graphs on separate axes:

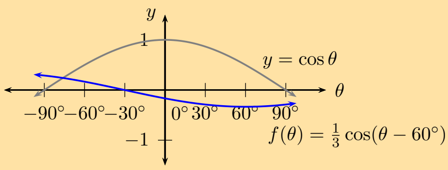
   a) $y = \cos(\theta + 15^\circ)$ for $-180^\circ \leq \theta \leq 180^\circ$
   b) $f(\theta) = \frac{1}{3}\cos(\theta - 60^\circ)$ for $-90^\circ \leq \theta \leq 90^\circ$
   c) $y = -2\cos\theta$ for $0^\circ \leq \theta \leq 360^\circ$
   d) $y = \cos(30^\circ - \theta)$ for $-360^\circ \leq \theta \leq 360^\circ$
   e) $g(\theta) = 1 + \cos(\theta - 90^\circ)$ for $0^\circ \leq \theta \leq 360^\circ$
   f) $y = \cos(2\theta + 60^\circ)$ for $-360^\circ \leq \theta \leq 360^\circ$

Solution:

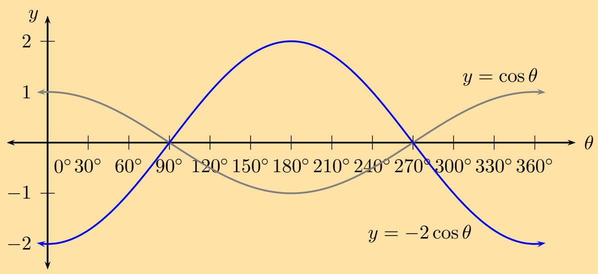
a)



b)

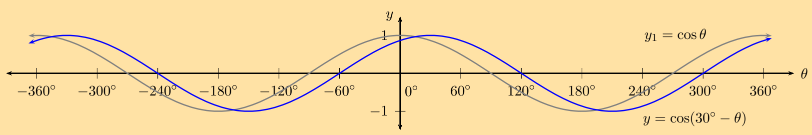


c)

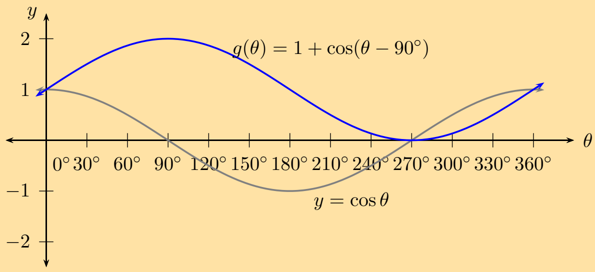


d)

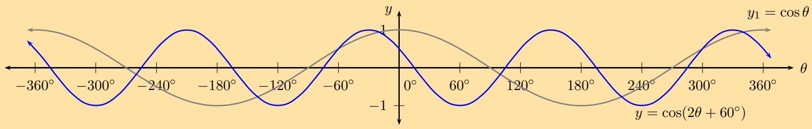
$$\begin{aligned} y &= \cos(30^\circ - \theta) \\ &= \cos\left(-(\theta - 30^\circ)\right) \\ &= \cos\left(\theta - 30^\circ\right) \end{aligned}$$



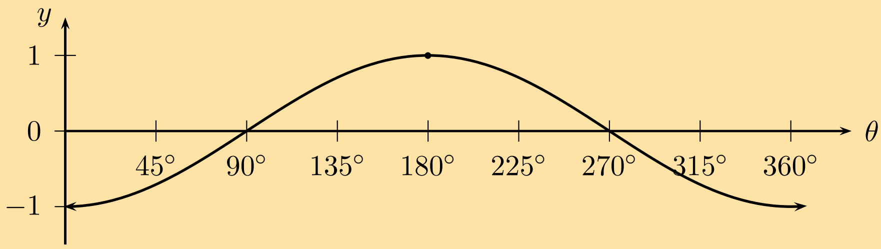
e)



f)



2. Two girls are given the following graph:



a) Audrey decides that the equation for the graph is a cosine function of the form $y = a \cos \theta$. Determine the value of a.

b) Megan thinks that the equation for the graph is a cosine function of the form $y = \cos(\theta + p)$. Determine the value of p.

c) What can they conclude?

Solution:

a) $a = -1$

b) $p = -180°$

c) $\cos(\theta - 180°) = -\cos \theta$

5.7 The tangent function

Revision

Functions of the form $y = \tan\theta$ for $0° \leq \theta \leq 360°$

Functions of the form $y = a\tan\theta + q$

Exercise 5 – 28: Revision

On separate axes, accurately draw each of the following functions for $0° \leq \theta \leq 360°$:

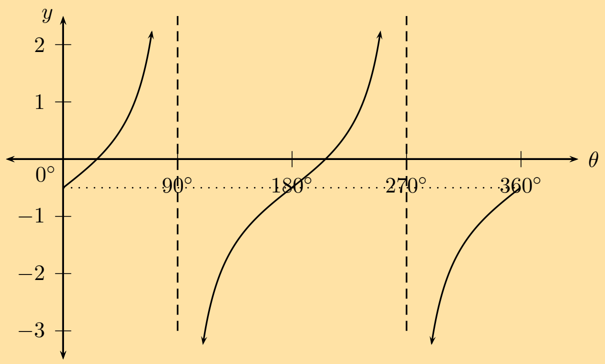
- Use tables of values if necessary.
- Use graph paper if available.

For each function determine the following:

- Period
- Domain and range
- x- and y-intercepts
- Asymptotes

1. $y_1 = \tan\theta - \frac{1}{2}$

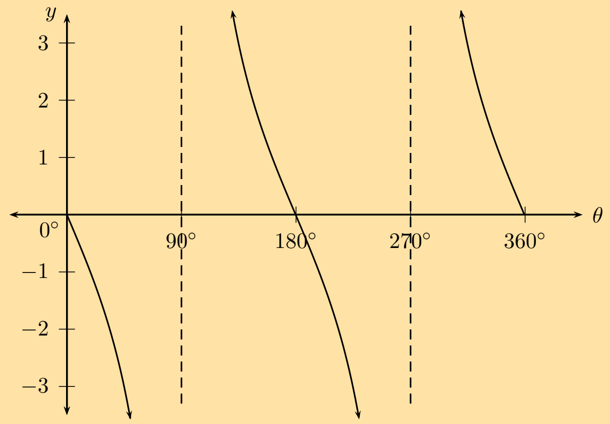
   Solution:



Period: $180°$
Domain: $[0°; 360°]$
Range: $[-\infty; \infty]$
x-intercepts: $(26{,}6°; 0); (206{,}6°; 0)$
y-intercepts: $(0°; -\frac{1}{2})$
Asymptotes: $90°; 270°$

2. $y_2 = -3\tan\theta$

Solution:



Period: $180°$

Domain: $[0°; 360°]$

Range: $[-\infty; \infty]$

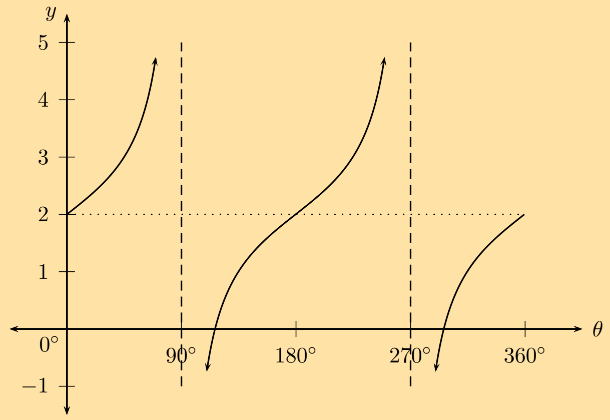
x-intercepts: $(0°; 0); (180°; 0); (360°; 0)$

y-intercepts: $(0°; 0)$

Asymptotes: $90°; 270°$

3. $y_3 = \tan\theta + 2$

Solution:



Period: $180°$

Domain: $[0°; 360°]$

Range: $[-\infty; \infty]$

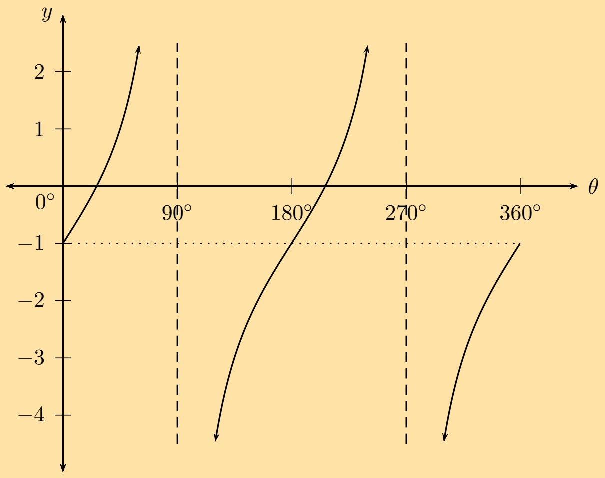
x-intercepts: $(116{,}6°; 0); (296{,}6°; 0)$

y-intercepts: $(0°; 2)$

Asymptotes: $90°; 270°$

4. $y_4 = 2\tan\theta - 1$

Solution:



Period: $180°$
Domain: $[0°; 360°]$
Range: $[-\infty; \infty]$
x-intercepts: $(26{,}6°; 0); (206{,}6°; 0)$
y-intercepts: $(0°; -1)$
Asymptotes: $90°; 270°$

Functions of the form $y = \tan(k\theta)$

Discovering the characteristics

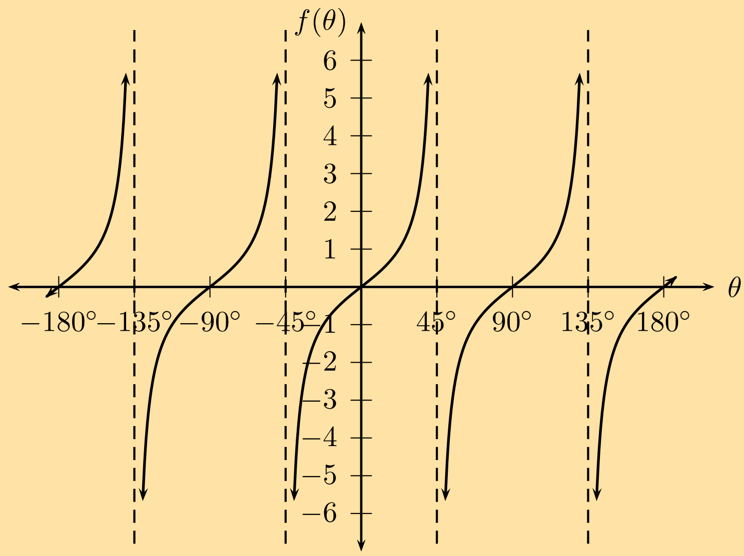
Exercise 5 – 29: Tangent functions of the form $y = \tan k\theta$

Sketch the following functions for $-180° \leq \theta \leq 180°$. For each graph determine:

- Period
- Domain and range
- x- and y-intercepts
- Asymptotes

1. $f(\theta) = \tan 2\theta$

Solution:



Period: $90°$

Domain: $[-180°; 180°]$

Range: $[-\infty; \infty]$

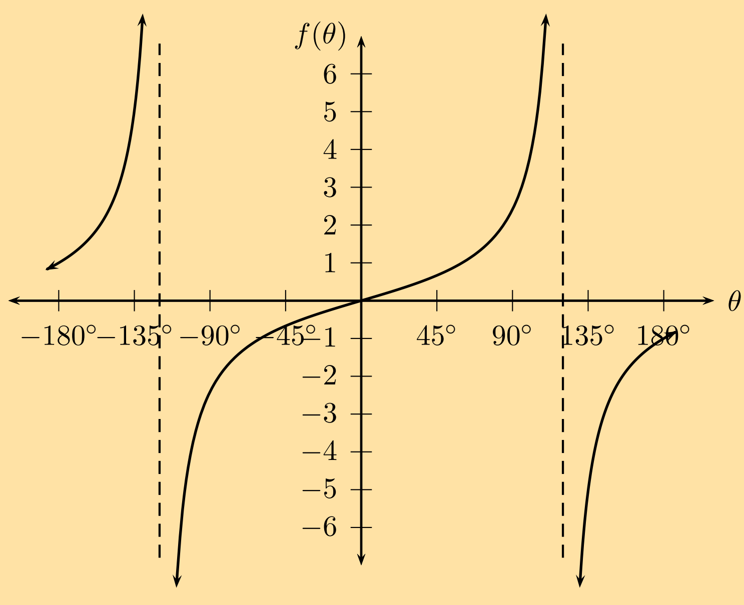
x-intercepts: $(-180°; 0); (-90°; 0); (0°; 0); (90°; 0); (180°; 0)$

y-intercepts: $(0°; 0)$

Asymptotes: $-135°; -45°; 45°; 135°$

2. $g(\theta) = \tan \frac{3\theta}{4}$

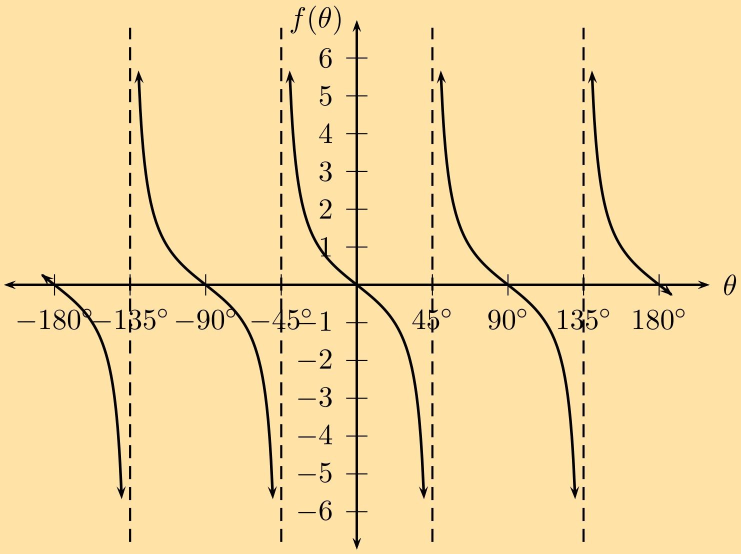
Solution:



Period: $240°$
Domain: $[-180°; 180°]$
Range: $[-\infty; \infty]$
x-intercepts: $(0°; 0)$
y-intercepts: $(0°; 0)$
Asymptotes: $-120°; 120°$

3. $h(\theta) = \tan(-2\theta)$

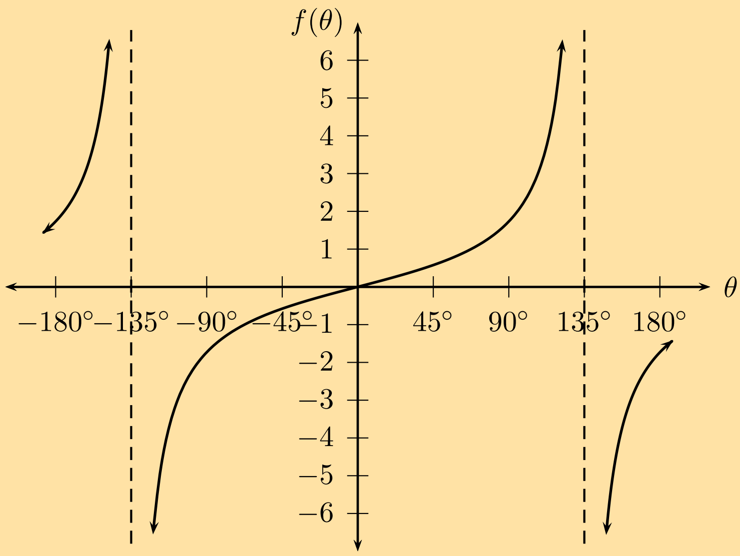
Solution:



Period: $90°$
Domain: $[-180°; 180°]$
Range: $[-\infty; \infty]$
x-intercepts: $(-180°; 0); (-90°; 0); (0°; 0); (90°; 0); (180°; 0)$
y-intercepts: $(0°; 0)$
Asymptotes: $-135°; -45°; 45°; 135°$

4. $k(\theta) = \tan \frac{2\theta}{3}$

Solution:



Period: $270°$

Domain: $[-180°; 180°]$

Range: $[-\infty; \infty]$

x-intercepts: $(0°; 0)$

y-intercepts: $(0°; 0)$

Asymptotes: $-135°; 135°$

Functions of the form $y = \tan(\theta + p)$

Discovering the characteristics

Exercise 5 – 30: Tangent functions of the form $y = \tan(\theta + p)$

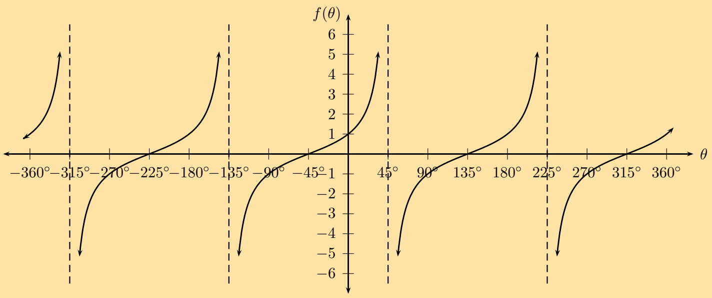
Sketch the following functions for $-360° \leq \theta \leq 360°$.

For each function, determine the following:

- Period
- Domain and range
- x- and y-intercepts
- Asymptotes

1. $f(\theta) = \tan(\theta + 45°)$

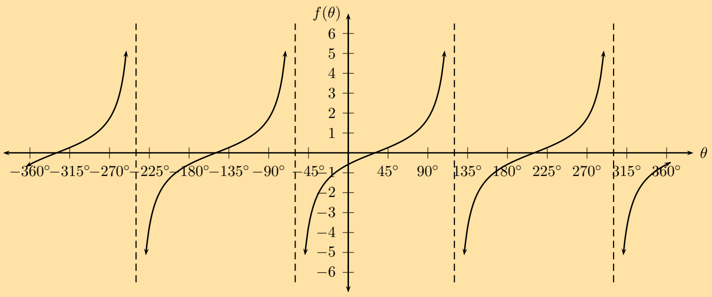
   Solution:



Period: $180°$
Domain: $[-360°; 360°]$
Range: $[-\infty; \infty]$
x-intercepts: $(-225°; 0); (-45°; 0); (135°; 0)(315°; 0)$
y-intercepts: $(0°; 0)$
Asymptotes: $-315°; -135°; 45°; 225°$

2. $g(\theta) = \tan(\theta - 30°)$

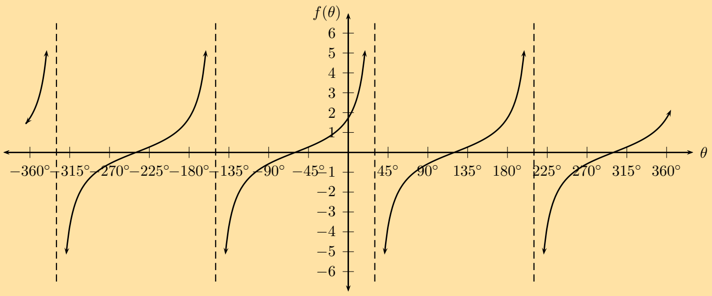
Solution:



Period: $180°$
Domain: $[-360°; 360°]$
Range: $[-\infty; \infty]$
x-intercepts: $(-330°; 0); (-150°; 0); (30°; 0)(210°; 0)$
y-intercepts: $(0°; -0{,}58)$
Asymptotes: $-315°; -135°; 45°; 225°$

3. $h(\theta) = \tan(\theta + 60°)$

Solution:



Period: $180°$
Domain: $[-360°; 360°]$
Range: $[-\infty; \infty]$
x-intercepts: $(-240°; 0); (-60°; 0); (120°; 0)(300°; 0)$
y-intercepts: $(0°; 1.73)$
Asymptotes: $-330°; -150°; 30°; 210°$

Sketching tangent graphs

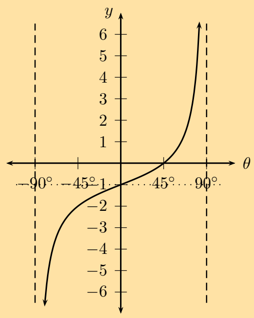
Exercise 5 – 31: The tangent function

1. Sketch the following graphs on separate axes:

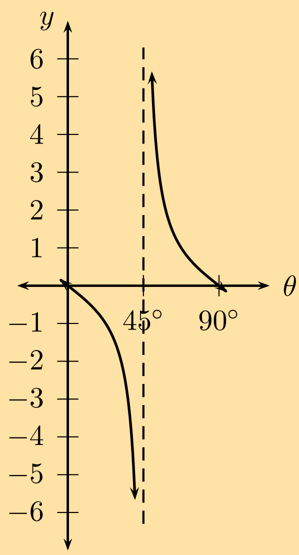
   a) $y = \tan\theta - 1$ for $-90° \leq \theta \leq 90°$
   b) $f(\theta) = -\tan 2\theta$ for $0° \leq \theta \leq 90°$
   c) $y = \frac{1}{2}\tan(\theta + 45°)$ for $0° \leq \theta \leq 360°$
   d) $y = \tan(30° - \theta)$ for $-180° \leq \theta \leq 180°$

Solution:

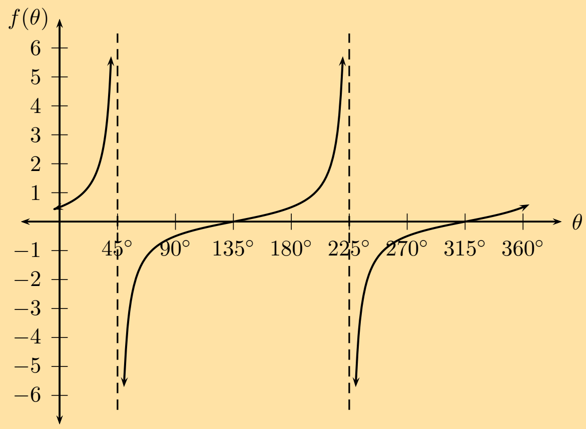
a)



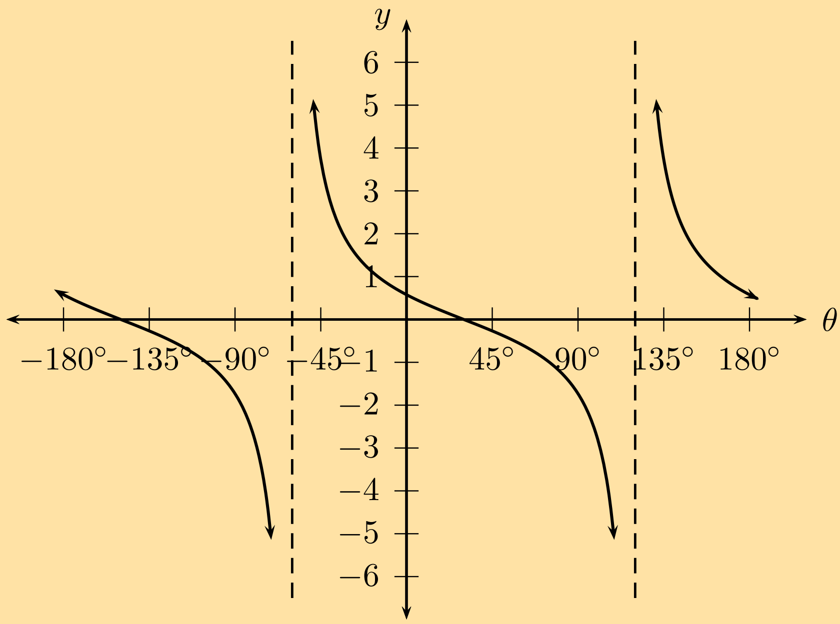
b)



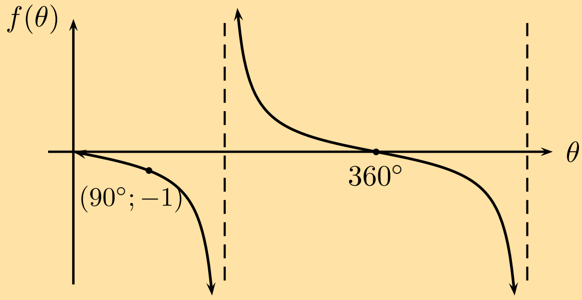
c)



d)



2. Given the graph of $y = a \tan k\theta$, determine the values of a and k.



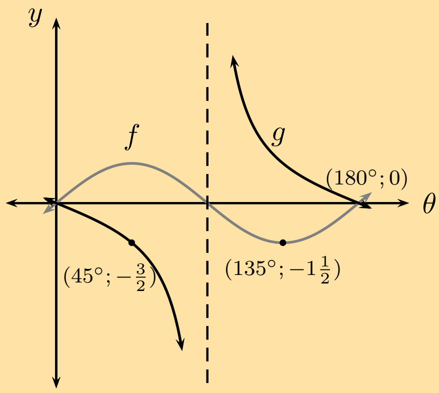
Solution:

$a = -1$; $k = \frac{1}{2}$

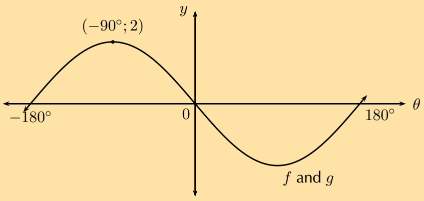
Exercise 5 – 32: Mixed exercises

1. Determine the equation for each of the following:

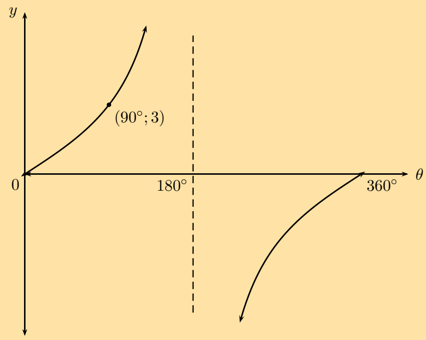
   a) $f(\theta) = a \sin k\theta$ and $g(\theta) = a \tan \theta$



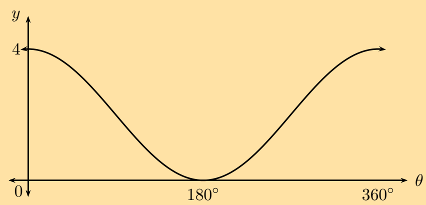
   b) $f(\theta) = a \sin k\theta$ and $g(\theta) = a \cos(\theta + p)$



   c) $y = a \tan k\theta$



   d) $y = a \cos \theta + q$



Solution:

   a) $f(\theta) = \frac{3}{2} \sin 2\theta$ and $g(\theta) = -\frac{3}{2} \tan \theta$

   b) $f(\theta) = -2 \sin \theta$ and $g(\theta) = 2 \cos(\theta + 360°)$

   c) $y = 3 \tan \frac{\theta}{2}$

   d) $y = y = 2 \cos \theta + 2$

2. Given the functions $f(\theta) = 2 \sin \theta$ and $g(\theta) = \cos \theta + 1$:

a) Sketch the graphs of both functions on the same system of axes, for $0° \leq \theta \leq 360°$. Indicate the turning points and intercepts on the diagram.

b) What is the period of f?

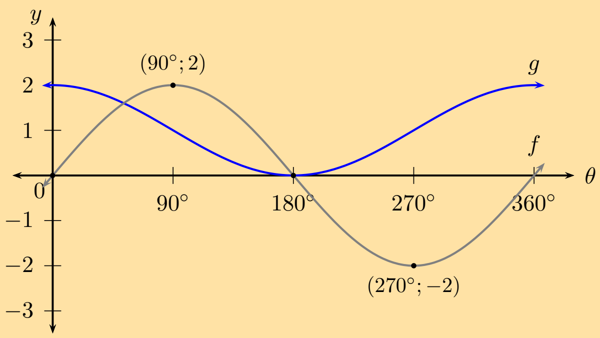
c) What is the amplitude of g?

d) Use your sketch to determine how many solutions there are for the equation $2\sin\theta - \cos\theta = 1$. Give one of the solutions.

e) Indicate on your sketch where on the graph the solution to $2\sin\theta = -1$ is found.

Solution:

a)



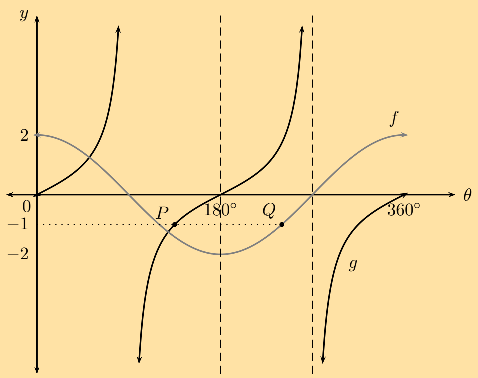
b) $360°$

c) 1

d) At $\theta = 180°$

e)

3. The sketch shows the two functions $f(\theta) = a\cos\theta$ and $g(\theta) = \tan\theta$ for $0° \leq \theta \leq 360°$. Points $P(135°; b)$ and $Q(c; -1)$ lie on $g(\theta)$ and $f(\theta)$ respectively.



a) Determine the values of a, b and c.

b) What is the period of g?

c) Solve the equation $\cos\theta = \frac{1}{2}$ graphically and show your answer(s) on the diagram.

d) Determine the equation of the new graph if g is reflected about the x-axis and shifted to the right by $45°$.

Solution:

a) $a = 2$, $b = -1$ and $c = 240°$

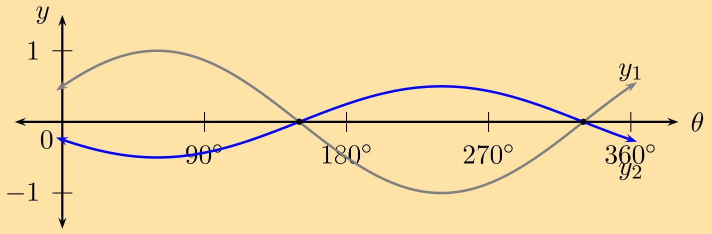
b) $180°$

c) $\theta = 60°; 300°$

d) $y = -\tan(\theta - 45°)$

4. Sketch the graphs of $y_1 = -\frac{1}{2}\sin(\theta + 30°)$ and $y_2 = \cos(\theta - 60°)$, on the same system of axes for $0° \leq \theta \leq 360°$.

Solution:



5.8 Summary

Exercise 5 – 33: End of chapter exercises

1. Show that if $a < 0$, then the range of $f(x) = a(x+p)^2 + q$ is $\{f(x) : f(x) \in (-\infty, q]\}$.

Solution:

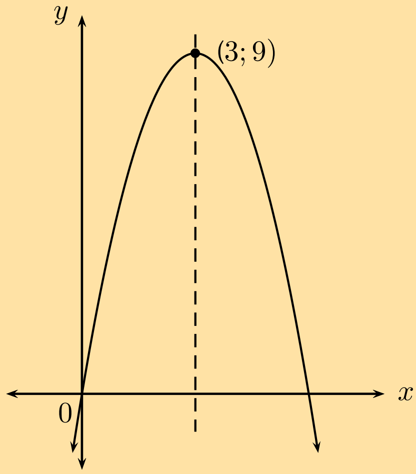
$$\begin{aligned}(x+p)^2 &\geq 0\\ a(x+p)^2 &\leq 0\\ a(x+p)^2 + q &< q\\ \therefore f(x) &< q\end{aligned}$$

2. If $(2; 7)$ is the turning point of $f(x) = -2x^2 - 4ax + k$, find the values of the constants a and k.

Solution:

$a = -2; k = -1$

3. The following graph is represented by the equation $f(x) = ax^2 + bx$. The coordinates of the turning point are $(3; 9)$. Show that $a = -1$ and $b = 6$.



Solution:

$$
\begin{aligned}
f(x) &= ax^2 + bx \\
x &= -b2a \\
\therefore 3 &= -\frac{b}{2a} \\
6a &= -b \\
\text{subst. } (3;9) \quad 9 &= a(3)^2 + b(3) \\
9 &= 9a + 3b \\
9 &= 9a + 3(-6a) \\
9 &= 9a - 18a \\
9 &= -9a \\
\therefore -1 &= a \\
\therefore b &= 6
\end{aligned}
$$

4. Given: $f(x) = x^2 - 2x + 3$. Give the equation of the new graph originating if:

   a) the graph of f is moved three units to the left.

   b) the x-axis is moved down three units.

   Solution:

   a) $y = x + 2^2 + 2$

   b) $y = x - 1^2 + 5$

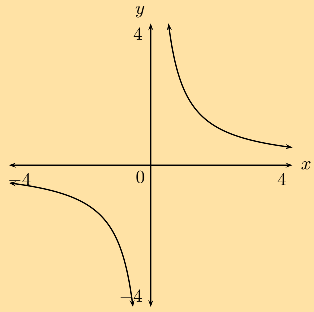
5. A parabola with turning point $(-1; -4)$ is shifted vertically by 4 units upwards. What are the coordinates of the turning point of the shifted parabola?

   Solution:

   $(-1; 0)$

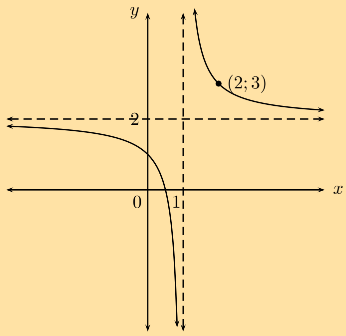
6. Plot the graph of the hyperbola defined by $y = \frac{2}{x}$ for $-4 \leq x \leq 4$. Suppose the hyperbola is shifted 3 units to the right and 1 unit down. What is the new equation then?

   Solution:



$y = \frac{2}{x-3} - 1$

7. Based on the graph of $y = \frac{k}{(x+p)} + q$, determine the equation of the graph with asymptotes $y = 2$ and $x = 1$ and passing through the point $(2;3)$.



Solution:

$y = \frac{1}{(x-1)} + 2$

8. The columns in the table below give the y-values for the following functions: $y = a^x$, $y = a^{x+1}$ and $y = a^x + 1$. Match each function to the correct column.

x	A	B	C
-2	7,25	6,25	2,5
-1	3,5	2,5	1
0	2	1	0,4
1	1,4	0,4	0,16
2	1,16	0,16	0,064

Solution:

Column A: $y = a^x + 1$; Column B: $y = a^x$; Column C: $y = a^{x+1}$

9. The graph of $f(x) = 1 + a \,.\, 2^x$ (a is a constant) passes through the origin.

   a) Determine the value of a.

   b) Determine the value of $f(-15)$ correct to five decimal places.

   c) Determine the value of x, if $P\,(x; 0{,}5)$ lies on the graph of f.

   d) If the graph of f is shifted 2 units to the right to give the function h, write down the equation of h.

Solution:

a)

$$\begin{aligned} 0 &= a \times 2^0 + 1 \\ \therefore -1 &= a \end{aligned}$$

b)

$$\begin{aligned} f(-15) &= -2^{-15} + 1 \\ &= 0{,}99997 \end{aligned}$$

c)

$$\begin{aligned} 0{,}5 &= -2^x + 1 \\ 0{,}5 &= 2^x \\ \therefore x &= -1 \end{aligned}$$

d) $h(x) = -2^{(x-2)} + 1$

10. The graph of $f(x) = a \, . \, b^x$ $(a \neq 0)$ has the point $P\,(2; 144)$ on f.

a) If $b = 0{,}75$, calculate the value of a.
b) Hence write down the equation of f.
c) Determine, correct to two decimal places, the value of $f(13)$.
d) Describe the transformation of the curve of f to h if $h(x) = f(-x)$.

Solution:

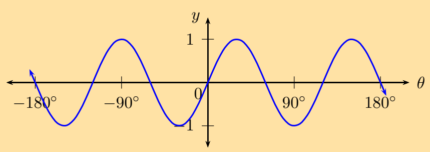
a) $a = 256$
b) $f(x) = 256\left(\frac{3}{4}\right)^x$
c) $f(13) = 6{,}08$
d) Shifted 2 units to the right

11. Using your knowledge of the effects of p and k draw a rough sketch of the following graphs without a table of values.

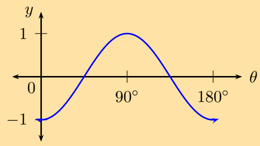
a) $y = \sin 3\theta$ for $-180° \leq \theta \leq 180°$
b) $y = -\cos 2\theta$ for $0° \leq \theta \leq 180°$
c) $y = \tan \frac{1}{2}\theta$ for $0° \leq \theta \leq 360°$
d) $y = \sin(\theta - 45°)$ for $-360° \leq \theta \leq 360°$
e) $y = \cos(\theta + 45°)$ for $0° \leq \theta \leq 360°$
f) $y = \tan(\theta - 45°)$ for $0° \leq \theta \leq 360°$
g) $y = 2\sin 2\theta$ for $-180° \leq \theta \leq 180°$
h) $y = \sin(\theta + 30°) + 1$ for $-360° \leq \theta \leq 0°$

Solution:

a)

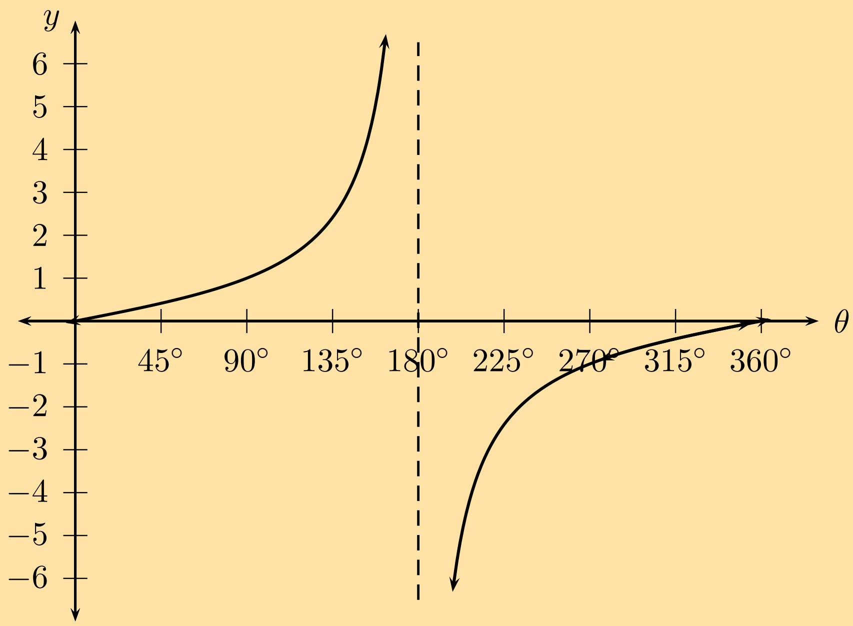


b)



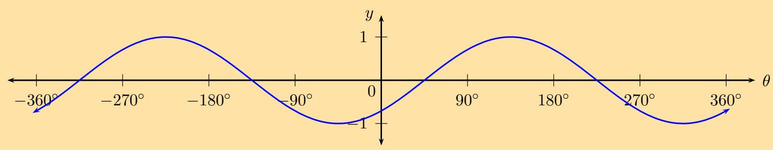
y
1
0
−1
90°
180°
θ

c)



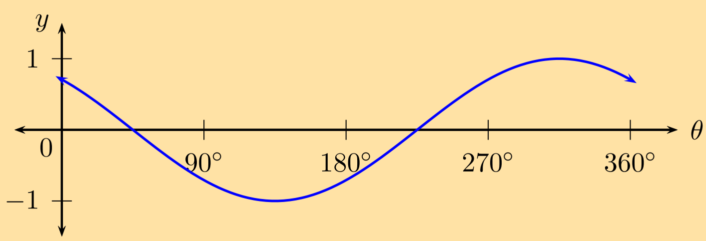
y
6
5
4
3
2
1
−1
−2
−3
−4
−5
−6
45°
90°
135°
180°
225°
270°
315°
360°
θ

d)



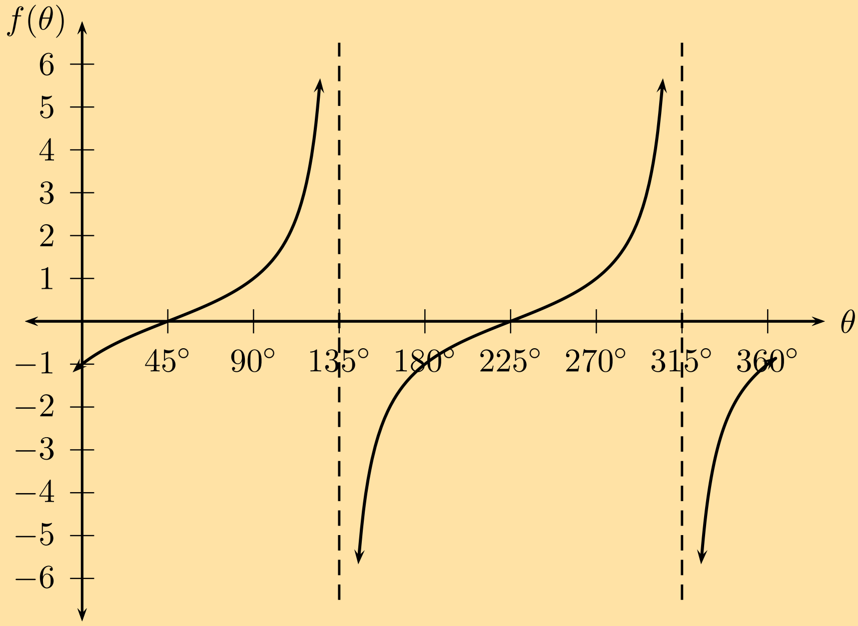
y
1
0
−1
−360°
−270°
−180°
−90°
90°
180°
270°
360°
θ

e)



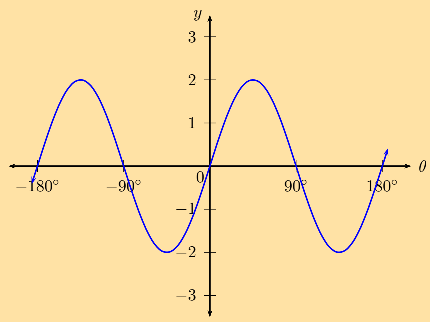
y
1
0
−1
90°
180°
270°
360°
θ

f)



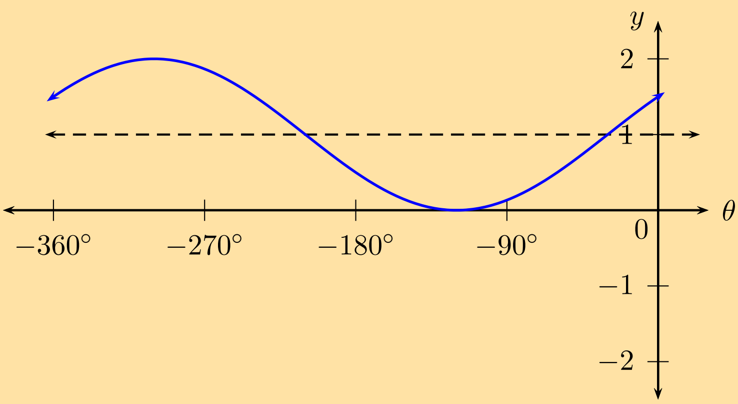
$f(\theta)$
6
5
4
3
2
1
−1
−2
−3
−4
−5
−6
45°
90°
135°
180°
225°
270°
315°
360°
θ

g)



y
3
2
1
0
−1
−2
−3
−180°
−90°
90°
180°
θ

h)



y
2
1
0
−1
−2
−360°
−270°
−180°
−90°
θ

CHAPTER 6

Trigonometry

- Emphasize that the area/sine/cosine rule does not require a right-angled triangle.
- Encourage learners to use the appropriate form of the area/sine/cosine rule.
- Remind learners that angles in the Cartesian plane are always measured from the positive x-axis.
- Important to note that:
  - $(270 \pm x)$ is not included in the section on reduction formulae.
  - co-function for tangent is not included.
- Remind learners to check that their answers are within the required interval.
- For the general solution, determine the solution in the correct quadrants and within the required interval.
- Do not label all triangles as ABC. Learners must be able to apply the formulae with different notations and in different contexts.
- Do not treat identities as equations: do NOT use = sign between the two expressions in an identity. Keep the LHS and RHS separate.
- To prove identities, we usually manipulate the more complicated expression until it looks the same as the more simple expression.

6.1 Revision

Exercise 6 – 1: Revision

1. If $p = 49°$ and $q = 32°$, use a calculator to determine whether the following statements are true of false:

   a) $\sin p + 3\sin p = 4\sin p$

   b) $\frac{\sin q}{\cos q} = \tan q$

   c) $\cos(p - q) = \cos p - \cos q$

   d) $\sin(2p) = 2\sin p\cos p$

   Solution:

a)

$$\begin{aligned}
\text{LHS} &= \sin p + 3\sin p \\
&= \sin 49° + 3\sin 49° \\
&= 0{,}754\ldots + 2{,}264\ldots \\
&= 3{,}012 \\
\text{RHS} &= 4\sin p \\
&= 3{,}012 \\
\therefore \text{LHS} &= \text{RHS}
\end{aligned}$$

b)

$$\begin{aligned}
\text{LHS} &= \frac{\sin q}{\cos q} \\
&= \frac{\sin 32°}{\cos 32°} \\
&= \frac{0{,}53\ldots}{0{,}84\ldots} \\
&= 0{,}62 \\
\text{RHS} &= \tan q \\
&= \tan 32° \\
&= 0{,}62 \\
\therefore \text{LHS} &= \text{RHS}
\end{aligned}$$

c)

$$\begin{aligned}
\text{LHS} &= \cos(p - q) \\
&= \cos(49° - 32°) \\
&= \cos 17° \\
&= 0{,}96 \\
\text{RHS} &= \cos p - \cos q \\
&= \cos 49° - \cos 32° \\
&= 0{,}65\ldots - 0{,}84\ldots \\
&= -0{,}19 \\
\therefore \text{LHS} &\neq \text{RHS}
\end{aligned}$$

d)

$$\begin{aligned}
\text{LHS} &= \sin(2p) \\
&= \sin 2(49°) \\
&= \sin 98° \\
&= 0{,}99 \\
\text{RHS} &= 2\sin p\cos p \\
&= 2\sin 49° \cos 49° \\
&= 2 \times 0{,}75\ldots \times 0{,}65\ldots \\
&= 0{,}99 \\
\therefore \text{LHS} &= \text{RHS}
\end{aligned}$$

2. Determine the following angles (correct to one decimal place):

a) $\cos\alpha = 0{,}64$

b) $\sin\theta + 2 = 2{,}65$

c) $\frac{1}{2}\cos 2\beta = 0{,}3$

d) $\tan\frac{\theta}{3} = \sin 48°$

e) $\cos 3p = 1{,}03$

f) $2\sin 3\beta + 1 = 2{,}6$

g) $\dfrac{\sin\theta}{\cos\theta} = 4\frac{2}{3}$

Solution:

a)

$$\begin{aligned}\cos\alpha &= 0{,}64\\ \therefore \alpha &= \cos^{-1} 0{,}64\\ &= 50{,}2°\end{aligned}$$

b)

$$\begin{aligned}\sin\theta + 2 &= 2{,}65\\ \sin\theta &= 0{,}65\\ \therefore \theta &= \sin^{-1} 0{,}65\\ &= 40{,}5°\end{aligned}$$

c)

$$\begin{aligned}\frac{1}{2}\cos 2\beta &= 0{,}3\\ \cos 2\beta &= 0{,}6\\ \therefore 2\beta &= \cos^{-1} 0{,}6\\ &= 53{,}1°\ldots\\ \therefore \beta &= \frac{1}{2} \times 53{,}1°\ldots\\ &= 26{,}6°\end{aligned}$$

d)

$$\begin{aligned}\tan\frac{\theta}{3} &= \sin 48°\\ &= 0{,}74\ldots\\ \therefore \frac{\theta}{3} &= \tan^{-1} 0{,}74\ldots\\ &= 36{,}6°\ldots\\ \therefore \theta &= 3 \times 36{,}6°\ldots\\ &= 109{,}8°\end{aligned}$$

e)

$$\begin{aligned}\cos 3p &= 1{,}03\\ \therefore 3p &= \cos^{-1} 1{,}03\\ &= \text{No solution}\end{aligned}$$

f)

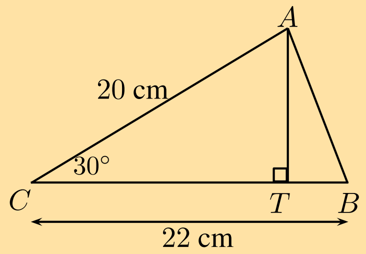
$$\begin{aligned}2\sin 3\beta + 1 &= 2{,}6\\ 2\sin 3\beta &= 1{,}6\\ \sin 3\beta &= 0{,}8\\ \therefore 3\beta &= \sin^{-1} 0{,}8\\ &= 53{,}1^\circ \ldots\\ \therefore \beta &= \frac{1}{3} \times 53{,}1^\circ \ldots\\ &= 17{,}7^\circ\end{aligned}$$

g)

$$\begin{aligned}\frac{\sin\theta}{\cos\theta} &= 4\frac{2}{3}\\ \therefore \tan\theta &= 2{,}66\ldots\\ \therefore \theta &= \tan^{-1} 2{,}66\ldots\\ &= 69{,}4^\circ\end{aligned}$$

3. In $\triangle ABC$, $A\hat{C}B = 30^\circ$, $AC = 20$ cm and $BC = 22$ cm. The perpendicular line from A intersects BC at T.

Determine:



a) the length TC

b) the length AT

c) the angle $B\hat{A}T$

Solution:

a)

$$\text{In } \triangle CAT: \quad \begin{aligned}\cos 30^\circ &= \frac{TC}{CA}\\ &= \frac{TC}{20}\\ \therefore TC &= 20 \times \cos 30^\circ\\ &= 17{,}3 \text{ cm}\end{aligned}$$

b)

$$\text{In } \triangle CAT: \quad \begin{aligned}\sin 30^\circ &= \frac{AT}{CA}\\ &= \frac{AT}{20}\\ \therefore AT &= 20 \times \sin 30^\circ\\ &= 10 \text{ cm}\end{aligned}$$

c)

$$\begin{aligned} CB &= CT + TB \\ 22 &= 17{,}3 + TB \\ \therefore TB &= 22 - 17{,}3 \\ &= 4{,}7 \text{ cm} \\ \text{In } \triangle BAT: \quad \tan \hat{B} &= \frac{AT}{BT} \\ &= \frac{10}{4{,}7} \\ \therefore \hat{B} &= \tan^{-1}\left(\frac{10}{4{,}7}\right) \\ &= 64{,}8° \end{aligned}$$

4. A rhombus has a perimeter of 40 cm and one of the internal angles is $30°$.

   a) Determine the length of the sides.
   b) Determine the lengths of the diagonals.
   c) Calculate the area of the rhombus.

Solution:

a)

$$\begin{aligned} \text{Length sides } &= \frac{40}{4} \\ &= 10 \text{ cm} \end{aligned}$$

b) Diagonals of a rhombus bisect each other at right-angles and also bisect interior angles. Let the shorter diagonal be $2x$:

$$\begin{aligned} \sin 15° &= \frac{x}{10} \\ \therefore x &= 10 \sin 15° \\ \therefore \text{Shorter diagonal } &= 2 \times 10 \sin 15° \\ &= 5{,}2 \text{ cm} \end{aligned}$$

Let the longer diagonal be $2y$:

$$\begin{aligned} \cos 15° &= \frac{y}{10} \\ \therefore y &= 10 \cos 15° \\ \therefore \text{Longer diagonal } &= 2 \times 10 \cos 15° \\ &= 19{,}3 \text{ cm} \end{aligned}$$

c)

$$\begin{aligned} \text{Area } &= b \times h_{\perp} \\ &= 10 \times 19{,}3 \times \sin 15° \\ &= 50 \text{ cm}^2 \end{aligned}$$

5. Simplify the following without using a calculator:

a) $2\sin 45^\circ \times 2\cos 45^\circ$

b) $\cos^2 30^\circ - \sin^2 60^\circ$

c) $\sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ - \tan 45^\circ$

d) $4\sin 60^\circ \cos 30^\circ - 2\tan 45^\circ + \tan 60^\circ - 2\sin 60^\circ$

e) $\sin 60^\circ \times \sqrt{2\tan 45^\circ + 1} - \sin 30^\circ$

Solution:

a)

$$\begin{aligned} 2\sin 45^\circ \times 2\cos 45^\circ &= 2 \times \frac{1}{\sqrt{2}} \times 2 \times \frac{1}{\sqrt{2}} \\ &= 4 \times \frac{1}{(\sqrt{2})} \\ &= \frac{4}{2} \\ &= 2 \end{aligned}$$

b)

$$\begin{aligned} \cos^2 30^\circ - \sin^2 60^\circ &= \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 \\ &= 0 \end{aligned}$$

c)

$$\begin{aligned} &\sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ - \tan 45^\circ \\ &= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} - \frac{1}{2} \times \frac{1}{2} - 1 \\ &= \frac{3}{4} - \frac{1}{4} - 1 \\ &= -1\frac{1}{2} \end{aligned}$$

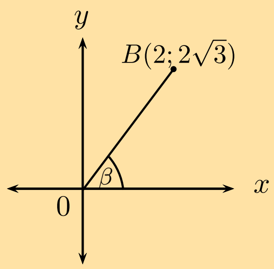
d)

$$\begin{aligned} &4\sin 60^\circ \cos 30^\circ - 2\tan 45^\circ + \tan 60^\circ - 2\sin 60^\circ \\ &= 4 \times \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} - 2 \times 1 + \sqrt{3} - 2 \times \frac{\sqrt{3}}{2} \\ &= 3 - 2 + \sqrt{3} - \sqrt{3} \\ &= 1 \end{aligned}$$

e)

$$\begin{aligned} \sin 60^\circ \times \sqrt{2\tan 45^\circ + 1} - \sin 30^\circ &= \frac{\sqrt{3}}{2} \times \sqrt{2(1) + 1} - \frac{1}{2} \\ &= \frac{\sqrt{3}}{2} \times \sqrt{3} - \frac{1}{2} \\ &= \frac{3}{2} - \frac{1}{2} \\ &= 1 \end{aligned}$$

6. Given the diagram below.



Determine the following without using a calculator:

a) β

b) $\cos\beta$

c) $\cos^2\beta + \sin^2\beta$

Solution:

a)

$$\begin{aligned} \tan\beta &= \frac{y}{x} \\ \tan\beta &= \frac{2\sqrt{3}}{2} \\ &= \sqrt{3} \\ \therefore \beta &= 60^\circ \end{aligned}$$

b)

$$\begin{aligned} OB^2 &= x^2 + y^2 \quad (\text{ Pythagoras}) \\ OB^2 &= 2^2 + (2\sqrt{3})^2 \\ &= 4 + 4(3) \\ &= 16 \\ \therefore OB &= 4 \\ \cos\beta &= \frac{x}{OB} \\ &= \frac{2}{4} \\ &= \frac{1}{2} \end{aligned}$$

c)

$$\begin{aligned} \cos^2\beta + \sin^2\beta &= \left(\frac{1}{2}\right)^2 + \left(\frac{2\sqrt{3}}{4}\right)^2 \\ &= \frac{1}{4} + \frac{3}{4} \\ &= 1 \end{aligned}$$

7. The 10 m ladder of a fire truck leans against the wall of a burning building at an angle of 60°. The height of an open window is 9 m from the ground. Will the ladder reach the window?

Solution:

$$\frac{h}{10} = \sin 60^\circ$$
$$= \frac{\sqrt{3}}{2}$$
$$\therefore h = \frac{10\sqrt{3}}{2}$$
$$= 5\sqrt{3}$$
$$= 8{,}66 \text{ m} \therefore \text{ No, the ladder is too short.}$$

6.2 Trigonometric identities

Exercise 6 – 2: Trigonometric identities

1. Reduce the following to one trigonometric ratio:

   a) $\dfrac{\sin\alpha}{\tan\alpha}$

   b) $\cos^2\theta\tan^2\theta + \tan^2\theta\sin^2\theta$

   c) $1 - \sin\theta\cos\theta\tan\theta$

   d) $\left(\dfrac{1-\cos^2\beta}{\cos^2\beta}\right) - \tan^2\beta$

Solution:

a)

$$\begin{aligned}\frac{\sin\alpha}{\tan\alpha} &= \frac{\sin\alpha}{\tan\alpha}\\ &= \frac{\sin\alpha}{\frac{\sin\alpha}{\cos\alpha}}\\ &= \sin\alpha \times \frac{\cos\alpha}{\sin\alpha}\\ &= \cos\alpha\end{aligned}$$

b)

$$\begin{aligned}\cos^2\theta\tan^2\theta + \tan^2\theta\sin^2\theta &= \tan^2\theta\left(\cos^2\theta + \sin^2\theta\right)\\ &= \tan^2\theta\,(1)\\ &= \tan^2\theta\end{aligned}$$

c)

$$\begin{aligned}1 - \sin\theta\cos\theta\tan\theta &= 1 - \sin\theta\cos\theta\left(\frac{\sin\theta}{\cos\theta}\right)\\ &= 1 - \sin^2\theta\\ &= \cos^2\theta\end{aligned}$$

d)

$$\left(\frac{1-\cos^2\beta}{\cos^2\beta}\right)-\tan^2\beta=\left(\frac{\sin^2\beta}{\cos^2\beta}\right)-\tan^2\beta$$
$$=\left(\tan^2\beta\right)-\tan^2\beta$$
$$=0$$

2. Prove the following identities and state restrictions where appropriate:

a) $\dfrac{1+\sin\theta}{\cos\theta}=\dfrac{\cos\theta}{1-\sin\theta}$

b) $\sin^2\alpha+(\cos\alpha-\tan\alpha)(\cos\alpha+\tan\alpha)=1-\tan^2\alpha$

c) $\dfrac{1}{\cos\theta}-\dfrac{\cos\theta\tan^2\theta}{1}=\cos\theta$

d) $\dfrac{2\sin\theta\cos\theta}{\sin\theta+\cos\theta}=\sin\theta+\cos\theta-\dfrac{1}{\sin\theta+\cos\theta}$

e) $\left(\dfrac{\cos\beta}{\sin\beta}+\tan\beta\right)\cos\beta=\dfrac{1}{\sin\beta}$

f) $\dfrac{1}{1+\sin\theta}+\dfrac{1}{1-\sin\theta}=d\frac{2\tan\theta}{\sin\theta\cos\theta}$

g) $\dfrac{(1+\tan^2\alpha)\cos\alpha}{(1-\tan\alpha)}=\dfrac{1}{\cos\alpha-\sin\alpha}$

Solution:

a)

$$\begin{aligned}\text{LHS} &= \frac{1+\sin\theta}{\cos\theta}\\ &= \frac{(1+\sin\theta)}{\cos\theta})\times\frac{(1-\sin\theta)}{1-\sin\theta}\\ &= \frac{1-\sin^2\theta}{\cos\theta(1-\sin\theta)}\\ &= \frac{\cos^2\theta}{\cos\theta(1-\sin\theta)}\\ &= \frac{\cos\theta}{(1-\sin\theta)}\\ &= \text{RHS}\end{aligned}$$

Restrictions: undefined where $\cos\theta=0, \sin\theta=1$ and where $\tan\theta$is undefined.

Therefore $\theta\neq 90°; 270°$.

b)

$$\begin{aligned}\text{LHS} &= \sin^2\alpha+(\cos\alpha-\tan\alpha)(\cos\alpha+\tan\alpha)\\ &= \sin^2\alpha+\cos^2\alpha-\tan^2\alpha\\ &= 1-\tan^2\alpha\\ &= \text{RHS}\end{aligned}$$

Restrictions: undefined where $\tan\theta$ is undefined.

Therefore $\theta\neq 90°; 270°$.

c)

$$\begin{aligned}\text{LHS} &= \frac{1}{\cos\theta} - \frac{\cos\theta\tan^2\theta}{1}\\ &= \frac{1-\cos^2\theta\times\tan^2\theta}{\cos\theta}\\ &= \frac{1-\cos^2\theta\times\dfrac{\sin^2\theta}{\cos^2\theta}}{\cos\theta}\\ &= \frac{1-\sin^2\theta}{\cos\theta}\\ &= \frac{\cos^2\theta}{\cos\theta}\\ &= \cos\theta\\ &= \text{RHS}\end{aligned}$$

Restrictions: undefined where $\cos\theta = 0$ and where $\tan\theta$is undefined. Therefore $\theta \neq 90°; 270°$.

d)

$$\begin{aligned}\text{RHS} &= \sin\theta + \cos\theta - \frac{1}{\sin\theta+\cos\theta}\\ &= \frac{\sin^2\theta + \sin\theta\cos\theta + \cos\theta\sin\theta + \cos^2\theta - 1}{\sin\theta+\cos\theta}\\ &= \frac{1+2\sin\theta\cos\theta - 1}{\sin\theta+\cos\theta}\\ &= \frac{2\sin\theta\cos\theta}{\sin\theta+\cos\theta}\\ &= \text{LHS}\end{aligned}$$

e)

$$\begin{aligned}\text{LHS} &= \left(\frac{\cos\beta}{\sin\beta} + \frac{\sin\beta}{\cos\beta}\right)\\ &= \left(\frac{\cos^2\beta + \sin^2\beta}{\sin\beta\cos\beta}\right)\cos\beta\\ &= \frac{1}{\sin\beta}\\ &= \text{RHS}\end{aligned}$$

f)

$$\begin{aligned}\text{LHS} &= \frac{1-\sin\theta+1+\sin\theta}{(1+\sin\theta)(1-\sin\theta)}\\ &= \frac{2}{1-\sin^2\theta}\\ &= \frac{2}{\cos^2\theta}\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \frac{2\tan\theta}{\sin\theta\cos\theta}\\ &= \frac{2\sin\theta}{\sin\theta\cos\theta\cos\theta}\\ &= \frac{2}{\cos^2\theta}\\ \therefore \text{LHS} &= \text{RHS}\end{aligned}$$

g)

$$\begin{aligned}
\text{LHS} &= \frac{(1+\tan^2\alpha)\cos\alpha}{(1-\tan\alpha)} \\
&= \frac{\left(1+\frac{\sin^2\alpha}{\cos^2\alpha}\right)\cos\alpha}{\left(1-\frac{\sin\alpha}{\cos\alpha}\right)} \\
&= \frac{\left(\frac{\cos^2\alpha+\sin^2\alpha}{\cos^2\alpha}\right)\cos\alpha}{\frac{\cos\alpha-\sin\alpha}{\cos\alpha}} \\
&= \frac{1}{\cos\alpha}\times\frac{\cos\alpha}{\cos\alpha-\sin\alpha} \\
&= \frac{1}{\cos\alpha-\sin\alpha} \\
&= \text{RHS}
\end{aligned}$$

6.3 Reduction formula

Deriving reduction formulae

Exercise 6 – 3: Reduction formulae for function values of $180^\circ \pm \theta$

1. Determine the value of the following expressions without using a calculator:

   a) $\tan 150^\circ \sin 30^\circ - \cos 210^\circ$

   b) $(1+\cos 120^\circ)(1-\sin^2 240^\circ)$

   c) $\cos^2 140^\circ + \sin^2 220^\circ$

   Solution:

a)

$$\begin{aligned}\tan 150^\circ \sin 30^\circ - \cos 210^\circ &= \tan(180^\circ - 30^\circ)\sin 30^\circ - \cos(180^\circ + 30^\circ)\\ &= -\tan 30^\circ \sin 30^\circ + \cos 30^\circ\\ &= -\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{2}\right) + \frac{\sqrt{3}}{2}\\ &= -\frac{1}{2\sqrt{3}} + \frac{\sqrt{3}}{2}\\ &= -\frac{1}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} + \frac{\sqrt{3}}{2}\\ &= -\frac{\sqrt{3}}{6} + \frac{\sqrt{3}}{2}\\ &= \frac{-\sqrt{3} + 3\sqrt{3}}{6}\\ &= \frac{2\sqrt{3}}{6}\\ &= \frac{\sqrt{3}}{3}\end{aligned}$$

b)

$$\begin{aligned}&(1 + \cos 120^\circ)(1 - \sin^2 240^\circ)\\ &= (1 + \cos(180^\circ - 60^\circ))\left(1 - \sin^2(180^\circ + 60^\circ)\right)\\ &= (1 - \cos 60^\circ)\left(1 - \sin^2 60^\circ\right)\\ &= \left(1 - \frac{1}{2}\right)\left(1 - \left(\frac{\sqrt{3}}{2}\right)^2\right)\\ &= \left(\frac{1}{2}\right)\left(1 - \frac{3}{4}\right)\\ &= \frac{1}{2} \times \frac{1}{4}\\ &= \frac{1}{8}\end{aligned}$$

c)

$$\begin{aligned}\cos^2 140^\circ + \sin^2 220^\circ &= \cos^2(180^\circ - 40^\circ) + \sin^2(180^\circ + 40^\circ)\\ &= \cos^2 40^\circ + \sin^2 40^\circ\\ &= 1\end{aligned}$$

2. Write the following in terms of a single trigonometric ratio:

a) $\tan(180^\circ - \theta) \times \sin(180^\circ + \theta)$

b) $\dfrac{\tan(180^\circ + \theta)\cos(180^\circ - \theta)}{\sin(180^\circ - \theta)}$

Solution:

a)

$$
\begin{aligned}
\tan(180° - \theta) \times \sin(180° + \theta) &= -\tan\theta \times (-\sin\theta) \\
&= -\frac{\sin\theta}{\cos\theta} \times (-\sin\theta) \\
&= \frac{\sin^2\theta}{\cos\theta} \\
&= \frac{1 - \cos^2\theta}{\cos\theta}
\end{aligned}
$$

b)

$$
\begin{aligned}
\frac{\tan(180° + \theta)\cos(180° - \theta)}{\sin(180° - \theta)} &= -\frac{\tan\theta\cos\theta}{\sin\theta} \\
&= -\frac{\frac{\sin\theta}{\cos\theta}\cos\theta}{\sin\theta} \\
&= -\frac{\sin\theta}{\sin\theta} \\
&= -1
\end{aligned}
$$

3. If $t = \tan 40°$, express the following in terms of t:

   a) $\tan 140° + 3\tan 220°$

   b) $\frac{\cos 220°}{\sin 140°}$

Solution:

a)

$$
\begin{aligned}
\tan 140° + 3\tan 220° &= \tan(180° - 40°) + 3\tan(180° + 40°) \\
&= -\tan 40° + 3\tan 40° \\
&= 2\tan 40° \\
&= 2t
\end{aligned}
$$

b)

$$
\begin{aligned}
\frac{\cos 220°}{\sin 140°} &= \frac{\cos(180° + 40°)}{\sin(180° - 40°)} \\
&= \frac{-\cos 40°}{\sin 40°} \\
&= -\left(\frac{\sin 40°}{\cos 40°}\right)^{-1} \\
&= -(\tan 40°)^{-1} \\
&= -\frac{1}{\tan 40°} \\
&= -\frac{1}{t}
\end{aligned}
$$

Exercise 6 – 4: Using reduction formula

1. Simplify the following:

   a) $\dfrac{\tan(180° - \theta)\sin(360° + \theta)}{\cos(180° + \theta)\tan(360° - \theta)}$

   b) $\cos^2(360° + \theta) + \cos(180° + \theta)\tan(360° - \theta)\sin(360° + \theta)$

   c) $\dfrac{\sin(360° + \alpha)\tan(180° + \alpha)}{\cos(360° - \alpha)\tan^2(360° + \alpha)}$

   Solution:

   a)
$$\begin{aligned}\frac{\tan(180° - \theta)\sin(360° + \theta)}{\cos(180° + \theta)\tan(360° - \theta)} &= \frac{-\tan\theta\sin\theta}{(-\cos\theta)(-\tan\theta)}\\ &= -\frac{\sin\theta}{\cos\theta}\\ &= -\tan\theta\end{aligned}$$

   b)
$$\begin{aligned}&\cos^2(360° + \theta) + \cos(180° + \theta)\tan(360° - \theta)\sin(360° + \theta)\\ &= \cos^2\theta + (-\cos\theta)(-\tan\theta)(\sin\theta)\\ &= \cos^2\theta + (\cos\theta)(\frac{\sin\theta}{\cos\theta})(\sin\theta)\\ &= \cos^2\theta + \sin^2\theta\\ &= 1\end{aligned}$$

   c)
$$\begin{aligned}\frac{\sin(360° + \alpha)\tan(180° + \alpha)}{\cos(360° - \alpha)\tan^2(360° + \alpha)} &= \frac{\sin\alpha\tan\alpha}{\cos\alpha\tan^2\alpha}\\ &= \frac{\sin\alpha\frac{\sin\alpha}{\cos\alpha}}{\cos\alpha\frac{\sin^2\alpha}{\cos^2\alpha}}\\ &= \frac{\frac{\sin^2\alpha}{\cos\alpha}}{\frac{\sin^2\alpha}{\cos\alpha}}\\ &= \frac{\sin^2\alpha}{\cos\alpha}\times\frac{\cos\alpha}{\sin^2\alpha}\\ &= 1\end{aligned}$$

2. Write the following in terms of $\cos\beta$:
   $\dfrac{\cos(360° - \beta)\cos(-\beta) - 1}{\sin(360° + \beta)\tan(360° - \beta)}$

   Solution:

$$\begin{aligned}\frac{\cos(360° - \beta)\cos(-\beta) - 1}{\sin(360° + \beta)\tan(360° - \beta)} &= \frac{\cos\beta\cos\beta - 1}{(-\sin\beta)(-\tan\beta)}\\ &= \frac{-(1 - \cos^2\beta)}{\sin\beta\frac{\sin\beta}{\cos\beta}}\\ &= \frac{-\sin^2\beta}{\sin^2\beta}\times\frac{\cos\beta}{1}\\ &= -\cos\beta\end{aligned}$$

3. Simplify the following without using a calculator:

a) $\dfrac{\cos 300^\circ \tan 150^\circ}{\sin 225^\circ \cos(-45^\circ)}$

b) $3\tan 405^\circ + 2\tan 330^\circ \cos 750^\circ$

c) $\dfrac{\cos 315^\circ \cos 405^\circ + \sin 45^\circ \sin 135^\circ}{\sin 750^\circ}$

d) $\tan 150^\circ \cos 390^\circ - 2\sin 510^\circ$

e) $\dfrac{2\sin 120^\circ + 3\cos 765^\circ - 2\sin 240^\circ - 3\cos 45^\circ}{5\sin 300^\circ + 3\tan 225^\circ - 6\cos 60^\circ}$

Solution:

a)

$$\begin{aligned}\frac{\cos 300^\circ \tan 150^\circ}{\sin 225^\circ \cos(-45^\circ)} &= \frac{\cos(360^\circ - 60^\circ)\tan(180^\circ - 30^\circ)}{\sin(180^\circ + 45^\circ)\cos(-45^\circ)} \\ &= \frac{\cos 60^\circ(-\tan 30^\circ)}{(-\sin 45^\circ)\cos 45^\circ} \\ &= \frac{\frac{1}{2}(\frac{1}{\sqrt{3}})}{\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)} \\ &= \frac{\frac{1}{2}(\frac{1}{\sqrt{3}})}{\frac{1}{2}} \\ &= \frac{1}{\sqrt{3}}\end{aligned}$$

b)

$$\begin{aligned}&3\tan 405^\circ + 2\tan 330^\circ \cos 750^\circ \\ &= 3\tan(360^\circ + 45^\circ) + 2\tan(360^\circ - 30^\circ)\cos\left(2(360^\circ) + 30^\circ\right) \\ &= 3\tan 45^\circ + 2(-\tan 30^\circ)\cos 30^\circ \\ &= 3\tan 45^\circ - 2(\frac{\sin 30^\circ}{\cos 30^\circ})\cos 30^\circ \\ &= 3\tan 45^\circ - 2\sin 30^\circ \\ &= 3(1) - 2\left(\frac{1}{2}\right) \\ &= 2\end{aligned}$$

c)

$$\begin{aligned}&\frac{\cos 315^\circ \cos 405^\circ + \sin 45^\circ \sin 135^\circ}{\sin 750^\circ} \\ &= \frac{\cos(360^\circ - 45^\circ)\cos(360^\circ + 45^\circ) + \sin 45^\circ \sin(180^\circ - 45^\circ)}{\sin\left(2(360^\circ) + 30^\circ\right)} \\ &= \frac{\cos 45^\circ \cos 45^\circ + \sin 45^\circ \sin -45^\circ}{\sin 30^\circ} \\ &= \frac{\cos^2 45^\circ + \sin^2 45^\circ}{\sin 30^\circ} \\ &= \frac{1}{\frac{1}{2}} \\ &= 2\end{aligned}$$

d)

$$\begin{aligned}
&\tan 150^\circ \cos 390^\circ - 2\sin 510^\circ \\
&= \tan(180^\circ - 30^\circ)\cos(360^\circ + 30^\circ) - 2\sin\left(2(360^\circ) + 150^\circ\right) \\
&= -\tan 30^\circ \cos 30^\circ - 2\sin 150^\circ \\
&= -\frac{\sin 30^\circ}{\cos 30^\circ}\cos 30^\circ - 2\sin(180^\circ - 30^\circ) \\
&= -\sin 30^\circ - 2\sin 30^\circ \\
&= -3\left(\frac{1}{2}\right) \\
&= -\frac{3}{2}
\end{aligned}$$

e)

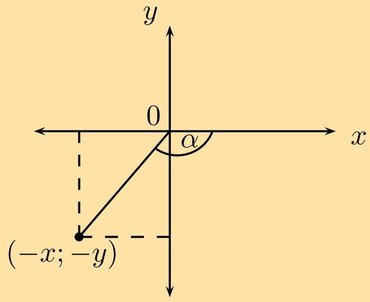
$$\begin{aligned}
&\frac{2\sin 120^\circ + 3\cos 765^\circ - 2\sin 240^\circ - 3\cos 45^\circ}{5\sin 330^\circ + 3\tan 225^\circ - 6\cos 60^\circ} \\
&= \frac{2\sin(180^\circ - 60^\circ) + 3\cos(2(360^\circ) + 45^\circ) - 2\sin 60^\circ - 3\cos 45^\circ}{5\sin(360^\circ - 30^\circ) + 3\tan(180^\circ + 45^\circ) - 6\cos 60^\circ} \\
&= \frac{2\sin 60^\circ + 3\cos 45^\circ + 2\sin 60^\circ - 3\cos 45^\circ}{-5\sin 30^\circ + 3\tan 45^\circ - 6\cos 60^\circ} \\
&= \frac{4\sin 60^\circ}{-5\sin 30^\circ + 3\tan 45^\circ - 6\cos 60^\circ} \\
&= \frac{4\left(\frac{\sqrt{3}}{2}\right)}{-5\left(\frac{1}{2}\right) + 3(1) - 6\left(\frac{1}{2}\right)} \\
&= \frac{2\sqrt{3}}{-\frac{5}{2}} \\
&= -2\sqrt{3} \times \frac{2}{5} \\
&= -\frac{4\sqrt{3}}{5}
\end{aligned}$$

4. Given $90^\circ < \alpha < 180^\circ$, use a sketch to help explain why:

   a) $\sin(-\alpha) = -\sin\alpha$
   b) $\cos(-\alpha) = -\cos\alpha$

Solution:

a)



$$\begin{aligned}
\sin(-\alpha) &= \frac{-y}{r} \qquad \text{(lies in III quad)} \\
&= -\sin\alpha
\end{aligned}$$

b)

$$\begin{aligned}\cos(-\alpha) &= \frac{-x}{r} \qquad \text{(lies in III quad)}\\ &= -\cos\alpha\end{aligned}$$

5. If $t = \sin 43°$, express the following in terms of t:

   a) $\sin 317°$
   b) $\cos^2 403°$
   c) $\tan(-43°)$

   Solution:

   a)

$$\begin{aligned}\sin 317° &= \sin(360° - 43°)\\ &= -\sin 43°\\ &= -t\end{aligned}$$

   b)

$$\begin{aligned}\cos^2 403° &= \cos^2(360° + 43°)\\ &= \cos^2 43°\\ &= 1 - \sin^2 43°\\ &= 1 - t^2\end{aligned}$$

   c)

$$\begin{aligned}\tan(-43°) &= -\tan 43°\\ &= -\frac{\sin 43°}{\cos 43°}\\ &= \pm\frac{t}{\sqrt{1 - t^2}}\end{aligned}$$

Exercise 6 – 5: Co-functions

1. Simplify the following:

   a) $\dfrac{\cos(90° + \theta)\sin(\theta + 90°)}{\sin(-\theta)}$
   b) $\dfrac{2\sin(90° - x) + \sin(90° + x)}{\sin(90° - x) + \cos(180° + x)}$

   Solution:

a)

$$\frac{\cos(90° + \theta)\sin(\theta + 90°)}{\sin(-\theta)} = \frac{-\sin\theta\cos\theta}{-\sin\theta}$$
$$= \cos\theta$$

b)

$$\frac{2\sin(90° - x) + \sin(90° + x)}{\sin(90° - x) + \cos(180° + x)} = \frac{2\cos x + \cos x}{\cos x + \cos x}$$
$$= \frac{3\cos x}{2\cos x}$$
$$= \frac{3}{2}$$

2. Given $\cos 36° = p$, express the following in terms on p:

a) $\sin 54°$

b) $\sin 36°$

c) $\tan 126°$

d) $\cos 324°$

Solution:

a)

$$\sin 54° = \sin(90° - 36°)$$
$$= \cos 36°$$
$$= p$$

b)

$$\sin 36° = \sqrt{1 - \cos^2 36°}$$
$$= \sqrt{1 - p^2}$$

c)

$$\tan 126° = \frac{\sin(90° + 36°)}{\cos(90° + 36°)}$$
$$= \frac{\cos 36°}{-\sin 36°}$$
$$= -\frac{p}{\sqrt{1 - p^2}}$$

d)

$$\cos 324° = \cos(360° - 36°)$$
$$= \cos 36°$$
$$= p$$

Exercise 6 – 6: Reduction formulae

1. Write A and B as a single trigonometric ratio:

   a) $A = \sin(360^\circ - \theta)\cos(180^\circ - \theta)\tan(360^\circ + \theta)$

   b) $B = \dfrac{\cos(360^\circ + \theta)\cos(-\theta)\sin(-\theta)}{\cos(90^\circ + \theta)}$

   c) Hence, determine:

      i. $A + B = \ldots$

      ii. $\frac{A}{B} = \ldots$

Solution:

a)

$$\begin{aligned} A &= \sin(360^\circ - \theta)\cos(180^\circ - \theta)\tan(360^\circ + \theta) \\ &= (-\sin\theta)(-\cos\theta)(\tan\theta) \\ &= \sin\theta\cos\theta\frac{\sin\theta}{\cos\theta} \\ &= \sin^2\theta \end{aligned}$$

b)

$$\begin{aligned} B &= \frac{\cos(360^\circ + \theta)\cos(-\theta)\sin(-\theta)}{\cos(90^\circ + \theta)} \\ &= \frac{\cos\theta\cos\theta(-\sin\theta)}{-\sin\theta} \\ &= \frac{\cos^2\theta\sin\theta}{\sin\theta} \\ &= \cos^2\theta \\ &= \cos^2\theta \end{aligned}$$

c) i.

$$\begin{aligned} A + B &= \cos^2\theta + \sin^2\theta \\ &= 1 \end{aligned}$$

ii.

$$\begin{aligned} \frac{A}{B} &= \frac{\sin^2\theta}{\cos^2\theta} \\ &= \tan^2\theta \end{aligned}$$

2. Write the following as a function of an acute angle:

   a) $\sin 163^\circ$

   b) $\cos 327^\circ$

   c) $\tan 248^\circ$

   d) $\cos(-213^\circ)$

Solution:

a)

$$\begin{aligned} \sin 163^\circ &= \sin(180^\circ - 17^\circ) \\ &= \sin 17^\circ \end{aligned}$$

b)

$$\begin{aligned}\cos 327^\circ &= \cos(360^\circ - 33^\circ)\\ &= \cos 33^\circ\end{aligned}$$

c)

$$\begin{aligned}\tan 248^\circ &= \tan(180^\circ + 68^\circ)\\ &= \tan 68^\circ\end{aligned}$$

d)

$$\begin{aligned}\cos(-213^\circ) &= \cos 213^\circ\\ &= \cos(180^\circ + 33^\circ)\\ &= -\cos 33^\circ\end{aligned}$$

3. Determine the value of the following, without using a calculator:

a) $\dfrac{\sin(-30^\circ)}{\tan(150^\circ)} + \cos 330^\circ$

b) $\tan 300^\circ \cos 120^\circ$

c) $(1 - \cos 30^\circ)(1 - \cos 210^\circ)$

d) $\cos 780^\circ - (\sin 315^\circ)(\cos 405^\circ)$

Solution:

a)

$$\begin{aligned}&\frac{\sin(-30^\circ)}{\tan(150^\circ)} + \cos 330^\circ\\ &= \frac{-\sin 30^\circ}{\tan(180^\circ - 30^\circ)} + \cos(360^\circ - 30^\circ)\\ &= \frac{-\sin 30^\circ}{-\tan 30^\circ} + \cos 30^\circ\\ &= \frac{\sin 30^\circ}{\frac{\sin 30^\circ}{\cos 30^\circ}} + \cos 30^\circ\\ &= \cos 30^\circ + \cos 30^\circ\\ &= 2\cos 30^\circ\\ &= 2\left(\frac{\sqrt{3}}{2}\right)\\ &= \sqrt{3}\end{aligned}$$

b)

$$\begin{aligned}&\tan 300^\circ \cos 120^\circ\\ &= \tan(360^\circ - 60^\circ)\cos(180^\circ - 60^\circ)\\ &= (-\tan 60^\circ)(-\cos 60^\circ)\\ &= \left(\sqrt{3}\right)\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

c)

$$
\begin{aligned}
&(1-\cos 30^\circ)(1-\cos 210^\circ)\\
&=(1-\cos 30^\circ)(1-\cos(180^\circ+30^\circ))\\
&=(1-\cos 30^\circ)(1+\cos 30^\circ)\\
&=1-\cos^2 30^\circ\\
&=\sin^2 30^\circ\\
&=\sin^2 30^\circ\\
&=\left(\frac{1}{2}\right)^2\\
&=\frac{1}{4}
\end{aligned}
$$

d)

$$
\begin{aligned}
&\cos 780^\circ-(\sin 315^\circ)(\cos 405^\circ)\\
&=\cos(2(360^\circ)+60^\circ)-(\sin(360^\circ-45^\circ))(\cos(360^\circ+45^\circ))\\
&=\cos 60^\circ+\sin 45^\circ\cos 45^\circ\\
&=\left(\frac{1}{2}\right)+\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\\
&=\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)\\
&=1
\end{aligned}
$$

4. Prove that the following identity is true and state any restrictions: $\dfrac{\sin(180^\circ+\alpha)\tan(360^\circ+\alpha)\cos\alpha}{\cos(90^\circ-\alpha)}=\sin\alpha$

Solution:

$$
\begin{aligned}
\text{LHS}&=\frac{\sin(180^\circ+\alpha)\tan(360^\circ+\alpha)\cos\alpha}{\cos(90^\circ-\alpha)}\\
&=\frac{(-\sin\alpha)\tan\alpha\cos\alpha}{\sin\alpha}\\
&=-\frac{\sin\alpha}{\cos\alpha}\cos\alpha\\
&=\sin\alpha\\
&=\text{RHS}
\end{aligned}
$$

6.4 Trigonometric equations

Exercise 6 – 7: Solving trigonometric equations

1. Determine the values of α for $\alpha \in [0^\circ; 360^\circ]$ if:

a) $4\cos\alpha = 2$

b) $\sin\alpha + 3{,}65 = 3$

c) $\tan\alpha = 5\frac{1}{4}$

d) $\cos\alpha + 0{,}939 = 0$

e) $5\sin\alpha = 3$

f) $\frac{1}{2}\tan\alpha = -1{,}4$

Solution:

a)

$$\begin{aligned} 4\cos\alpha &= 2 \\ \cos\alpha &= \frac{1}{2} \\ \text{ref}\angle &= \cos^{-1} 0{,}5 \\ &= 60° \\ \therefore \alpha &= 60° \end{aligned}$$

In the fourth quadrant:

$$\begin{aligned} \alpha &= 360° - 60° \\ &= 300° \end{aligned}$$

$\alpha = 60°; 300°$

b)

$$\begin{aligned} \sin\alpha + 3{,}65 &= 3 \\ \sin\alpha &= -0{,}65 \\ \text{ref}\angle &= \sin^{-1} 0{,}65 \\ &= 40{,}5° \end{aligned}$$

In the third quadrant:

$$\begin{aligned} \alpha &= 180° + 40{,}5° \\ &= 220{,}5° \end{aligned}$$

In the fourth quadrant:

$$\begin{aligned} \alpha &= 360° - 40{,}5° \\ &= 319{,}5° \end{aligned}$$

$\alpha = 220{,}5°; 319{,}5°$

c)

$$\begin{aligned} \tan\alpha &= 5\frac{1}{4} \\ \tan\alpha &= -5{,}25 \\ \text{ref}\angle &= \tan^{-1} 5{,}25 \\ &= 79{,}2° \\ \therefore \alpha &= 79{,}2° \end{aligned}$$

In the third quadrant:

$$
\begin{aligned}
\alpha &= 180° + 79{,}2° \\
&= 259{,}2°
\end{aligned}
$$

$\alpha = 79{,}2°; 259{,}2°$

d)

$$
\begin{aligned}
\cos\alpha + 0{,}939 &= 0 \\
\cos\alpha &= -0{,}939 \\
\text{ref } \angle &= \cos^{-1} 0{,}939 \\
&= 20{,}1°
\end{aligned}
$$

In the third quadrant:

$$
\begin{aligned}
\alpha &= 180° + 20{,}1° \\
&= 200{,}1°
\end{aligned}
$$

In the fourth quadrant:

$$
\begin{aligned}
\alpha &= 360° - 20{,}1° \\
&= 339{,}9°
\end{aligned}
$$

$\alpha = 200{,}1°; 339{,}9°$

e)

$$
\begin{aligned}
5\sin\alpha &= 3 \\
\sin\alpha &= \frac{3}{5} \\
\text{ref } \angle &= \sin^{-1}\left(\frac{3}{5}\right) \\
&= 36{,}9° \\
\therefore \alpha &= 36{,}9°
\end{aligned}
$$

In the second quadrant:

$$
\begin{aligned}
\alpha &= 180° - 20{,}1° \\
&= 143{,}1°
\end{aligned}
$$

$\alpha = 36{,}9°; 143{,}1°$

f)

$$
\begin{aligned}
\frac{1}{2}\tan\alpha &= -1{,}4 \\
\tan\alpha &= -2{,}8 \\
\text{ref } \angle &= \tan^{-1} 2{,}8 \\
&= 70{,}3°
\end{aligned}
$$

In the second quadrant:

$$\begin{aligned} \alpha &= 180° - 70{,}3° \\ &= 109{,}7° \end{aligned}$$

In the fourth quadrant:

$$\begin{aligned} \alpha &= 360° - 70{,}3° \\ &= 289{,}7° \end{aligned}$$

$\alpha = 109{,}7°; 289{,}7°$

2. Determine the values of θ for $\theta \in [-360°; 360°]$ if:

   a) $\sin\theta = 0{,}6$
   b) $\cos\theta + \frac{3}{4} = 0$
   c) $3\tan\theta = 20$
   d) $\sin\theta = \cos 180°$
   e) $2\cos\theta = \frac{4}{5}$

Solution:

a)

$$\begin{aligned} \sin\theta &= 0{,}6 \\ \text{ref}\,\angle &= \sin^{-1} 0{,}6 \\ &= 36{,}9° \\ \therefore \theta &= 36{,}9° \end{aligned}$$

In the second quadrant:

$$\begin{aligned} \theta &= 180° - 36{,}9° \\ &= 143{,}1° \end{aligned}$$

Negative angles:

$$\begin{aligned} \theta &= 36{,}9° - 360° \\ &= -323{,}1° \\ \text{And } \theta &= 143{,}1° - 360° \\ &= -216{,}9° \end{aligned}$$

$\theta = -323{,}1°; -216{,}9°; 36{,}9°; 143{,}1°$

b)

$$\begin{aligned} \cos\theta + \frac{3}{4} &= 0 \\ \cos\theta &= -\frac{3}{4} \\ \text{ref}\,\angle &= \cos^{-1}\left(\frac{3}{4}\right) \\ &= 41{,}4° \end{aligned}$$

In the second quadrant:

$$\begin{aligned}\theta &= 180° - 41{,}4° \\ &= 138{,}6°\end{aligned}$$

In the third quadrant:

$$\begin{aligned}\theta &= 180° + 41{,}4° \\ &= 221{,}4°\end{aligned}$$

Negative angles:

$$\begin{aligned}\theta &= 138{,}6° - 360° \\ &= -221{,}4° \\ \text{And } \theta &= 221{,}4° - 360° \\ &= -138{,}6°\end{aligned}$$

$\theta = -221{,}4°; -138{,}6°; 138{,}6°; 221{,}4°$

c)

$$\begin{aligned}3\tan\theta &= 20 \\ \tan\theta &= \frac{20}{3} \\ \text{ref}\,\angle &= \tan^{-1}\left(\frac{20}{3}\right) \\ &= 81{,}5° \\ \theta &= 81{,}5°\end{aligned}$$

In the third quadrant:

$$\begin{aligned}\theta &= 180° + 81{,}5° \\ &= 261{,}5°\end{aligned}$$

Negative angles:

$$\begin{aligned}\theta &= 81{,}5° - 360° \\ &= -278{,}5° \\ \text{And } \theta &= 261{,}5° - 360° \\ &= -98{,}5°\end{aligned}$$

$\theta = -278{,}5°; -98{,}5°; 81{,}5°; 261{,}5°$

d)

$$\begin{aligned}\sin\theta &= \cos 180° \\ \sin\theta &= -1 \\ \text{ref}\,\angle &= \sin^{-1}(1) \\ &= 90°\end{aligned}$$

In the third quadrant:

$$\begin{aligned} \theta &= 180° + 90° \\ &= 270° \end{aligned}$$

Negative angles:

$$\begin{aligned} \theta &= 270° - 360° \\ &= -90° \end{aligned}$$

$\theta = -90°; 270°$

e)

$$\begin{aligned} 2\cos\theta &= \frac{4}{5} \\ \cos\theta &= \frac{2}{5} \\ \text{ref } \angle &= \cos^{-1}\left(\frac{2}{5}\right) \\ &= 66{,}4° \\ \theta &= 66{,}4° \end{aligned}$$

In the fourth quadrant:

$$\begin{aligned} \theta &= 360° - 66{,}4° \\ &= 293{,}6° \end{aligned}$$

Negative angles:

$$\begin{aligned} \theta &= 66{,}4° - 360° \\ &= -293{,}6° \\ \text{And } \theta &= 293{,}6° - 360° \\ &= -66{,}4° \end{aligned}$$

$\theta = -293{,}6°; -66{,}4°; 66{,}4°; 293{,}6°$

The general solution

Exercise 6 – 8: General solution

1.
   - Find the general solution for each equation.
   - Hence, find all the solutions in the interval $[-180°; 180°]$.

a) $\cos(\theta + 25°) = 0{,}231$

b) $\sin 2\alpha = -0{,}327$

c) $2\tan\beta = -2{,}68$

d) $\cos\alpha = 1$

e) $4\sin\theta = 0$

f) $\cos\theta = -1$

g) $\tan\frac{\theta}{2} = 0{,}9$

h) $4\cos\theta + 3 = 1$

i) $\sin 2\theta = -\frac{\sqrt{3}}{2}$

Solution:

a)

$$\begin{aligned} \cos(\theta + 25°) &= 0{,}231 \\ \text{ref } \angle &= 76{,}64° \\ \text{I quad:} \quad \theta + 25° &= 76{,}64° + n\,.\,360°, n \in \mathbb{Z} \\ \therefore \theta &= 51{,}64° + n\,.\,360° \\ \therefore \theta &= -128{,}36°; 51{,}64° \\ \text{IV quad:} \quad \theta + 25° &= 360° - 76{,}64° + n\,.\,360°, n \in \mathbb{Z} \\ \therefore \theta &= 258{,}36° + n\,.\,360° \\ \therefore \theta &= -101{,}64° \end{aligned}$$

$\theta = -128{,}36°; -101{,}64°; 51{,}64°$

b)

$$\begin{aligned} \sin 2\alpha &= -0{,}327 \\ \text{ref } \angle &= 19{,}09° \\ \text{III quad:} \quad 2\alpha &= 180° + 19{,}09° + n\,.\,360°, n \in \mathbb{Z} \\ 2\alpha &= 199{,}09° + n\,.\,360° \\ \therefore \alpha &= 99{,}55° + n\,.\,180° \\ \therefore \alpha &= -80{,}45°; 99{,}55° \\ \text{IV quad:} \quad 2\alpha &= 360° - 19{,}09° + n\,.\,360°, n \in \mathbb{Z} \\ 2\alpha &= 340{,}91° + n\,.\,360° \\ \therefore \alpha &= 170{,}46° + n\,.\,180° \\ \therefore \alpha &= -9{,}54°; 170{,}46° \end{aligned}$$

$\theta = -80{,}45°; -9{,}54°; 99{,}55°; 170{,}46°$

c)

$$\begin{aligned} 2\tan\theta &= -2{,}68 \\ \tan\theta &= -1{,}34 \\ \text{ref } \angle &= 53{,}27° \\ \text{II quad:} \quad \theta &= 180° - 53{,}27° + n\,.\,180°, n \in \mathbb{Z} \\ \therefore \theta &= 126{,}73° + n\,.\,180° \\ \therefore \theta &= -53{,}27°; 126{,}73° \end{aligned}$$

$\theta = -53{,}27°; 126{,}73°$

d)

$$\begin{aligned} \cos\alpha &= 1 \\ \text{ref } \angle &= 0° \\ \text{I/IV quad:} \quad \alpha &= 0° + n\,.\,360°, n \in \mathbb{Z} \\ \therefore \alpha &= 0° \end{aligned}$$

$\alpha = 0°$

e)

$$\begin{aligned} 4\sin\theta &= 0 \\ \text{ref } \angle &= 0^\circ \\ \text{I quad:} \quad \theta &= 0^\circ + n \ . \ 360^\circ, n \in \mathbb{Z} \\ \text{II quad:} \quad \theta &= 180^\circ + n \ . \ 360^\circ, n \in \mathbb{Z} \end{aligned}$$

$\theta = -180^\circ; 0^\circ; 180^\circ$

f)

$$\begin{aligned} \cos\theta &= -1 \\ \text{ref } \angle &= 0^\circ \\ \text{I/III quad:} \quad \theta &= 180^\circ + n \ . \ 360^\circ, n \in \mathbb{Z} \end{aligned}$$

$\theta = -180^\circ; 180^\circ$

g)

$$\begin{aligned} \tan\frac{\theta}{2} &= 0{,}9 \\ \text{ref } \angle &= 42^\circ \\ \text{I quad:} \quad \frac{\theta}{2} &= 42^\circ + n \ . \ 180^\circ, n \in \mathbb{Z} \\ \theta &= 84^\circ + n \ . \ 360^\circ \\ \text{III quad:} \quad \frac{\theta}{2} &= 180^\circ + 42^\circ + n \ . \ 180^\circ, n \in \mathbb{Z} \\ \frac{\theta}{2} &= 222^\circ + n \ . \ 180^\circ \\ \therefore \theta &= 444^\circ + n \ . \ 360^\circ \end{aligned}$$

$\theta = 84^\circ$

h)

$$\begin{aligned} 4\cos\theta + 3 &= 1 \\ 4\cos\theta &= -2 \\ \cos\theta &= -\frac{1}{2} \\ \text{ref } \angle &= 60^\circ \\ \text{II quad:} \quad \theta &= 180^\circ - 60^\circ + n \ . \ 360^\circ, n \in \mathbb{Z} \\ \therefore \theta &= 120^\circ + n \ . \ 360^\circ \\ \text{III quad:} \quad \theta &= 180^\circ + 60^\circ + n \ . \ 360^\circ, n \in \mathbb{Z} \\ \therefore \theta &= 240^\circ + n \ . \ 360^\circ \end{aligned}$$

$\theta = -120^\circ; 120^\circ$

i)

$$\begin{aligned} \sin 2\theta &= -\frac{\sqrt{3}}{2} \\ \text{ref } \angle &= 60^\circ \\ \text{III quad:} \quad 2\theta &= 180^\circ + 60^\circ + n \ . \ 360^\circ, n \in \mathbb{Z} \\ 2\theta &= 240^\circ + n \ . \ 360^\circ \\ \therefore \theta &= 120^\circ + n \ . \ 180^\circ \\ \text{IV quad:} \quad \theta &= 360^\circ - 60^\circ + n \ . \ 360^\circ, n \in \mathbb{Z} \\ 2\theta &= 300^\circ + n \ . \ 360^\circ \\ \therefore \theta &= 150^\circ + n \ . \ 180^\circ \end{aligned}$$

$\theta = -60°; -30°; 120°; 150°$

2. Find the general solution for each equation.

a) $\cos(\theta + 20°) = 0$

b) $\sin 3\alpha = -1$

c) $\tan 4\beta = 0{,}866$

d) $\cos(\alpha - 25°) = 0{,}707$

e) $2\sin\frac{3\theta}{2} = -1$

f) $5\tan(\beta + 15°) = \frac{5}{\sqrt{3}}$

Solution:

a)

$$\begin{aligned}\cos(\theta + 20°) &= 0\\ \text{ref } \angle &= 0°\\ \therefore \theta + 20° &= 0° + n\,.\,360°, n \in \mathbb{Z}\\ \therefore \theta &= -20° + n\,.\,360°\end{aligned}$$

$\theta = -20° + n\,.\,360°$

b)

$$\begin{aligned}\sin 3\alpha &= -1\\ \text{ref } \angle &= 90°\\ \therefore 3\alpha &= 90° + n\,.\,360°, n \in \mathbb{Z}\\ \therefore \alpha &= 30° + n\,.\,120°\end{aligned}$$

$\alpha = 30° + n\,.\,120°$

c)

$$\begin{aligned}\tan 4\beta = 0{,}866\\ \text{ref } \angle &= 41°\\ \therefore 4\beta &= 41° + n\,.\,180°, n \in \mathbb{Z}\\ \therefore \beta &= 10{,}25° + n\,.\,45°\\ \text{III quad:} \quad 4\beta &= 180° + 41° + n\,.\,180°, n \in \mathbb{Z}\\ &= 221° + n\,.\,180°\\ \therefore \beta &= 55{,}25° + n\,.\,45°\end{aligned}$$

$\beta = 10{,}25° + n\,.\,45°$ or $\beta = 55{,}25° + n\,.\,45°$

d)

$$\begin{aligned}\cos(\alpha - 25°) &= 0{,}707\\ \text{ref } \angle &= 45°\\ \therefore \alpha - 25° &= 45° + n\,.\,360°, n \in \mathbb{Z}\\ \therefore \alpha &= 70° + n\,.\,360°\\ \text{IV quad: } \alpha - 25° &= 360° - 45° + n\,.\,360°, n \in \mathbb{Z}\\ &= 340° + n\,.\,360°\end{aligned}$$

$\alpha = 70° + n\,.\,360°$ or $\alpha = 340° + n\,.\,360°$

e)

$$2\sin\frac{3\theta}{2} = -1$$
$$\text{ref}\,\angle = 30^\circ$$
$$\text{III quad:}\quad \frac{3\theta}{2} = 180^\circ + 30^\circ + n\,.\,360^\circ, n \in \mathbb{Z}$$
$$\therefore \theta = \frac{2}{3}(210^\circ + \times n\,.\,360^\circ)$$
$$= 140^\circ + n\,.\,240^\circ$$
$$\text{IV quad:}\quad \frac{3\theta}{2} = 360^\circ - 30^\circ + n\,.\,360^\circ, n \in \mathbb{Z}$$
$$\therefore \theta = \frac{2}{3}(330^\circ + n\,.\,360^\circ)$$
$$= 220^\circ + n\,.\,240^\circ$$

$\theta = 140^\circ + n\,.\,240^\circ$ or $\theta = 220^\circ + n\,.\,240^\circ$

f)

$$5\tan(\beta + 15^\circ) = \frac{5}{\sqrt{3}}$$
$$\tan(\beta + 15^\circ) = \frac{1}{\sqrt{3}}$$
$$\text{ref}\,\angle = 30^\circ$$
$$\therefore \beta + 15^\circ = 30^\circ$$
$$\therefore \beta = 15^\circ + n\,.\,180^\circ$$

Exercise 6 – 9: Solving trigonometric equations

1. Find the general solution for each of the following equations:

a) $\cos 2\theta = 0$

b) $\sin(\alpha + 10^\circ) = \frac{\sqrt{3}}{2}$

c) $2\cos\frac{\theta}{2} - \sqrt{3} = 0$

d) $\frac{1}{2}\tan(\beta - 30^\circ) = -1$

e) $5\cos\theta = \tan 300^\circ$

f) $3\sin\alpha = -1{,}5$

g) $\sin 2\beta = \cos(\beta + 20^\circ)$

h) $0{,}5\tan\theta + 2{,}5 = 1{,}7$

i) $\sin(3\alpha - 10^\circ) = \sin(\alpha + 32^\circ)$

j) $\sin 2\beta = \cos 2\beta$

Solution:

a)

$$\begin{aligned} \cos 2\theta &= 0 \\ \text{ref } \angle &= 90^\circ \\ \text{I/II quad:} \quad 2\theta &= 90^\circ + k\ .\ 360^\circ, k \in \mathbb{Z} \\ \therefore \theta &= 45^\circ + \times k\ .\ 180^\circ \\ &= 140^\circ + k\ .\ 240^\circ \\ \text{III/IV quad:} \quad 2\theta &= 270^\circ + k\ .\ 360^\circ, k \in \mathbb{Z} \\ \therefore \theta &= 135^\circ + k\ .\ 180^\circ \end{aligned}$$

$\theta = 45^\circ + k\ .\ 180^\circ$ or $\theta = 135^\circ + k\ .\ 180^\circ$

b)

$$\begin{aligned} \sin(\alpha + 10^\circ) &= \frac{\sqrt{3}}{2} \\ \text{ref } \angle &= 60^\circ \\ \text{I quad:} \quad \alpha + 10^\circ &= 60^\circ + k\ .\ 360^\circ, k \in \mathbb{Z} \\ \therefore \alpha &= 50^\circ + \times k\ .\ 360^\circ \\ \text{III quad:} \ \alpha + 10^\circ &= 120^\circ + k\ .\ 360^\circ, k \in \mathbb{Z} \\ \therefore \alpha &= 110^\circ + k\ .\ 360^\circ \end{aligned}$$

$\alpha = 50^\circ + k\ .\ 360^\circ$ or $\alpha = 110^\circ + k\ .\ 360^\circ$

c)

$$\begin{aligned} 2\cos\frac{\theta}{2} - \sqrt{3} &= 0 \\ \cos\frac{\theta}{2} &= \frac{\sqrt{3}}{2} \\ \text{ref } \angle &= 30^\circ \\ \text{I quad:} \quad \frac{\theta}{2} &= 30^\circ + k\ .\ 360^\circ, k \in \mathbb{Z} \\ \therefore \theta &= 60^\circ + \times k\ .\ 720^\circ \\ \text{IV quad:} \ \frac{\theta}{2} &= 330^\circ + k\ .\ 360^\circ, k \in \mathbb{Z} \\ \therefore \theta &= 660^\circ + k\ .\ 720^\circ \end{aligned}$$

$\theta = 60^\circ + k\ .\ 720^\circ$ or $\theta = 660^\circ + k\ .\ 720^\circ$

d)

$$\begin{aligned} \frac{1}{2}\tan(\beta - 30^\circ) &= -1 \\ \tan(\beta - 30^\circ) &= -2 \\ \text{ref } \angle &= 63{,}4^\circ \\ \text{II quad:} \quad \beta - 30^\circ &= 116{,}6^\circ + k\ .\ 180^\circ, k \in \mathbb{Z} \\ \therefore \beta &= 146{,}6^\circ + k\ .\ 180^\circ \end{aligned}$$

$\beta = 146{,}6^\circ + k\ .\ 180^\circ$

e)

$$\begin{aligned} 5\cos\theta &= \tan 300^\circ \\ \cos\theta &= -0{,}3464 \\ \text{ref } \angle &= 69{,}73^\circ \\ \text{II quad:} \quad \theta &= 110{,}27^\circ + k\ .\ 360^\circ, k \in \mathbb{Z} \\ \text{III quad:} \quad \theta &= 249{,}73^\circ + k\ .\ 360^\circ, k \in \mathbb{Z} \end{aligned}$$

$\theta = 110{,}27° + k\ .\ 360°$ or $\theta = 249{,}73° + k\ .\ 360°$

f)

$$\begin{aligned} 3\sin\alpha &= -1{,}5 \\ \sin\alpha &= -0{,}5 \\ \text{ref}\ \angle &= 30° \\ \text{III quad:} \quad \alpha &= 210° + k\ .\ 360°, k \in \mathbb{Z} \\ \text{IV quad:} \quad \alpha &= 330° + k\ .\ 360°, k \in \mathbb{Z} \end{aligned}$$

$\alpha = 210° + k\ .\ 360°$ or $\alpha = 330° + k\ .\ 360°$

g)

$$\begin{aligned} \sin 2\beta &= \cos(\beta + 20°) \\ 2\beta + \beta + 20° &= 90° + k\ .\ 360°, k \in \mathbb{Z} \\ 3\beta &= 70° + k\ .\ 360° \\ \beta &= 23{,}3° + k\ .\ 120° \end{aligned}$$

$\beta = 23{,}3° + k\ .\ 120°$

h)

$$\begin{aligned} 0{,}5\tan\theta + 2{,}5 &= 1{,}7 \\ \tan\theta &= -1{,}6 \\ \text{ref}\ \angle &= 58° \\ \text{III quad:} \quad \theta &= 122° + k\ .\ 180°, k \in \mathbb{Z} \end{aligned}$$

$\theta = 122° + k\ .\ 180°$

i)

$$\begin{aligned} \sin(3\alpha - 10°) &= \sin(\alpha + 32°) \\ 3\alpha - 10° &= \alpha + 32° + k\ .\ 360°, k \in \mathbb{Z} \\ 2\alpha &= 42° + k\ .\ 360° \\ \alpha &= 21° + k\ .\ 180° \\ \text{Or} \quad 3\alpha - 10° &= 180° - (\alpha + 32°) + k\ .\ 360°, k \in \mathbb{Z} \\ 4\alpha &= 158° + k\ .\ 360° \\ \therefore \alpha &= 39{,}5° + k\ .\ 90° \end{aligned}$$

$\alpha = 21° + k\ .\ 180°$ or $\alpha = 39{,}5° + k\ .\ 90°$

j)

$$\begin{aligned} \sin 2\beta &= \cos 2\beta \\ \tan 2\beta &= 1 \\ \text{ref}\ \angle &= 45° \\ 2\beta &= 45° + k\ .\ 180°, k \in \mathbb{Z} \\ \therefore \beta &= 22{,}5° + k\ .\ 90° \end{aligned}$$

$\beta = 22{,}5° + k\ .\ 90°$

2. Find θ if $\sin^2\theta + \frac{1}{2}\sin\theta = 0$ for $\theta \in [0°; 360°]$.

Solution:

$$\sin^2\theta + \frac{1}{2}\sin\theta = 0$$
$$\sin\theta(\sin\theta + \frac{1}{2}) = 0$$
$$\therefore \sin\theta = 0 \text{ or } \sin\theta = -\frac{1}{2}$$
$$\therefore \theta = 0°, 180° \text{ or } 360°$$
$$\text{Or } \sin\theta = -\frac{1}{2}$$
$$\text{ref } \angle = 30°$$
$$\therefore \theta = 210° \text{ or } 330°$$

$\theta = 0°, 180°, 210°, 330°$ or $360°$

3. Determine the general solution for each of the following:

a) $2\cos^2\theta - 3\cos\theta = 2$

b) $3\tan^2\theta + 2\tan\theta = 0$

c) $\cos^2\alpha = 0{,}64$

d) $\sin(4\beta + 35°) = \cos(10° - \beta)$

e) $\sin(\alpha + 15°) = 2\cos(\alpha + 15°)$

f) $\sin^2\theta - 4\cos^2\theta = 0$

g) $\dfrac{\cos(2\theta + 30°)}{2} + 0{,}38 = 0$

Solution:

a)

$$2\cos^2\theta - 3\cos\theta = 2$$
$$(2\cos\theta + 1)(\cos\theta - 2) = 0$$
$$\therefore \cos\theta = -\frac{1}{2} \text{ or } \cos\theta = 2$$
$$\text{For } \cos\theta = 2$$
$$\therefore \text{No solution}$$
$$\text{For } \cos\theta = -\frac{1}{2}$$
$$\text{ref } \angle = 60°$$
$$\text{II quad: } \theta = 180° - 60° + k\ .\ 360°, k \in \mathbb{Z}$$
$$\therefore \theta = 120° + \times k\ .\ 360°$$
$$\text{III quad: } \theta = 180° + 60° + k\ .\ 360°, k \in \mathbb{Z}$$
$$\therefore \theta = 240° + k\ .\ 360°$$

$\theta = 120° + k\ .\ 360°$ or $\theta = 240° + k\ .\ 360°$

b)

$$
\begin{aligned}
3\tan^2\theta + 2\tan\theta &= 0 \\
\tan\theta(3\tan\theta + 2) &= 0 \\
\therefore \tan\theta &= -\frac{2}{3} \text{ or } \tan\theta = 0 \\
\text{For } \tan\theta &= -\frac{1}{2} \\
\theta &= 0° + k \ . \ 180°, k \in \mathbb{Z} \\
\text{For } \tan\theta &= -\frac{2}{3} \\
\text{ref} \angle &= 33{,}7° \\
\theta &= 180° - 33{,}7° + k \ . \ 180°, k \in \mathbb{Z} \\
\therefore \theta &= 146{,}3° + \times k \ . \ 180°
\end{aligned}
$$

$\theta = 0° + k \ . \ 180°$ or $\theta = 146{,}3° + k \ . \ 180°$

c)

$$
\begin{aligned}
\cos^2\alpha &= 0{,}64 \\
\therefore \cos\alpha &= \pm 0{,}8 \\
\text{For } \cos\alpha &= -0{,}8 \\
\text{ref} \angle &= 36{,}9° \\
\text{II quad: } \alpha &= 180° - 36{,}9° + k \ . \ 360°, k \in \mathbb{Z} \\
\therefore \alpha &= 143{,}1° + \times k \ . \ 360° \\
\text{For } \cos\alpha &= 0{,}8 \\
\text{III quad: } \alpha &= 180° + 36{,}9° + k \ . \ 360°, k \in \mathbb{Z} \\
\therefore \alpha &= 216{,}9° + k \ . \ 360° \\
\text{IV quad: } \alpha &= 360° - 36{,}9° + k \ . \ 360°, k \in \mathbb{Z} \\
\therefore \alpha &= 323{,}1° + k \ . \ 360°
\end{aligned}
$$

$\alpha = 36{,}9° + k \ . \ 360°$ or $\alpha = 143{,}1° + k \ . \ 360°$ or $\alpha = 216{,}9° + k \ . \ 360°$ or $\alpha = 323{,}1° + k \ . \ 360°$

d)

$$
\begin{aligned}
\sin(4\beta + 35°) &= \cos(10° - \beta) \\
\sin(4\beta + 35°) &= \sin\left(90°(10° - \beta)\right) \\
\sin(4\beta + 35°) &= \sin\left(80° + \beta\right) \\
4\beta + 35° &= 80° + \beta + k \ . \ 360°, k \in \mathbb{Z} \\
3\beta &= 45° + k \ . \ 360° \\
\beta &= 15° + k \ . \ 120° \\
\text{Or} \quad (4\beta + 35°) + (10° - \beta) &= 270° + k \ . \ 360°, k \in \mathbb{Z} \\
3\beta &= 225° + k \ . \ 360° \\
\beta &= 75° + k \ . \ 120°
\end{aligned}
$$

$\beta = 15° + k \ . \ 120°$ or $\beta = 75° + k \ . \ 120°$

e)

$$\begin{aligned}
\sin(\alpha + 15^\circ) &= 2\cos(\alpha + 15^\circ)\\
\frac{\sin(\alpha + 15^\circ)}{\cos(\alpha + 15^\circ)} &= 2\\
\tan(\alpha + 15^\circ) &= 2\\
\text{ref}\,\angle &= 63{,}4^\circ\\
\alpha + 15^\circ &= 63{,}4^\circ + k\,.\,180^\circ, k \in \mathbb{Z}\\
\alpha &= 48{,}4^\circ + k\,.\,180^\circ
\end{aligned}$$

$\alpha = 48{,}4^\circ + k\,.\,180^\circ$

f)

$$\begin{aligned}
\sin^2\theta - 4\cos^2\theta &= 0\\
(\sin\theta - 2\cos^2\theta)(\sin\theta + 2\cos^2\theta) &= 0\\
\therefore \sin\theta - 2\cos^2\theta &= 0\\
\sin\theta &= 2\cos^2\theta\\
\frac{\sin\theta}{\cos\theta} &= 2\\
\tan\theta &= 2\\
\text{ref}\,\angle &= 63{,}4^\circ\\
\theta &= 63{,}4^\circ + k\,.\,180^\circ, k \in \mathbb{Z}\\
\text{Or } \sin\theta + 2\cos^2\theta &= 0\\
\sin\theta &= -2\cos^2\theta\\
\frac{\sin\theta}{\cos\theta} &= -2\\
\tan\theta &= -2\\
\text{ref}\,\angle &= -63{,}4^\circ + 180^\circ + k\,.\,180^\circ, k \in \mathbb{Z}\\
\theta &= 116{,}6^\circ + k\,.\,180^\circ
\end{aligned}$$

$\theta = 63{,}4^\circ + k\,.\,180^\circ$ or $\theta = 116{,}6^\circ + k\,.\,180^\circ$

g)

$$\begin{aligned}
\frac{\cos(2\theta + 30^\circ)}{2} + 0{,}38 &= 0\\
\frac{\cos(2\theta + 30^\circ)}{2} &= -0{,}38\\
\cos(2\theta + 30^\circ) &= -0{,}76\\
\text{ref}\,\angle &= 40{,}5^\circ\\
2\theta + 30^\circ) &= 180^\circ - 40{,}5^\circ + k\,.\,360^\circ, k \in \mathbb{Z}\\
2\theta &= 109{,}5^\circ + k\,.\,360^\circ\\
\theta &= 54{,}8^\circ + k\,.\,180^\circ\\
\text{Or} \quad 2\theta + 30^\circ) &= 180^\circ + 40{,}5^\circ + k\,.\,360^\circ, k \in \mathbb{Z}\\
2\theta &= 190{,}5^\circ + k\,.\,360^\circ\\
\theta &= 95{,}25^\circ + k\,.\,180^\circ
\end{aligned}$$

$\theta = 54{,}8^\circ + k\,.\,180^\circ$ or $\theta = 95{,}25^\circ + k\,.\,180^\circ$

4. Find β if $\frac{1}{3}\tan\beta = \cos 200^\circ$ for $\beta \in [-180^\circ; 180^\circ]$.

Solution:

$$\begin{aligned} \frac{1}{3}\tan\beta &= \cos 200^\circ \\ \frac{1}{3}\tan\beta &= -0{,}9396 \\ \tan\beta &= -2{,}819 \\ \text{ref } \angle &= 70{,}5^\circ \\ \beta &= 180^\circ - 70{,}47^\circ + k\,.\,180^\circ, k \in \mathbb{Z} \\ \beta &= 109{,}5^\circ + k\,.\,180^\circ \\ \therefore \beta &= -70{,}5^\circ \text{ or } \beta = 109{,}5^\circ \end{aligned}$$

$\beta = -70{,}5^\circ$ or $\beta = 109{,}5^\circ$

6.5 Area, sine, and cosine rules

The area rule

Exercise 6 – 10: The area rule

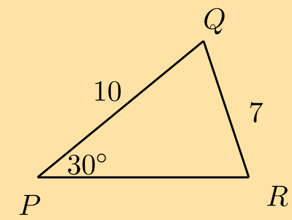
1. Draw a sketch and calculate the area of $\triangle PQR$ given:

   a) $\hat{Q} = 30^\circ$; $r = 10$ and $p = 7$

   b) $\hat{R} = 110^\circ$; $p = 8$ and $q = 9$

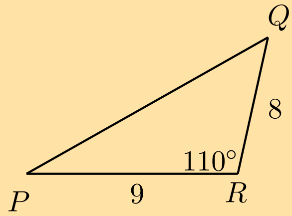
Solution:

a)



$$\begin{aligned} \text{Area } \triangle PQR &= \frac{1}{2}(7)(10)\sin 30^\circ \\ &= 17{,}5 \text{ square units} \end{aligned}$$

b)



$$\begin{aligned}\text{Area } \triangle PQR &= \frac{1}{2}(8)(9)\sin 110^\circ \\ &= 33{,}8 \text{ square units}\end{aligned}$$

2. Find the area of $\triangle XYZ$ given $XZ = 52$ cm, $XY = 29$ cm and $\hat{X} = 58{,}9^\circ$.

   Solution:

$$\begin{aligned}\text{Area } \triangle XYZ &= \frac{1}{2}(52)(29)\sin 58{,}9^\circ \\ &= 645{,}6 \text{ square units}\end{aligned}$$

3. Determine the area of a parallelogram in which two adjacent sides are 10 cm and 13 cm and the angle between them is 55°.

   Solution:

$$\begin{aligned}\text{Area } &= 2\left(\frac{1}{2}(13)(10)\sin 55^\circ\right) \\ &= 106{,}5 \text{ square units}\end{aligned}$$

4. If the area of $\triangle ABC$ is 5000 m^2 with $a = 150$ m and $b = 70$ m, what are the two possible sizes of $\hat{C}$?

   Solution:

$$\begin{aligned}\text{Area } &= \frac{1}{2}(150)(70)\sin \hat{C} \\ \therefore 5000 &= 5250 \sin \hat{C} \\ \therefore \sin \hat{C} &= \frac{5000}{5250} \\ \therefore \hat{C} &= 72{,}2^\circ \\ \text{Or } \hat{C} &= 180^\circ - 72{,}2^\circ \\ &= 107{,}8^\circ\end{aligned}$$

The sine rule

Exercise 6 – 11: Sine rule

1. Find all the unknown sides and angles of the following triangles:

   a) $\triangle PQR$ in which $\hat{Q} = 64^\circ$; $\hat{R} = 24^\circ$ and $r = 3$

   b) $\triangle KLM$ in which $\hat{K} = 43^\circ$; $\hat{M} = 50^\circ$ and $m = 1$

   c) $\triangle ABC$ in which $\hat{A} = 32{,}7^\circ$; $\hat{C} = 70{,}5^\circ$ and $a = 52{,}3$

d) $\triangle XYZ$ in which $\hat{X} = 56°$; $\hat{Z} = 40°$ and $x = 50$

Solution:

a)

$$\begin{aligned}
\frac{q}{\sin \hat{Q}} &= \frac{r}{\sin \hat{R}} \\
\frac{q}{\sin 64°} &= \frac{3}{\sin 24°} \\
q &= \frac{3 \sin 64°}{\sin 24°} \\
&= 6{,}6 \text{ units} \\
\hat{P} &= 180° - 64° - 24° \\
&= 92° \\
\frac{p}{\sin \hat{P}} &= \frac{r}{\sin \hat{R}} \\
\frac{p}{\sin 92°} &= \frac{3}{\sin 24°} \\
p &= \frac{3 \sin 92°}{\sin 24°} \\
&= 7{,}4 \text{ units}
\end{aligned}$$

b)

$$\begin{aligned}
\frac{m}{\sin \hat{M}} &= \frac{l}{\sin \hat{L}} \\
\frac{1}{\sin 50°} &= \frac{l}{\sin 87°} \\
l &= \frac{\sin 87°}{\sin 50°} \\
&= 1{,}3 \text{ units} \\
\hat{L} &= 180° - 50° - 43° \\
&= 87° \\
\frac{k}{\sin \hat{K}} &= \frac{m}{\sin \hat{M}} \\
\frac{k}{\sin 43°} &= \frac{1}{\sin 50°} \\
k &= \frac{\sin 43°}{\sin 50°} \\
&= 0{,}89 \text{ units}
\end{aligned}$$

c)

$$\frac{c}{\sin \hat{C}} = \frac{a}{\sin \hat{A}}$$
$$\frac{c}{\sin 70{,}5^\circ} = \frac{52{,}3}{\sin 32{,}7^\circ}$$
$$c = \frac{52{,}3 \sin 70{,}5^\circ}{\sin 32{,}7^\circ}$$
$$= 91{,}3 \text{ units}$$
$$\hat{B} = 180^\circ - 70{,}5^\circ - 32{,}7^\circ$$
$$= 76{,}8^\circ$$
$$\frac{b}{\sin \hat{B}} = \frac{a}{\sin \hat{A}}$$
$$\frac{b}{\sin 76{,}8^\circ} = \frac{52{,}3}{\sin 32{,}7^\circ}$$
$$b = \frac{52{,}3 \sin 76{,}8^\circ}{\sin 32{,}7^\circ}$$
$$= 94{,}3 \text{ units}$$

d)

$$\frac{x}{\sin \hat{X}} = \frac{z}{\sin \hat{Z}}$$
$$\frac{50}{\sin 56^\circ} = \frac{z}{\sin 40^\circ}$$
$$z = \frac{50 \sin 40^\circ}{\sin 56^\circ}$$
$$= 38{,}8 \text{ units}$$
$$\hat{Y} = 180^\circ - 56^\circ - 40^\circ$$
$$= 84^\circ$$
$$\frac{y}{\sin \hat{Y}} = \frac{x}{\sin \hat{X}}$$
$$\frac{y}{\sin 84^\circ} = \frac{50}{\sin 40^\circ}$$
$$y = \frac{50 \sin 84^\circ}{\sin 56^\circ}$$
$$= 60 \text{ units}$$

2. In $\triangle ABC$, $\hat{A} = 116^\circ$; $\hat{C} = 32^\circ$ and $AC = 23$ m. Find the lengths of the sides AB and BC.

Solution:

$$\hat{B} = 180^\circ - 116^\circ - 32^\circ$$
$$= 32^\circ$$
$$\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}}$$
$$\frac{a}{\sin 116^\circ} = \frac{23}{\sin 32^\circ}$$
$$a = \frac{23 \sin 116^\circ}{\sin 32^\circ}$$
$$= 39{,}01 \text{ units}$$
$$b = 23 \text{ units}$$

3. In $\triangle RST$, $\hat{R} = 19°$; $\hat{S} = 30°$ and $RT = 120$ km. Find the length of the side ST.

Solution:

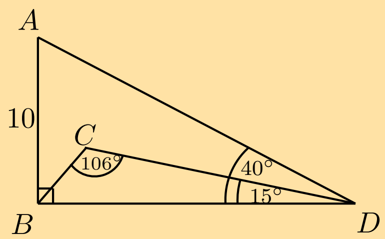
$$\begin{aligned}
\frac{s}{\sin \hat{S}} &= \frac{r}{\sin \hat{R}} \\
\frac{120}{\sin 30°} &= \frac{ST}{\sin 19°} \\
ST &= \frac{120 \sin 19°}{\sin 30°} \\
&= 78{,}1 \text{ km}
\end{aligned}$$

4. In $\triangle KMS$, $\hat{K} = 20°$; $\hat{M} = 100°$ and $s = 23$ cm. Find the length of the side m.

Solution:

$$\begin{aligned}
\hat{S} &= 180° - 100° - 20° \\
&= 60° \\
\frac{m}{\sin \hat{M}} &= \frac{s}{\sin \hat{S}} \\
\frac{m}{\sin 100°} &= \frac{23}{\sin 60°} \\
m &= \frac{23 \sin 100°}{\sin 60°} \\
&= 26{,}2 \text{ units}
\end{aligned}$$

5. In $\triangle ABD$, $\hat{B} = 90°$, $AB = 10$ cm and $A\hat{D}B = 40°$. In $\triangle BCD$, $\hat{C} = 106°$ and $C\hat{D}B = 15°$. Determine BC.



Solution:

$$\begin{aligned}
\frac{BD}{\sin 50°} &= \frac{10}{\sin 40°} \\
BD &= \frac{10 \sin 50°}{\sin 40°} \\
&= 11{,}92 \text{ units} \\
\frac{BC}{\sin 15°} &= \frac{11{,}92}{\sin 106°} \\
BC &= \frac{11{,}92 \sin 15°}{\sin 106°} \\
&= 3{,}2 \text{ units}
\end{aligned}$$

6. In $\triangle ABC$, $\hat{A} = 33°$, $AC = 21$ mm and $AB = 17$ mm. Can you determine BC?

Solution:

No, not enough information is given

Exercise 6 – 12: The cosine rule

1. Solve the following triangles (that is, find all unknown sides and angles):

   a) $\triangle ABC$ in which $\hat{A} = 70°$; $b = 4$ and $c = 9$
   b) $\triangle RST$ in which $RS = 14$; $ST = 26$ and $RT = 16$
   c) $\triangle KLM$ in which $KL = 5$; $LM = 10$ and $KM = 7$
   d) $\triangle JHK$ in which $\hat{H} = 130°$; $JH = 13$ and $HK = 8$
   e) $\triangle DEF$ in which $d = 4$; $e = 5$ and $f = 7$

Solution:

a)

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos \hat{A} \\ a^2 &= 4^2 + 9^2 - 2(9)(4)\cos \hat{70°} \\ &= 72{,}374\ldots \\ \therefore a &= 8{,}5 \end{aligned}$$

$$\begin{aligned} c^2 &= b^2 + a^2 - 2ba \cos \hat{C} \\ 9^2 &= 4^2 + 8{,}5^2 - 2(4)(8{,}5)\cos \hat{C} \\ \therefore \hat{C} &= 83{,}9° \end{aligned}$$

$$\begin{aligned} \hat{B} &= 180° - 70° - 83{,}9° \\ \therefore \hat{B} &= 26{,}1° \end{aligned}$$

b)

$$\begin{aligned} r^2 &= s^2 + t^2 - 2st \cos \hat{R} \\ 26^2 &= 16^2 + 14^2 - 2(16)(14)\cos \hat{R} \\ \therefore \hat{R} &= 120° \end{aligned}$$

$$\begin{aligned} s^2 &= r^2 + t^2 - 2rt \cos \hat{S} \\ 16^2 &= 26^2 + 14^2 - 2(26)(14)\cos \hat{S} \\ \therefore \hat{S} &= 32{,}2° \end{aligned}$$

$$\begin{aligned} \hat{T} &= 180° - 120° - 32{,}2° \\ \therefore \hat{T} &= 27{,}8° \end{aligned}$$

c)

$$\begin{aligned} m^2 &= k^2 + l^2 - 2kl \cos \hat{M} \\ 5^2 &= 10^2 + 7^2 - 2(10)(7)\cos \hat{M} \\ \therefore \hat{M} &= 27{,}7° \end{aligned}$$

$$l^2 = k^2 + m^2 - 2km\cos\hat{L}$$
$$7^2 = 10^2 + 5^2 - 2(10)(5)\cos\hat{L}$$
$$\therefore \hat{L} = 40{,}5°$$

$$\hat{K} = 180° - 27{,}7° - 40{,}5°$$
$$\therefore \hat{K} = 111{,}8°$$

d)

$$h^2 = j^2 + k^2 - 2jk\cos\hat{H}$$
$$h^2 = 13^2 + 8^2 - 2(13)(8)\cos 1\hat{3}0°$$
$$= 366{,}69\ldots$$
$$\therefore h = 19{,}1$$

$$k^2 = j^2 + h^2 - 2jh\cos\hat{K}$$
$$13^2 = 8^2 + 19{,}1^2 - 2(8)(19{,}1)\cos\hat{K}$$
$$\therefore \hat{K} = 31{,}8°$$

$$\hat{J} = 180° - 130° - 31{,}8°$$
$$\therefore \hat{J} = 18{,}2°$$

e)

$$e^2 = d^2 + f^2 - 2df\cos\hat{E}$$
$$5^2 = 4^2 + 7^2 - 2(4)(7)\cos\hat{E}$$
$$\therefore \hat{E} = 44{,}4°$$

$$d^2 = e^2 + f^2 - 2ef\cos\hat{D}$$
$$4^2 = 5^2 + 7^2 - 2(7)(5)\cos\hat{D}$$
$$\therefore \hat{D} = 34{,}0°$$

$$\hat{F} = 180° - 44{,}4° - 34{,}0°$$
$$\therefore \hat{F} = 101{,}6°$$

2. Find the length of the third side of the $\triangle XYZ$ where:

   a) $\hat{X} = 71{,}4°$; $y = 3{,}42$ km and $z = 4{,}03$ km
   b) $x = 103{,}2$ cm; $\hat{Y} = 20{,}8°$ and $z = 44{,}59$ cm

   Solution:

   a)

$$x^2 = y^2 + z^2 - 2yz\cos\hat{X}$$
$$x^2 = 3{,}42^2 + 4{,}03^2 - 2(3{,}42)(4{,}03)\cos 71\hat{,}4°$$
$$= 19{,}14\ldots$$
$$\therefore x = 4{,}4 \text{ km}$$

b)

$$y^2 = x^2 + z^2 - 2xz \cos \hat{Y}$$
$$y^2 = 103{,}2^2 + 44{,}59^2 - 2(103{,}2)(44{,}59) \cos 20\hat{,}8^\circ$$
$$= 4034{,}95\ldots$$
$$\therefore y = 63{,}5 \text{ cm}$$

3. Determine the largest angle in:

a) $\triangle JHK$ in which $JH = 6$; $HK = 4$ and $JK = 3$

b) $\triangle PQR$ where $p = 50$; $q = 70$ and $r = 60$

Solution:

a)

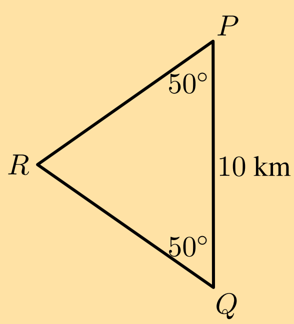
$$k^2 = j^2 + h^2 - 2jh \cos \hat{K}$$
$$6^2 = 4^2 + 3^2 - 2(4)(3) \cos \hat{K}$$
$$\therefore \hat{K} = 117{,}3^\circ$$

b)

$$q^2 = p^2 + r^2 - 2pr \cos \hat{Q}$$
$$70^2 = 60^2 + 50^2 - 2(60)(50) \cos \hat{Q}$$
$$\therefore \hat{Q} = 78{,}5^\circ$$

Exercise 6 – 13: Area, sine and cosine rule

1. Q is a ship at a point 10 km due south of another ship P. R is a lighthouse on the coast such that $\hat{P} = \hat{Q} = 50^\circ$.



Determine:

a) the distance QR

b) the shortest distance from the lighthouse to the line joining the two ships (PQ).

Solution:

a)

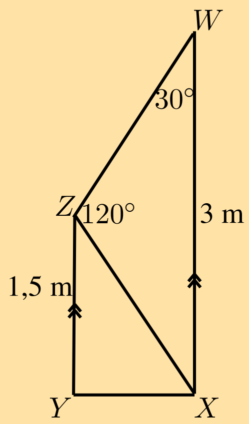
$$\frac{QR}{\sin 50°} = \frac{10}{\sin 80°}$$
$$QR = \frac{10 \sin 50°}{\sin 80°}$$
$$\therefore QR = 7{,}78 \text{ km}$$

b)

$$\frac{h}{5} = \tan 50°$$
$$\therefore h = 6 \text{ km}$$

2. $WXYZ$ is a trapezium, $WX \parallel YZ$ with $WX = 3$ m; $YZ = 1{,}5$ m; $\hat{Z} = 120°$ and $\hat{W} = 30°$.

   Determine the distances XZ and XY.



Solution:

$$\text{In } \triangle XWZ, \quad \frac{ZX}{\sin 30°} = \frac{3}{\sin 120°}$$
$$XZ = \frac{3 \sin 30°}{\sin 120°}$$
$$\therefore XZ = 1{,}73 \text{ km}$$

$$X\hat{Z}Y = 180° - 120° - 30°$$
$$\therefore X\hat{Z}Y = 30°$$

$$\text{In } \triangle XZY, XY^2 = (1{,}73)^2 + (1{,}5)^2 - 2(1{,}73)(1{,}5)\cos 30°$$
$$\therefore XY = 0{,}87 \text{ km}$$

3. On a flight from Johannesburg to Cape Town, the pilot discovers that he has been flying $3°$ off course. At this point the plane is 500 km from Johannesburg. The direct distance between Cape Town and Johannesburg airports is 1552 km. Determine, to the nearest km:

   a) The distance the plane has to travel to get to Cape Town and hence the extra distance that the plane has had to travel due to the pilot's error.

b) The correction, to one hundredth of a degree, to the plane's heading (or direction).

Solution:

a)

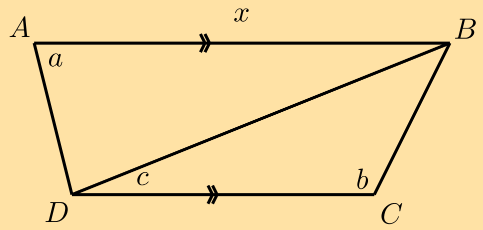
$$x^2 = (500)^2 + (1552)^2 - 2(500)(1552)\cos 3^\circ$$
$$\therefore x = 1053 \text{ km}$$

b)

$$\frac{\sin\theta}{1552} = \frac{\sin 3^\circ}{1053}$$
$$\sin\theta = \frac{1552\sin 3^\circ}{1053}$$
$$\therefore \theta = 4{,}42^\circ$$

4. $ABCD$ is a trapezium (meaning that $AB \parallel CD$). $AB = x$; $B\hat{A}D = a$; $B\hat{C}D = b$ and $B\hat{D}C = c$.

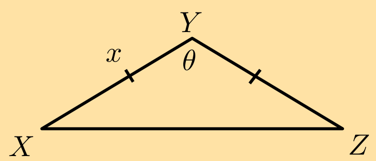
Find an expression for the length of CD in terms of x, a, b and c.



Solution:

$$\frac{DB}{\sin a} = \frac{x}{\sin[180^\circ - (a+c)]}$$
$$DB = \frac{x\sin a}{\sin(a+c)}$$
$$\text{And } \frac{DC}{\sin[180^\circ - (b+c)]} = \frac{DB}{\sin b}$$
$$\therefore \frac{DC}{\sin(b+c)} = \frac{x\sin a}{\sin(a+c)\sin b}$$
$$\text{Therefore } DC = \frac{x\sin a\sin(b+c)}{\sin(a+c)\sin b}$$

5. A surveyor is trying to determine the distance between points X and Z. However the distance cannot be determined directly as a ridge lies between the two points. From a point Y which is equidistant from X and Z, he measures the angle $X\hat{Y}Z$.



a) If $XY = x$ and $X\hat{Y}Z = \theta$, show that $XZ = x\sqrt{2(1-\cos\theta)}$.

b) Calculate XZ (to the nearest kilometre) if $x = 240$ km and $\theta = 132°$.

Solution:

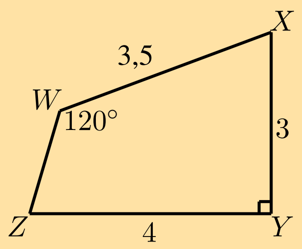
a)

$$XZ^2 = 2x^2 - 2x^2\cos\theta$$
$$= 2x^2(1 - \cos\theta)$$
$$\therefore XZ = \sqrt{2x^2(1-\cos\theta)}$$
$$= x\sqrt{2(1-\cos\theta)}$$

b)

$$XZ^2 = x\sqrt{2(1-\cos\theta)}$$
$$= 240\sqrt{2(1-\cos 132°)}$$
$$= 438{,}5 \text{ km}$$

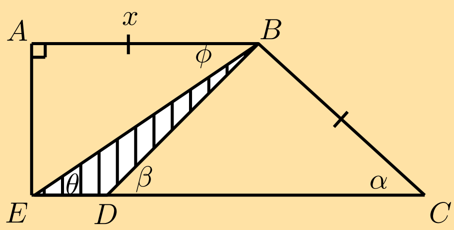
6. Find the area of $WXYZ$ (to two decimal places):



Solution:

$$\frac{\sin W\hat{Z}X}{3{,}5} = \frac{\sin 120°}{5}$$
$$\therefore W\hat{Z}X = 37{,}3°$$
$$\therefore W\hat{X}Z = 180° - 120° - 37{,}3°$$
$$= 22{,}7°$$
$$\therefore \text{Area} WXYZ = \frac{1}{2}(3{,}5)(5)\sin 22{,}7° + \frac{1}{2}(4)(3)$$
$$= 9{,}38 \text{ m}^2$$

7. Find the area of the shaded triangle in terms of x, α, β, θ and ϕ:



Solution:

$$\text{In } \triangle ABE, \quad \frac{x}{EB} = \cos\theta$$
$$\therefore EB = \frac{x}{\cos\theta}$$
$$\text{In } \triangle BDC, \quad \frac{DB}{\sin\alpha} = \frac{x}{\sin\beta}$$
$$\therefore DB = \frac{x\sin\alpha}{\sin\beta}$$
$$\begin{aligned}\therefore \text{Area}\triangle BED &= \frac{1}{2}(BE)(BD)\sin(\beta-\theta)\\ &= \frac{1}{2}\times\frac{x}{\cos\theta}\times\frac{x\sin\alpha}{\sin\beta}\times\sin(\beta-\theta)\\ &= \frac{x^2\sin\alpha\sin(\beta-\theta)}{2\cos\theta\sin\beta}\end{aligned}$$

6.6 Summary

Exercise 6 – 14: End of chapter exercises

1. Write the following as a single trigonometric ratio:
$$\frac{\cos(90° - A)\sin 20°}{\sin(180° - A)\cos 70°} + \cos(180° + A)\sin(90° + A)$$

   Solution:

$$\begin{aligned}&\frac{\sin A\ .\ \sin 20°}{\sin A\ .\ \sin(90° - 70°)} + (-\cos A)\cos(A)\\ &= \frac{\sin A\ .\ \sin 20°}{\sin A\ .\ \sin 20°} + (-\cos A)\cos A\\ &= 1 - \cos^2 A\\ &= \sin^2 A\end{aligned}$$

2. Determine the value of the following expression without using a calculator: $\sin 240° \cos 210° - \tan^2 225° \cos 300° \cos 180°$

   Solution:

$$\begin{aligned}&\sin(180° + 60°)\ .\ \cos(180° + 30°) - \tan^2(180° + 45°)\ .\ \cos(360° - 60°)\ .\ (-1)\\ &= (-\sin 60°)\ .\ (-\cos 30°) - (\tan^2 45°)\ .\ (\cos 60°)\ .\ (-1)\\ &= \frac{\sqrt{3}}{2}\ .\ \frac{\sqrt{3}}{2} - (1)^2\ .\ (\frac{1}{2})\ .\ (-1)\\ &= \frac{3}{4} + \frac{1}{2}\\ &= \frac{3+2}{4}\\ &= 1\frac{1}{4}\end{aligned}$$

3. Simplify: $\dfrac{\sin(180° + \theta)\sin(\theta + 360°)}{\sin(-\theta)\tan(\theta - 360°)}$

Solution:

$$\begin{aligned} & \frac{\sin(180° + \theta)\ .\ \sin(\theta + 360°)}{\sin(-\theta)\ .\ \tan(\theta - 360°)} \\ &= \frac{(-\sin\theta)\ .\ \sin\theta}{(-\sin\theta)\ .\ (-\tan(360° - \theta))} \\ &= \frac{\sin\theta}{\tan\theta} \\ &= \frac{\sin\theta}{1} \cdot \frac{\cos\theta}{\sin\theta} \\ &= \cos\theta \end{aligned}$$

4. Without the use of a calculator, evaluate: $\dfrac{3\sin 55° \sin^2 325°}{\cos(-145°)} - 3\cos 395° \sin 125°$

Solution:

$$\begin{aligned} & \frac{3\sin 55°.\sin^2(360° - (35°))}{\cos(145°} - 3\cos((360° + (35°).\sin(90° + 35°) \\ &= \frac{3\sin 55°.(-\sin 35°)^2}{\cos(90° + 55°)} + 3\cos 35°.\cos 35° \\ &= \frac{-3\sin 55°.\sin^2 35°}{-\sin 55°} + 3\cos^2 35° \\ &= 3(\sin^2 35° + \cos^2 35°) \\ &= 3 \end{aligned}$$

5. Prove the following identities:

a) $\dfrac{1}{(\cos x - 1)(\cos x + 1)} = \dfrac{-1}{\tan^2 x \cos^2 x}$

b) $(1 - \tan\alpha)\cos\alpha = \sin(90 + \alpha) + \cos(90 + \alpha)$

Solution:

a)

$$\begin{aligned} \text{RHS} &= \frac{-1}{\frac{\sin^2 x}{\cos^2 x} \cdot \frac{\cos^2 x}{1}} \\ &= \frac{-1}{\sin^2 x} \\ \text{LHS} &= \frac{1}{\cos^2 x - 1} \\ &= \frac{1}{-(1 - \cos^2 x)} \\ &= -\frac{1}{\sin^2 x} \\ &= \text{RHS} \end{aligned}$$

b)

$$\begin{aligned} \text{LHS} &= (1 - \frac{\sin\alpha}{\cos\alpha}).\cos\alpha \\ &= (\frac{\cos\alpha - \sin\alpha}{\cos\alpha})\cos\alpha \\ &= \cos\alpha - \sin\alpha \\ \text{RHS} &= \cos(-\alpha) + \sin(-\alpha) \\ &= \cos\alpha - \sin\alpha \\ &= \text{LHS} \end{aligned}$$

6. a) Prove: $\tan y + \frac{1}{\tan y} = \frac{1}{\cos^2 y \tan y}$

   b) For which values of $y \in [0°; 360°]$ is the identity above undefined?

Solution:

a)

$$\begin{aligned} \text{LHS} &= \frac{\sin y}{\cos y} + \frac{1}{\frac{1}{\sin y}\cos y} \\ &= \frac{\sin y}{\cos y} + \frac{\cos y}{\sin y} \\ &= \frac{\sin^2 y + \cos^2 y}{\cos y.\sin y} \\ &= \frac{1}{\cos y.\sin y} \\ \text{RHS} &= \frac{1}{\cos^2 y \frac{\sin y}{\cos y}} \\ &= \frac{\cos y}{\cos^2 y.\sin y} \\ &= \frac{1}{\cos y \sin y} \\ &= \text{LHS} \end{aligned}$$

b)

$$\begin{aligned} \tan y &= 0° \\ y &= 0°; 180°; 360° \\ \cos y &= 0° \\ y &= 90°; 270° \end{aligned}$$

7. a) Simplify: $\frac{\sin(180° + \theta)\tan(360° - \theta)}{\sin(-\theta)\tan(180° + \theta)}$

   b) Hence, solve the equation $\frac{\sin(180° + \theta)\tan(360° - \theta)}{\sin(-\theta)\tan(180° + \theta)} = \tan\theta$ for $\theta \in [0°; 360°]$.

Solution:

a)

$$\begin{aligned} &\frac{(-\sin\theta).(-\tan\theta)}{(-\sin\theta).\tan\theta} \\ &= -1 \end{aligned}$$

b)

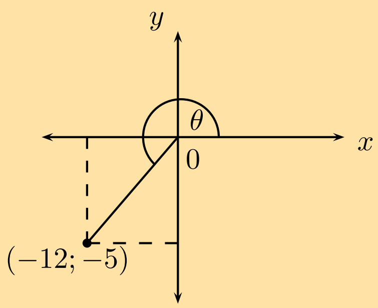
$$\tan\theta = -1$$
$$\therefore \theta = 180° - 45° \text{ or } \theta = 360° - 45°$$
$$= 135° \text{ or } 315°$$

8. Given $12\tan\theta = 5$ and $\theta > 90°$.

   a) Draw a sketch.
   b) Determine without using a calculator $\sin\theta$ and $\cos(180° + \theta)$.
   c) Use a calculator to find θ (correct to two decimal places).

   Solution:

   a)



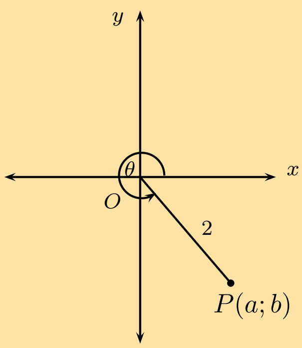
   b)

$$OP = \sqrt{(-12)^2 + (-5)^2}$$
$$= \sqrt{144 + 25}$$
$$= \sqrt{169}$$
$$= 13$$
$$\sin\theta = -\frac{5}{13}$$
$$\cos(180° + \theta) = -\cos\theta$$
$$= -(-\frac{12}{13})$$
$$= \frac{12}{13}$$

   c)

$$\tan\theta = \frac{5}{12}$$
$$\theta = 180° + 22{,}62°$$
$$= 202{,}62°$$

9.



In the figure, P is a point on the Cartesian plane such that $OP = 2$ units and $\theta = 300°$. Without the use of a calculator, determine:

a) the values of a and b

b) the value of $\sin(180° - \theta)$

Solution:

a)

$$\begin{aligned} \cos 300° &= \cos 60° \\ &= \frac{1}{2} \\ \therefore a &= 1 \\ b &= -\sqrt{4-1} \\ &= -\sqrt{3} \end{aligned}$$

b)

$$\begin{aligned} \sin(180° - \theta) &= \sin\theta \\ &= \sin 300° \\ &= \sin(360° - 60°) \\ &= \sin(-60°) \\ &= -\sin 60° \\ &= -\frac{\sqrt{3}}{2} \end{aligned}$$

10. Solve for x with $x \in [-180°; 180°]$ (correct to one decimal place):

a) $2\sin\frac{x}{2} = 0{,}86$

b) $\tan(x + 10°) = \cos 202{,}6°$

c) $\cos^2 x - 4\sin^2 x = 0$

Solution:

a)

$$\begin{aligned} \sin\frac{1}{2}x &= 0{,}43 \\ \frac{1}{2}x &= 25{,}467° \\ x &= 50{,}9° \\ \text{Or} \quad \frac{1}{2}x &= 180° - 25{,}467° \\ x &= 309{,}1° \end{aligned}$$

b)

$$\begin{aligned} \tan(x + 10°) &= -0{,}92 \\ x + 10° &= 180° - 42{,}7° \\ &= 137{,}3° \\ x &= 127{,}3° \\ \text{Or} \quad x + 10° &= 360° - 42{,}7° \\ &= 317{,}3° \\ \therefore x &= 307{,}3° \end{aligned}$$

c)

$$\begin{aligned}
\cos x - 2\sin x &= 0 \\
\cos x &= 2\sin x \\
\frac{\sin x}{\cos x} &= \frac{1}{2} \\
\tan x &= \frac{1}{2} \\
x &= 26{,}6^\circ \\
\text{or } x &= -180^\circ + 26{,}6^\circ \\
&= 206{,}6^\circ
\end{aligned}$$

$$\begin{aligned}
\text{Or} \quad \cos x + 2\sin x &= 0 \\
\cos x &= -2\sin x \\
\frac{\sin x}{\cos x} &= -\frac{1}{2} \\
\tan x &= -\frac{1}{2} \\
x &= 180^\circ - 26{,}6^\circ \\
&= 153{,}4^\circ \\
\text{or } x &= 360^\circ - 26{,}6^\circ \\
&= 333{,}4^\circ
\end{aligned}$$

11. Find the general solution for the following equations:

    a) $\frac{1}{2}\sin(x - 25^\circ) = 0{,}25$
    b) $\sin^2 x + 2\cos x = -2$

    Solution:

    a)

$$\begin{aligned}
\sin(x - 25^\circ) &= 0.5 \\
x - 25^\circ &= 30^\circ \\
\text{or } x - 25^\circ &= 180^\circ - 30^\circ \\
x &= 55^\circ + 360^\circ n, n \in \mathbb{Z} \\
\text{or } x &= 175^\circ + 360^\circ n, n \in \mathbb{Z}
\end{aligned}$$

    b)

$$\begin{aligned}
\sin^2 x + 2\cos x + 2 &= 0 \\
1 - \cos^2 x + 2\cos x + 2 &= 0 \\
-\cos^2 x + 2\cos x + 3 &= 0 \\
\cos^2 x - 2\cos x - 3 &= 0 \\
(\cos x - 3)(\cos x + 1) &= 0 \\
\cos x &= 3 \\
&\text{no solution} \\
\text{OR } \cos x &= -1 \\
x &= 180^\circ + 360^\circ n, n \in \mathbb{Z}
\end{aligned}$$

12. Given the equation: $\sin 2\alpha = 0{,}84$

a) Find the general solution of the equation.

b) Illustrate how this equation could be solved graphically for $\alpha \in [0°; 360°]$.

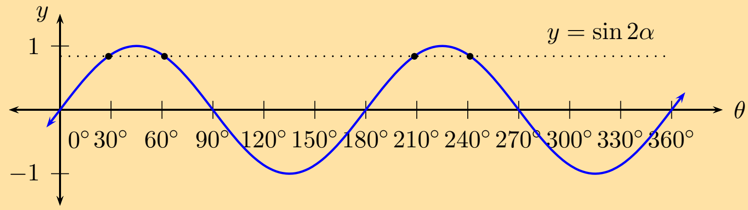
c) Write down the solutions for $\sin 2\alpha = 0{,}84$ for $\alpha \in [0°; 360°]$.

Solution:

a)

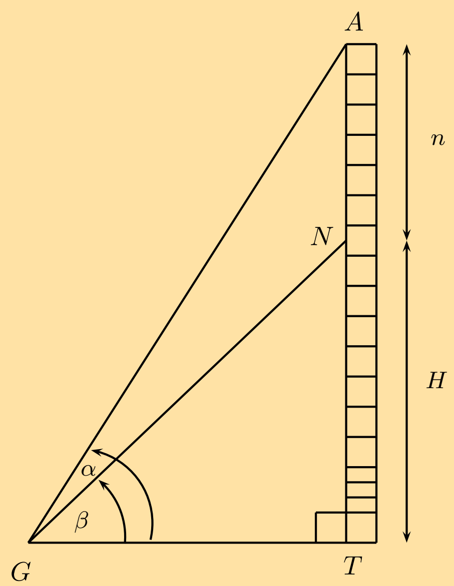
$$\begin{aligned} \sin 2\alpha &= 0{,}84 \\ 2\alpha &= 57{,}14° + 360°n, n \in \mathbb{Z} \\ \alpha &= 28{,}6° + 180°n \\ \text{Or } 2\alpha &= 180° - 57{,}14° + 360°n, n \in \mathbb{Z} \\ 2\alpha &= 122{,}86° + 360°n \\ \alpha &= 61{,}43° + 180°n \end{aligned}$$

b)



c) $28{,}6°; 61{,}4°; 208{,}6°; 241{,}4°$

13.



A is the highest point of a vertical tower AT. At point N on the tower, n metres from the top of the tower, a bird has made its nest. The angle of inclination from G to point A is α and the angle of inclination from G to point N is β.

a) Express $A\hat{G}N$ in terms of α and β.

b) Express $\hat{A}$ in terms of α and/or β.

c) Show that the height of the nest from the ground (H) can be determined by the formula

$$H = \frac{n \cos \alpha \sin \beta}{\sin(\alpha - \beta)}$$

d) Calculate the height of the nest H if $n = 10$ m, $\alpha = 68°$ and $\beta = 40°$ (give your answer correct to the nearest metre).

Solution:

a)

$$A\hat{G}N = \alpha - \beta$$

b)

$$\hat{A} = 90° - \alpha$$

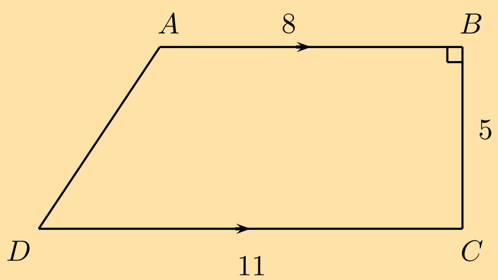
c)

$$\begin{aligned}
&\text{In } \triangle GNT, \\
&\frac{H}{GN} = \sin\beta \\
&\therefore H = GN\sin\beta \\
&\text{In } \triangle AGN, \\
&\frac{n}{\sin(\alpha-\beta)} = \frac{GN}{\sin(90° - \alpha)} \\
&\therefore GN = \frac{n\sin(90° - \alpha)}{\sin(\alpha - \beta)} \\
&= \frac{n\cos\alpha}{\sin(\alpha-\beta)} \\
&\text{Substitute for } GN, \\
&H = \frac{n\cos\alpha\sin\beta}{\sin(\alpha-\beta)}
\end{aligned}$$

d)

$$\begin{aligned}
H &= \frac{n\cos\alpha\sin\beta}{\sin(\alpha-\beta)} \\
&= \frac{10\cos 68°\sin 40°}{\sin 28°} \\
&= 5{,}1 \text{ m}
\end{aligned}$$

14.



Mr. Collins wants to pave his trapezium-shaped backyard, $ABCD$. $AB \parallel DC$ and $\hat{B} = 90°$. $DC = 11$ m, $AB = 8$ m and $BC = 5$ m.

a) Calculate the length of the diagonal AC.

b) Calculate the length of the side AD.

c) Calculate the area of the patio using geometry.

d) Calculate the area of the patio using trigonometry.

Solution:

a)

$$\begin{aligned} AC^2 &= 8^2 + 5^2 \\ &= 64 + 25 = 89 \\ AC &= 9{,}43 \text{ m} \end{aligned}$$

b)

$$\begin{aligned} &\text{In } \triangle ABC, \\ &\tan B\hat{A}C = \frac{5}{8} \\ &\therefore B\hat{A}C = 32^\circ \\ &A\hat{C}D = B\hat{A}C = 32^\circ (AB \parallel DC) \\ &\text{In } \triangle ADC, \\ &AD^2 = AC^2 + DC^2 - 2.AC.AD.\cos(A\hat{C}D) \\ &= (9{,}4)^2 + (11)^2 - 2(9{,}4)(11)\cos(32^\circ) \\ &= 89 + 121 - 176 \\ &= 34 \\ &\therefore AD = 6{,}2 \text{ m} \end{aligned}$$

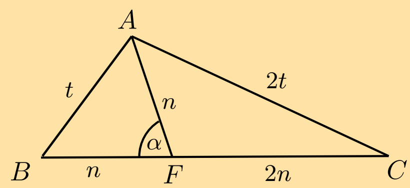
c)

$$\begin{aligned} \text{Area } &= \frac{1}{2} \times (8 + 11) \times 5 \\ &= 47{,}5 \text{ m}^2 \end{aligned}$$

d)

$$\begin{aligned} \text{Area } ABCD &= \text{area } \triangle ABC + \text{area } \triangle ADC \\ &= \frac{1}{2}(8)(5) + \frac{1}{2}(9.43)(11)\sin 32^\circ \\ &= 20 + 27{,}5 \\ &= 47{,}5 \text{ m}^2 \end{aligned}$$

15.



In $\triangle ABC$, $AC = 2A$, $AF = BF$, $A\hat{F}B = \alpha$ and $FC = 2AF$. Prove that $\cos\alpha = \frac{1}{4}$.

Solution:

In $\triangle ABF$

$$\begin{aligned}
t^2 &= n^2 + n^2 - 2n^2 \cos\alpha \\
t^2 &= 2n^2(1 - \cos\alpha) \\
1 - \cos\alpha &= \frac{t^2}{2n^2} \\
\cos\alpha &= 1 - \frac{t^2}{2n^2}
\end{aligned}$$

In $\triangle AFC$

$$\begin{aligned}
(2t)^2 &= n^2 + (2n)^2 - 4n^2 \cos(A\hat{F}C) \\
4t^2 &= n^2 + 4n^2 - 4n^2 \cos(180^\circ - \alpha) \\
&= 5n^2 + 4n^2 \cos\alpha \\
t^2 &= \frac{5}{4}n^2 + n^2 \cos\alpha
\end{aligned}$$

Substitute for t^2

$$\begin{aligned}
\cos\alpha &= 1 - \frac{\frac{5}{4}n^2 + n^2\cos\alpha}{2n^2} \\
&= \frac{2n^2 - \frac{5}{4}n^2 - n^2\cos\alpha}{2n^2} \\
&= \frac{\frac{3}{4}n^2 - n^2\cos\alpha}{2n^2} \\
&= \frac{3}{8} - \frac{\cos\alpha}{2} \\
\frac{3}{2}\cos\alpha &= \frac{3}{8} \\
\therefore \cos\alpha &= \frac{1}{4}
\end{aligned}$$

CHAPTER 7

Measurement

- Use paper or cardboard for the net of solids to help learners see the different heights, particularly perpendicular and slanted heights.
- Units are compulsory when working with real life contexts.
- Sketches are valuable and important tools.
- Rounding off should only be done in the last step and level of accuracy should be relevant to the context.

7.1 Area of a polygon

Exercise 7 – 1: Area of a polygon

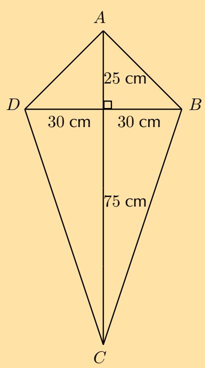
1. Vuyo and Banele are having a competition to see who can build the best kite using balsa wood (a lightweight wood) and paper. Vuyo decides to make his kite with one diagonal 1 m long and the other diagonal 60 cm long. The intersection of the two diagonals cuts the longer diagonal in the ratio $1 : 3$.

   Banele also uses diagonals of length 60 cm and 1 m, but he designs his kite to be rhombus-shaped.

   a) Draw a sketch of Vuyo's kite and write down all the known measurements.
   b) Determine how much balsa wood Vuyo will need to build the outside frame of the kite (give answer correct to the nearest cm).
   c) Calculate how much paper he will need to cover the frame of the kite.
   d) Draw a sketch of Banele's kite and write down all the known measurements.
   e) Determine how much wood and paper Banele will need for his kite.
   f) Compare the two designs and comment on the similarities and differences. Which do you think is the better design? Motivate your answer.

Solution:

a)



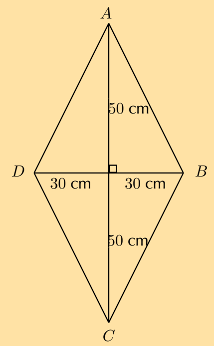
b)

$$\begin{aligned} AD &= AB \\ &= \sqrt{30^2 + 25^2} \quad \text{(Pythagoras)} \\ AD &= 39 \text{ cm} \\ DC &= BC \\ &= \sqrt{30^2 + 75^2} \quad \text{(Pythagoras)} \\ &= 81 \text{ cm} \\ \therefore \text{Balsa wood: } &= 2(81 + 39) \\ &= 240 \text{ cm} \end{aligned}$$

c)

$$\begin{aligned} \text{Area } &= 60 \times 100 \\ &= 6000 \text{ cm}^2 \\ &= 0{,}6 \text{ m}^2 \end{aligned}$$

d)

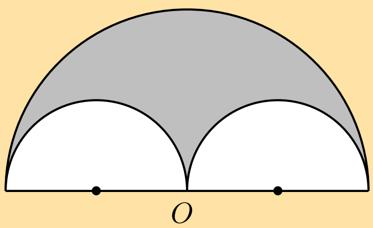


e)

$$\begin{aligned} \text{Side length } &= \sqrt{50^2 + 30^2} \\ &= 58{,}3 \text{ cm} \\ \therefore \text{Wood for frame } &= 4 \times 58{,}3 \text{ cm} \\ &= 233{,}2 \text{ cm} \\ \text{Area } &= 60 \times 100 \\ &= 6000 \text{ cm}^2 \\ &= 0{,}6 \text{ m}^2 \end{aligned}$$

f) Same amount of paper is required for both designs. Vuyo's designs uses more balsa wood.

2. O is the centre of the bigger semi-circle with a radius of 10 units. Two smaller semi-circles are inscribed into the bigger one, as shown on the diagram. Calculate the following (in terms of π):



a) The area of the shaded figure.

b) The perimeter enclosing the shaded area.

Solution:

a)

$$\begin{aligned}\text{Area } &= \frac{1}{2}\pi(10)^2 - 2\left(\frac{1}{2}\pi(5)^2\right)\\ &= 50\pi - 25\pi\\ &= 25\pi\text{units}^2\end{aligned}$$

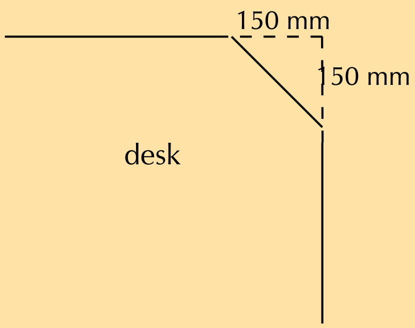
b)

$$\begin{aligned}\text{Perimeter } &= \frac{1}{2}(2\pi(10)) + 2\pi(5)\\ &= 10\pi + 10\pi\\ &= 20\pi\text{units}^2\end{aligned}$$

3. Karen's engineering textbook is 30 cm long and 20 cm wide. She notices that the dimensions of her desk are in the same proportion as the dimensions of her textbook.

   a) If the desk is 90 cm wide, calculate the area of the top of the desk.

   b) Karen uses some cardboard to cover each corner of her desk with an isosceles triangle, as shown in the diagram:



   Calculate the new perimeter and area of the visible part of the top of her desk.

   c) Use this new area to calculate the dimensions of a square desk with the same desk top area.

Solution:

a)

$$\begin{aligned}\text{Ratio } &= \frac{\text{table width}}{\text{book width}}\\ &= \frac{90}{20}\\ &= 4{,}5\\ \therefore \text{ Length of table } &= 30 \times 4{,}5\text{cm}\\ &= 135 \text{ cm}\\ \text{Area table } &= 135 \times 90\\ &= 12\ 150 \text{ cm}^2\\ &= 1{,}2 \text{ m}^2\end{aligned}$$

b)

$$x^2 = 15^2 + 15^2$$
$$x = 21{,}2 \text{ cm}$$
$$\text{New length} = 135 - 2(15)$$
$$= 105 \text{ cm}$$
$$\text{New breadth} = 90 - 2(15)$$
$$= 60 \text{ cm}$$
$$\text{New perimeter} = 2(105) + 2(60) + 4(21{,}2)$$
$$= 414{,}8 \text{ cm}$$

$$\text{Area cut off} = 2 \times (15^2)$$
$$= 450 \text{ m}^2$$
$$\text{New area} = 12\ 150 \text{ cm}^2 - 450 \text{ m}^2$$
$$= 11\ 700 \text{ cm}^2$$

c)

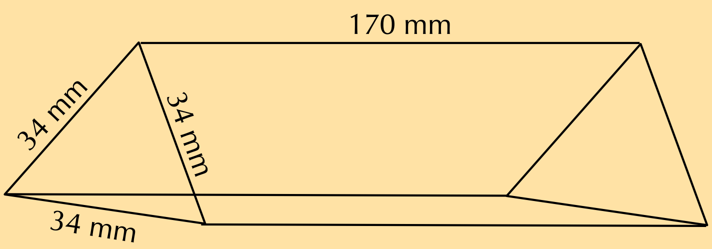
$$s^2 = 11\ 700 \text{ cm}^2$$
$$\therefore s = 108{,}2 \text{ cm}$$
$$\text{Square table of length} \approx 108 \times 108\text{cm}^2$$

7.2 Right prisms and cylinders

Surface area of prisms and cylinders

Exercise 7 – 2: Calculating surface area

1. A popular chocolate container is an equilateral right triangular prism with sides of 34 mm. The box is 170 mm long. Calculate the surface area of the box (to the nearest square centimetre).



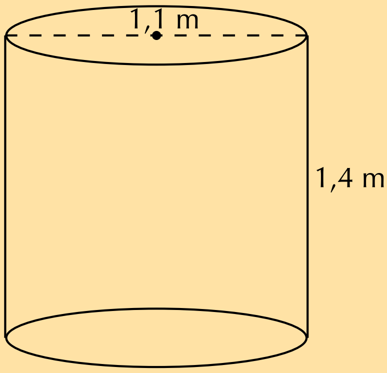
Solution:

$$h = \sin 60°$$
$$\therefore h = 29{,}4 \text{ mm}$$
$$\text{Area } \triangle = 2 \times \frac{1}{2} \times 34 \times 29{,}4$$
$$= 1000 \text{ mm}^2$$
$$\text{Area rectangular sides} = 3 \times 170 \times 34$$
$$= 17\ 340 \text{ mm}^2$$
$$\text{Total surface area} = 27\ 340 \text{ mm}^2$$
$$= 273 \text{ cm}^2$$

2. Gordon buys a cylindrical water tank to catch rain water off his roof. He discovers a full 2 ℓ tin of green paint in his garage and decides to paint the tank (not the base). If he uses 250 ml to cover 1 m^2, will he have enough green paint to cover the tank with one layer of paint?

   Dimensions of the tank:

$$\text{diameter} = 1{,}1 \text{ m}$$
$$\text{height} = 1{,}4 \text{ m}$$



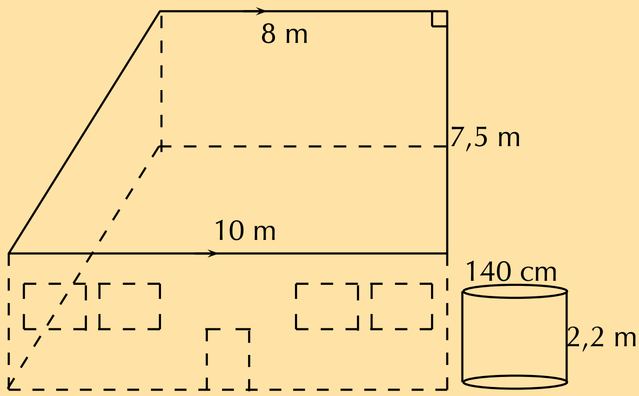
Solution:

$$\text{Surface area} = 2 \times \pi \times (0{,}55) \times 1{,}4 + \pi(0{,}55)^2$$
$$= 5{,}788 \text{ m}^2$$
$$\text{So, he needs} \approx 6 \text{ m}^2 \times 250 \text{ ml paint}$$
$$= 1500 \text{ ml}$$
$$= 1{,}5\ \ell$$

Yes, he will have enough paint for 1 layer.

Exercise 7 – 3: Calculating volume

1. The roof of Phumza's house is the shape of a right-angled trapezium. A cylindrical water tank is positioned next to the house so that the rain on the roof runs into the tank. The diameter of the tank is 140 cm and the height is $2{,}2$ m.



a) Determine the area of the roof.

b) Determine how many litres of water the tank can hold.

Solution:

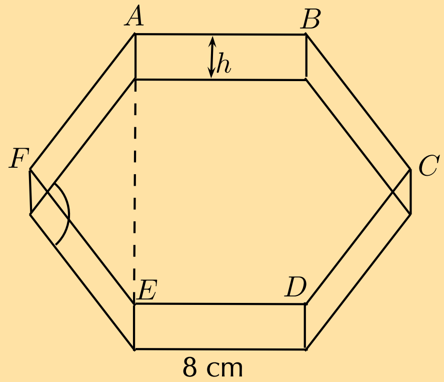
a)

$$\begin{aligned}\text{Area of roof} &= \frac{1}{2}(8 + 10) \times 7{,}5 \\ &= 67{,}5 \text{ m}^2\end{aligned}$$

b)

$$\begin{aligned}\text{Volume of tank} &= \pi(0{,}7)^2 \times 2{,}2 \\ &= 3{,}39 \text{ m}^3 \\ &= 3{,}39\ \ell\end{aligned}$$

2. The length of a side of a hexagonal sweet tin is 8 cm and its height is equal to half of the side length.



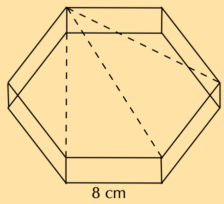
a) Show that the interior angles are equal to $120°$.

b) Determine the length of the line AE.

c) Calculate the volume of the tin.

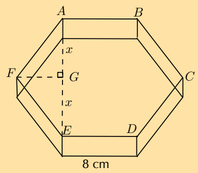
Solution:

a)



$$\begin{aligned} \text{Sum interior angles} &= 180^\circ \times 4 \\ \therefore \text{Each angle} &= \frac{180^\circ \times 4}{6} \\ &= 120^\circ \end{aligned}$$

b)



Construct line FG perpendicular to AE such that G lies on AE. Let $AG = x$.

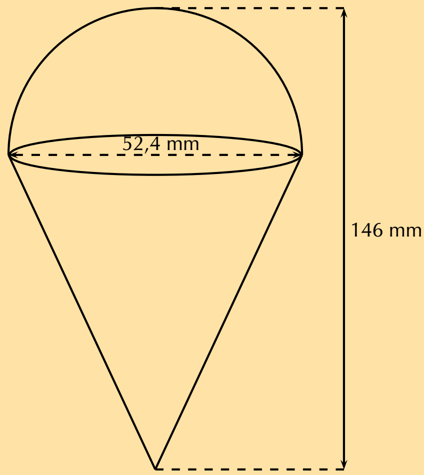
$$\begin{aligned} AG = GE &= x \\ \text{In } \triangle AGF: \quad \frac{x}{8} &= \sin 60^\circ \\ \therefore x &= 6{,}93 \text{ cm} \\ \therefore AE &= 2 \times x \\ &= 13{,}86 \text{ cm} \end{aligned}$$

c)

$$\begin{aligned} \text{Volume } &= \text{area} \times h \\ &= (2 \triangleleft' s + \square) \times 4 \\ FG^2 &= 8^2 - (8 \sin 60^\circ)^2 \\ \therefore FG &= 4 \text{ cm} \\ &\text{OR} \\ \cos 60^\circ &= \frac{FG}{8} \\ \frac{1}{2} &= \frac{FG}{8} \\ \therefore FG &= 4 \text{ cm} \\ \text{Volume of box } &= \left[2\left(\frac{1}{2} \times 8 \sin 60^\circ \times 4\right) + (2(8 \sin 60^\circ) \times 8)\right] \times (4) \\ &= 138{,}56 \times 4 \\ &= 554{,}24 \text{ m}^3 \end{aligned}$$

Exercise 7 – 4: Finding surface area and volume

1. An ice-cream cone has a diameter of $52{,}4$ mm and a total height of 146 mm.

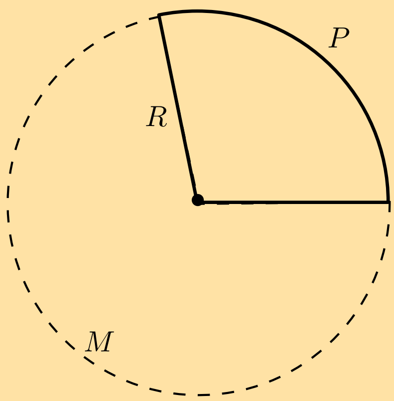


a) Calculate the surface area of the ice-cream and the cone.

b) Calculate the total volume of the ice-cream and the cone.

c) How many ice-cream cones can be made from a $5\ \ell$ tub of ice-cream (assume the cone is completely filled with ice-cream)?

d) Consider the net of the cone given below. R is the length from the tip of the cone to its perimeter, P.



i. Determine the value of R.

ii. Calculate the length of arc P.

iii. Determine the length of arc M.

Solution:

a)

$$\begin{aligned}
\text{Radius} &= \frac{52{,}4}{2} \\
&= 26{,}2 \text{ mm} \\
\text{Height of cone} &= 146 - 26{,}2 \\
&= 119{,}8 \text{ mm} \\
&\approx 120 \text{ mm} \\
\text{Surface area cone} &= \pi \times 26{,}2 \times (26{,}2 + 119{,}8) \\
&= 12{,}0 \text{ mm}^2 \\
&= 120 \text{ cm}^2
\end{aligned}$$

b)

$$\begin{aligned}
\text{Volume} &= \text{volume(cone)} + \text{volume}(\frac{1}{2}\text{ sphere}) \\
&= \frac{1}{3}\pi r^2 H + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) \\
&= \frac{1}{3}\pi(26{,}2)^2 \times 120 + \frac{2}{3}\pi(26{,}2)^3 \\
&= 86\ 260{,}59\ldots + 37\ 667{,}12\ldots \\
&= 123\ 928 \text{ mm}^3 \\
&= 124 \text{ cm}^3
\end{aligned}$$

c)

$$\begin{aligned}
1000 \text{ cm}^3 &= 1\ \ell \\
\therefore 5\ \ell &= 5000 \text{ cm}^3 \\
\therefore \frac{5000}{124} &\approx 40 \text{ cones}
\end{aligned}$$

d) i.

$$\begin{aligned}
R &= 146 - 26{,}2 \\
&= 119{,}8 \text{ mm} \\
&\approx 120 \text{ mm}
\end{aligned}$$

ii.

$$\begin{aligned}
P &= \text{circumference of cone} \\
&= 2\pi(26{,}2) \\
&\approx 165 \text{ mm}
\end{aligned}$$

iii.

$$\begin{aligned}
M &= 2\pi(120) - 165 \\
&= 589 \text{ mm}
\end{aligned}$$

Exercise 7 – 5: The effects of k

1. Complete the following sentences:

   a) If one dimension of a cube is multiplied by a factor $\frac{1}{2}$, the volume of the cube ...

   b) If two dimensions of a cube are multiplied by a factor 7, the volume of the cube ...

   c) If three dimensions of a cube are multiplied by a factor 3, then:

      i. each side of the cube will ...

      ii. the outer surface area of the cube will ...

      iii. the volume of the cube will ...

   d) If each side of a cube is halved, then:

      i. the outer surface area of the cube will ...

      ii. the volume of the cube will ...

   Solution:

   a) Is halved

   b)

   $$\text{multiplied by } 7^2 = 49$$
   $$\therefore \approx \text{ 50 times bigger}$$

   Approx. 50 times bigger

   c) i. become 3 times longer.

      ii. become $3^2 = 9$ times bigger.

      iii. become $3^3 = 27$ times bigger.

   d) i. be multiplied by a factor of $\frac{1}{2}^2 = \frac{1}{4}$, therefore surface area become 4 times smaller.

      ii. be multiplied by a factor of $\frac{1}{2}^3 = \frac{1}{8}$, therefore surface area become 8 times smaller.

2. The municipality intends building a swimming pool of volume W^3 cubic metres. However, they realise that it will be very expensive to fill the pool with water, so they decide to make the pool smaller.

   a) The length and breadth of the pool are reduced by a factor of $\frac{7}{10}$. Express the new volume in terms of W.

   b) The dimensions of the pool are reduced so that the volume of the pool decreases by a factor of $0{,}8$. Determine the new dimensions of the pool in terms of W (remember that the pool must be a cube).

   Solution:

a)

$$\begin{aligned} \text{Volume} &= W^3 \\ \text{New volume} &= 0{,}7W \times 0{,}7W \times W \\ &= (0{,}7)^2 W^3 \\ &= 0{,}49W^3 \\ &\approx 0{,}5W^3 \end{aligned}$$

b)

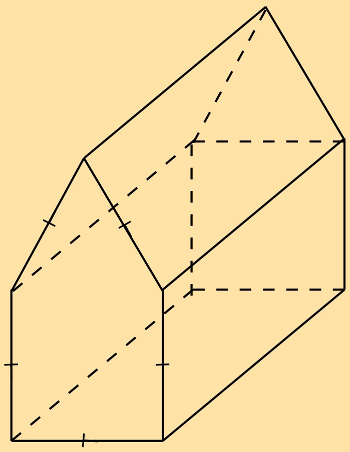
$$\begin{aligned} \text{Volume} &= W^3 \\ \text{New volume} &= 0{,}8W^3 \\ \therefore \text{side length} &= \sqrt[3]{0{,}8} \times W \\ &= 0{,}93 \times W \end{aligned}$$

Each side should now be slightly more than $\frac{9}{10}$ of original length .

7.5 Summary

Exercise 7 – 6: End of chapter exercises

1.



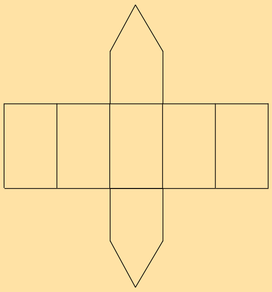
a) Describe this figure in terms of a prism.

b) Draw a net of this figure.

Solution:

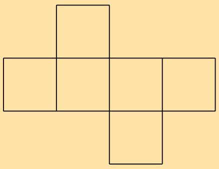
a) Square prism with a triagonal prism on one side. Note that this is not a pentagonal prism since not all the angles are the same size.

b)

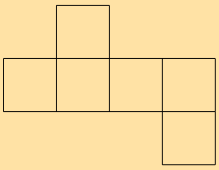


2. Which of the following is a net of a cube?

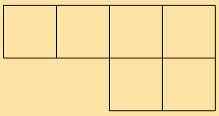
   a)



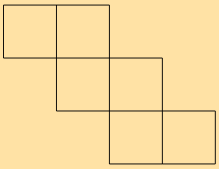
   b)



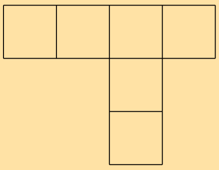
   c)



   d)



   e)



Solution:

a and d

3. Name and draw the following figures:

   a) A prism with the least number of sides.
   b) A pyramid with the least number of vertices.
   c) A right prism with a kite base.

Solution:

   a) Triangular prism

b) Triangular pyramid

c) Rhombic prism

4. a) i. Determine how much paper is needed to make a box of width 16 cm, height 3 cm and length 20 cm (assume no overlapping at corners).

ii. Give a mathematical name for the shape of the box.

iii. Calculate the volume of the box.

b) Determine how much paper is needed to make a cube with a capacity of $1\ \ell$.

c) Compare the box and the cube. Which has the greater volume and which requires the most paper to make?

Solution:

a) i.

$$\begin{aligned} \text{Box surface area} &= 2[3 \times 16 + 3 \times 20 + 16 \times 20] \\ &= 2[428] \\ &= 856\ \text{cm}^2 \end{aligned}$$

ii. Rectangular prism

iii.

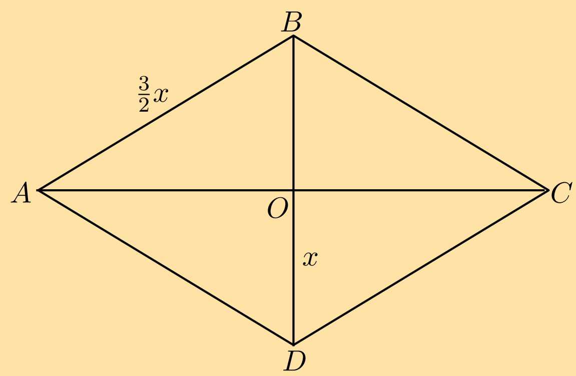
$$\begin{aligned} \text{Volume} &= 3 \times 16 \times 20 \\ &= 960\ \text{cm}^3 \end{aligned}$$

b)

$$\begin{aligned} 1\ \text{liter} &= 1000\ \text{cm}^3 \\ \therefore \text{Side length} &= 10\ \text{cm} \\ \text{Surface area} &= 6 \times (10 \times 10) \\ &= 600\ \text{cm}^2 \end{aligned}$$

c) The cube has a greater volume and the box requires more paper to make.

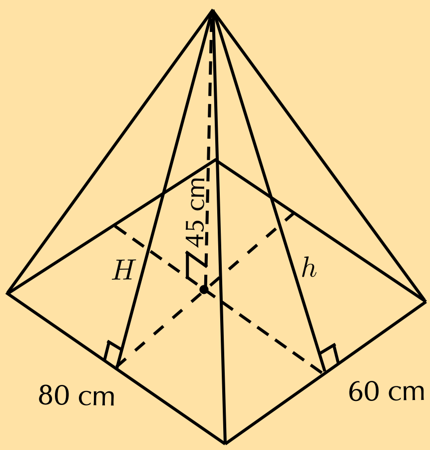
5. $ABCD$ is a rhombus with sides of length $\frac{3}{2}x$ millimetres. The diagonals intersect at O and length $DO = x$ millimetres. Express the area of $ABCD$ in terms of x.



Solution:

$$AD = \frac{3}{2}x$$
$$DO = x$$
$$AO^2 = \left(\frac{3}{2}x\right)^2 - x^2 \quad (\text{ Pythagoras})$$
$$= \frac{9}{4}x^2 - x^2$$
$$= \frac{5}{4}x^2$$
$$\therefore AO = \frac{x\sqrt{5}}{2}$$
$$\therefore AC = x\sqrt{5}$$
$$\text{Area } = \frac{1}{2}AC \times BD$$
$$= \frac{1}{2} \times x\sqrt{5} \times 2x$$
$$= \sqrt{5}x^2$$

6. The diagram shows a rectangular pyramid with a base of length 80 cm and breadth 60 cm. The vertical height of the pyramid is 45 cm.



a) Calculate the volume of the pyramid.

b) Calculate H and h.

c) Calculate the surface area of the pyramid.

Solution:

a)

$$\text{Volume } = \frac{1}{3} \times \text{area of base} \times \text{height}$$
$$= \frac{1}{3}(80 \times 60) \times 45$$
$$= 72\ 000 \text{ cm}^3$$

b)

$$\begin{aligned} H^2 &= 30^2 + 45^2 \quad (\text{ Pythagoras}) \\ &= 2925 \\ \therefore H &= 54 \text{ cm} \\ h^2 &= 45^2 + 40^2 \quad (\text{ Pythagoras}) \\ h &= 60{,}2 \text{ cm} \end{aligned}$$

c)

$$\begin{aligned} \text{Surface area } &= 2\left(\frac{1}{2} \times 80 \times H\right) + 2\left(\frac{1}{2} \times 60 \times h\right) + 80 \times 60 \\ &= (80 \times 54) + (60 \times 60{,}2) + 4800 \\ &= 12\ 732 \text{ cm}^2 \end{aligned}$$

7. A group of children are playing soccer in a field. The soccer ball has a capacity of 5000 cc (cubic centimetres). A drain pipe in the corner of the field has a diameter of 20 cm. Is it possible for the children to lose their ball down the pipe? Show your calculations.

   Solution:

$$\begin{aligned} \text{Volume of ball } &= \frac{4}{3}\pi r^3 \\ &= 5000 \\ \therefore r &= 10{,}6 \\ \text{And diameter } &= 21{,}2 \text{ cm} \\ \text{Diameter of pipe} &= 20 \text{ cm} \end{aligned}$$

   No, the ball is too big to go down the drain pipe.

8. A litre of washing powder goes into a standard cubic container at the factory.

   a) Determine the length of the sides of the container.

   b) Determine the dimensions of the cubic container required to hold double the volume of washing powder.

   Solution:

   a)

$$\text{Dimensions: } 10 \text{ cm} \times 10 \text{ cm} \times 10 \text{ cm}$$

   b)

$$\begin{aligned} 2l &= 2000 \text{ cm}^3 \\ \therefore s &= \sqrt[3]{2000} \\ &= 12{,}6 \text{ cm} \end{aligned}$$

9. A cube has sides of length k units.

   a) Describe the effect on the volume of the cube if the height is tripled.

   b) If all three dimensions of the cube are tripled, determine the effect on the outer surface area.

c) If all three dimensions of the cube are tripled, determine the effect on the volume.

Solution:

a)

$$
\begin{aligned}
\text{Volume} &= k \times k \times k \\
&= k^3 \\
\text{New volume} &= k \times k \times 3k \\
&= 3k^3
\end{aligned}
$$

b)

$$
\begin{aligned}
\text{Surface area} &= 6 \times k^2 \\
&= 6k^2 \\
\text{Surface area} &= 6 \times (3k)^2 \\
&= 54k^2
\end{aligned}
$$

c)

$$
\begin{aligned}
\text{Volume} &= k^3 \\
\text{New volume} &= (3k)^3 \\
&= 27k^3
\end{aligned}
$$

CHAPTER 8

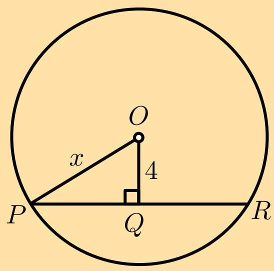
Euclidean geometry

- Discuss terminology.
- Converses are not examinable.
- Good practice for answering questions:
  - make a neat and accurate drawing
  - state triangle/figure being considered
  - give statement and appropriate reason
  - give a conclusion

8.1 Circle geometry

Exercise 8 – 1: Perpendicular line from center bisects chord

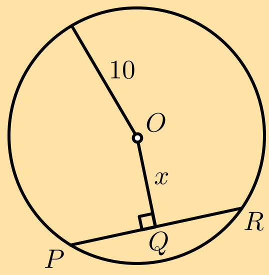
1. In the circle with centre O, $OQ \perp PR$, $OQ = 4$ units and $PR = 10$. Determine x.



Solution:

$$\begin{aligned} PR &= 10 && (\perp \text{ given }) \\ \therefore PQ &= 5 && (\perp \text{ from centre bisects chord}) \\ OP^2 &= OQ^2 + QP^2 && (\text{Pythagoras}) \\ x^2 &= 4^2 + 5^2 \\ \therefore x^2 &= 25 + 16 \\ x^2 &= 41 \\ x &= \sqrt{41} \end{aligned}$$

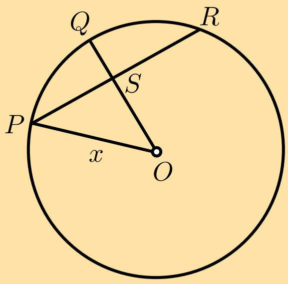
2. In the circle with centre O and radius $= 10$ units, $OQ \perp PR$ and $PR = 8$. Determine x.



Solution:

$$\begin{aligned} PR &= 8 && (\perp \text{ given }) \\ \therefore PQ &= 4 && (\perp \text{ from centre bisects chord}) \\ OP^2 &= OQ^2 + QP^2 && \text{(Pythagoras)} \\ 10^2 &= x^2 + 4^2 \\ \therefore x^2 &= 100 - 16 \\ x^2 &= 84 \\ x &= \sqrt{84} \end{aligned}$$

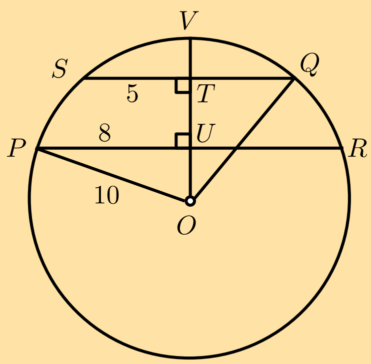
3. In the circle with centre O, $OQ \perp PR$, $PR = 12$ units and $SQ = 2$ units. Determine x.



Solution:

$$\begin{aligned} SO &= x - 2 \\ OP^2 &= OS^2 + SP^2 && \text{(Pythagoras)} \\ x^2 &= (x-2)^2 + 6^2 \\ x^2 &= x^2 - 4x + 4 + 6^2 \\ 4x &= 40 \\ \therefore x &= 10 \end{aligned}$$

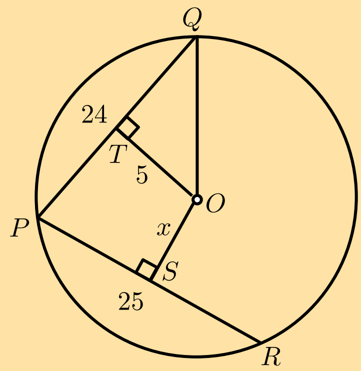
4. In the circle with centre O, $OT \perp SQ$, $OT \perp PR$, $OP = 10$ units, $ST = 5$ units and $PU = 8$ units. Determine TU.



Solution:

$$\begin{aligned}
\text{In } \triangle POU, \quad OP^2 &= OU^2 + PU^2 \qquad \text{(Pythagoras)}\\
10^2 &= OU^2 + 8^2\\
100 - 64 &= OU^2\\
\therefore OU^2 &= 36\\
\therefore OU &= 6\\
\text{In } \triangle QTO, \quad QO^2 &= OT^2 + TQ^2 \qquad \text{(Pythagoras)}\\
10^2 &= OT^2 + 5^2\\
100 - 25 &= OT^2\\
\therefore OT^2 &= 75\\
\therefore OT &= \sqrt{75}\\
\therefore TU &= OT - OU\\
&= \sqrt{75} - 6\\
&= 2{,}66
\end{aligned}$$

5. In the circle with centre O, $OT \perp QP$, $OS \perp PR$, $OT = 5$ units, $PQ = 24$ units and $PR = 25$ units. Determine $OS = x$.



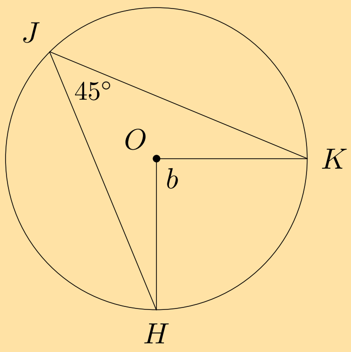
Solution:

$$\begin{aligned}
\text{In } \triangle QTO, \quad QO^2 &= OT^2 + QT^2 \qquad \text{(Pythagoras)}\\
QO^2 &= 5^2 + 12^2\\
&= 25 + 144\\
\therefore QO^2 &= 169\\
\therefore QO &= 13\\
\text{In } \triangle OSR, \quad OR^2 &= SR^2 + OS^2 \qquad \text{(Pythagoras)}\\
13^2 &= 12{,}5^2 + OS^2\\
\therefore OS^2 &= 12{,}75\\
\therefore OS &= 3{,}6
\end{aligned}$$

Exercise 8 – 2: Angle at the centre of circle is twice angle at circumference

Given O is the centre of the circle, determine the unknown angle in each of the following diagrams:

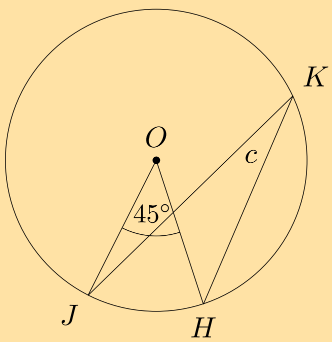
1.



Solution:

$$b = 2 \times 45° \qquad (\angle \text{ at centre } = 2\angle \text{ at circum.})$$
$$\therefore b = 90°$$

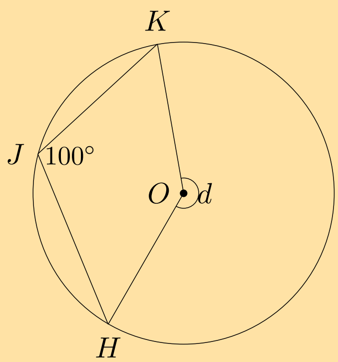
2.



Solution:

$$c = \frac{1}{2} \times 45° \qquad (\angle \text{ at centre } = 2\angle \text{ at circum.})$$
$$\therefore c = 22{,}5°$$

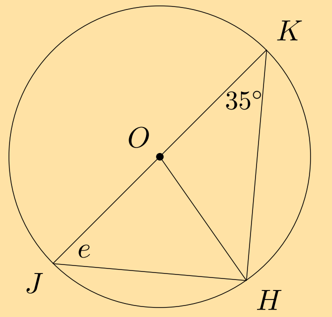
3.



Solution:

$$d = 2 \times 100° \qquad (\angle \text{ at centre } = 2\angle \text{ at circum.})$$
$$\therefore d = 200°$$

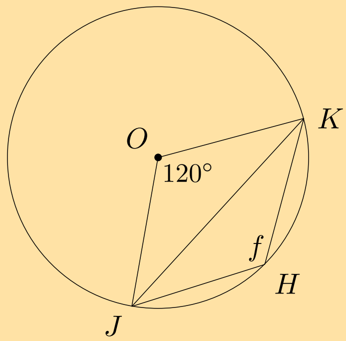
4.



Solution:

$$e = 100° - 90° - 35° \quad (\angle \text{ in semi circle})$$
$$\therefore e = 55°$$

5.



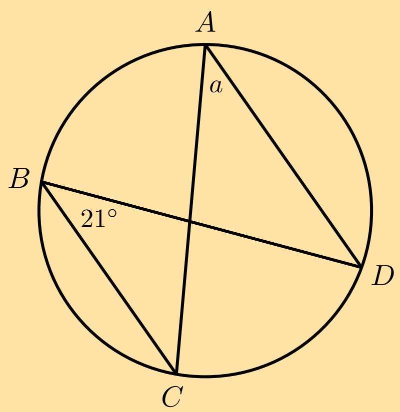
Solution:

$$f = \frac{1}{2} \times 240° \quad (\angle \text{ at centre } = 2\angle \text{ at circum.})$$
$$\therefore f = 120°$$

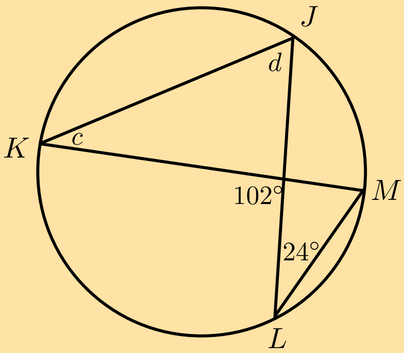
Exercise 8 – 3: Subtended angles in the same segment

1. Find the values of the unknown angles.

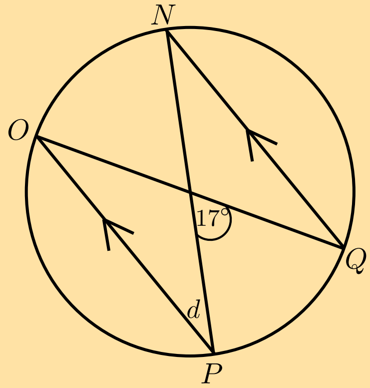
a)



b)



c)



Solution:

a)

$$a = 21^\circ \qquad \text{(}\angle\text{s in same seg.)}$$

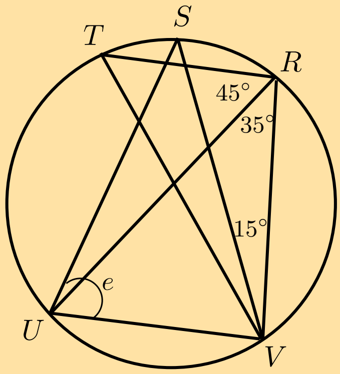
b)

$$\begin{aligned} c &= 24^\circ && \text{(}\angle\text{s in same seg.)} \\ d &= 102^\circ - 24^\circ && \text{(ext. } \angle\triangle = \text{sum int. opp.)} \\ \therefore d &= 78^\circ \end{aligned}$$

c)

$$\begin{aligned} d &= \hat{N} && \text{(alt. } \angle, PO \parallel QN\text{)} \\ \hat{N} &= \tfrac{1}{2} \times 17^\circ && \text{(}\angle \text{ at centre } = 2\angle \text{ at circum.)} \\ \hat{O} &= \hat{N} && \text{(}\angle\text{s in same seg.)} \\ 17^\circ &= \hat{O} + d && \text{(ext. angle of } \triangle\text{)} \\ \therefore 2d &= 17^\circ \\ \therefore d &= 8{,}5^\circ && \text{(alt. } \angle, PO \parallel QN\text{)} \end{aligned}$$

2.



a) Given $T\hat{V}S = S\hat{V}R$, determine the value of e.

b) Is TV a diameter of the circle? Explain your answer.

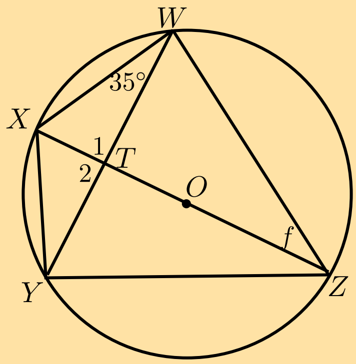
Solution:

a)

$$\begin{aligned} \text{In } \triangle TRV, \hat{T} &= 180° - (80° + 30°) && (\angle\text{s sum of } \triangle) \\ \therefore \hat{T} &= 70° \\ \therefore e &= 15° + 70° \\ &= 85° \end{aligned}$$

b) No, since $45° + 35° \neq 90°$

3. Given circle with centre O, $WT = TY$ and $X\hat{W}T = 35°$. Determine f.



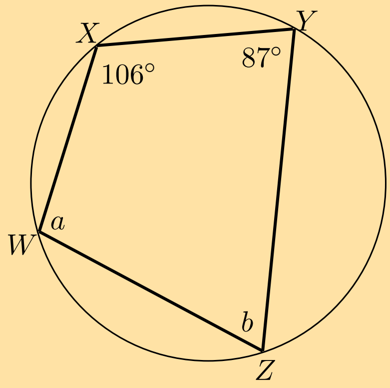
Solution:

$$\begin{aligned} &\text{In } \triangle WTZ \text{ and in } \triangle YTZ, \\ WT &= YT && (\text{ given }) \\ ZT &= ZT && (\text{ common side }) \\ Y\hat{T}Z = W\hat{T}Z &= 90° && (\text{ line from circle centre to mid-point }) \\ \therefore T\hat{Z}Y &= T\hat{Z}W && (\text{ SAS }) \\ T\hat{Z}Y = T\hat{Z}W &= f \\ \text{And } T\hat{Z}Y &= 35° && (\angle\text{s in same seg.}) \\ \therefore T\hat{Z}W = f &= 35° \end{aligned}$$

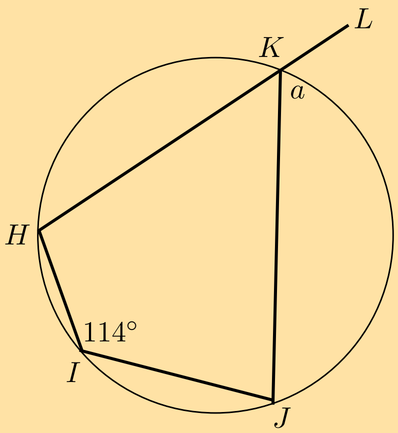
Exercise 8 – 4: Cyclic quadrilaterals

1. Find the values of the unknown angles.

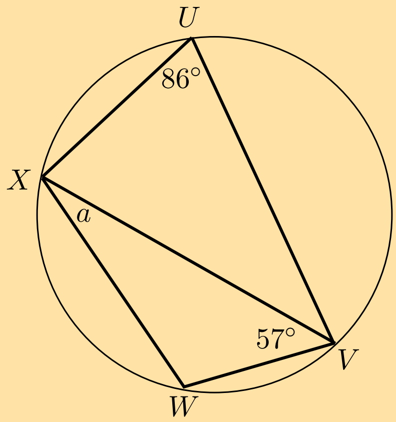
a)



b)



c)



Solution:

a)

$$
\begin{aligned}
a + 87° &= 180° && \text{(opp. angles of cyclic quad. supp.)}\\
\therefore a &= 93°\\
b + 106° &= 180° && \text{(opp. angles of cyclic quad. supp.)}\\
\therefore b &= 74°
\end{aligned}
$$

b)

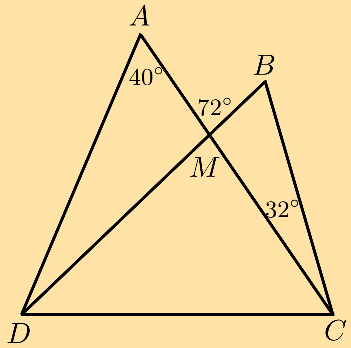
$$
\begin{aligned}
a &= H\hat{I}J && \angle(\text{ ext. angle cyclic quad = int. opp })\\
&= 114°
\end{aligned}
$$

c)

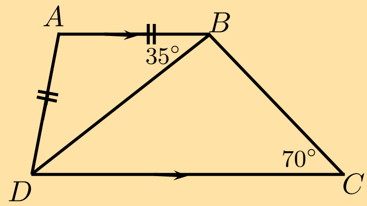
$$
\begin{aligned}
\hat{W} + 86° &= 180° && \text{(opp. angles of cyclic quad. supp.)}\\
\therefore \hat{W} &= 94°\\
a + \hat{W} + 57° &= 180° && \text{(angles sum of } \triangle)\\
\therefore a &= 29°
\end{aligned}
$$

2. Prove that $ABCD$ is a cyclic quadrilateral:

a)



b)



Solution:

a)

$A\hat{M}B = 32° + D\hat{B}C$	(ext.$\angle\triangle$ = sum int. opp.$\angle$)
$\therefore 72° = 32° + D\hat{B}C$	
$\therefore D\hat{B}C = 40°$	(angles sum of $\triangle$)
$\therefore D\hat{B}C = D\hat{A}C$	
Therefore $ABCD =$ is cyclic quad.	(angles in same seg.)

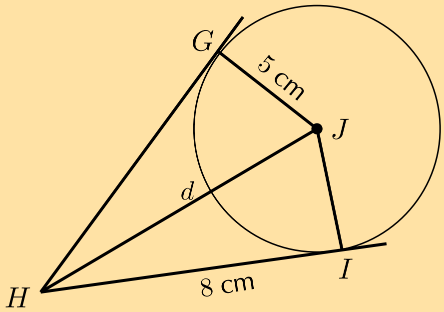
b)

In $\triangle ABD$, $A\hat{B}D = A\hat{D}B = 35°$	($\angle$s opp. = sides)
$\therefore 35° + 35° + D\hat{A}B = 180°$	($\angle$s sum of $\triangle$)
$\therefore D\hat{A}B = 110°$	(angles sum of $\triangle$)
And $D\hat{A}B + D\hat{C}B = 180°$	
Therefore $ABCD =$ is cyclic quad.	(opp. int. angles supp.)

Exercise 8 – 5: Tangents to a circle

Find the values of the unknown lengths.

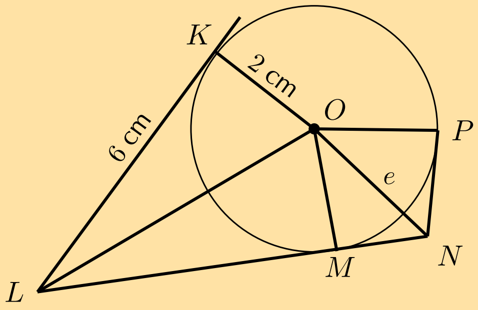
1.



Solution:

$HI = HG$	(tangents same pt.)
In $\triangle HIJ$, $d^2 = 8^2 + 5^2$	(Pythagoras, radius perp. tangent)
$\therefore d^2 = 89$	
$\therefore d = 9{,}4$ cm	

2.

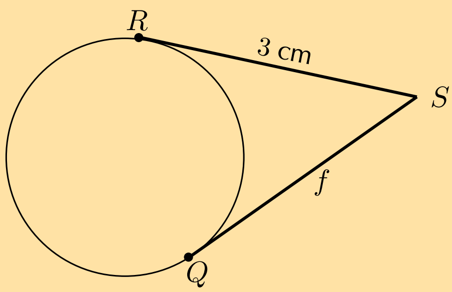


$LN = 7{,}5$ cm

Solution:

$$
\begin{aligned}
LM = LK &= 6 && \text{(tangents same pt.)} \\
LN &= 7{,}5 \text{ cm} && \text{(given)} \\
\therefore MN = 7{,}5 - 6 &= 1{,}5 \text{ cm} && \\
OM &= 2 \text{ cm} && \text{(radius)} \\
\text{In } \triangle OMN, \quad e^2 &= 2^2 + (1{,}5)^2 && \text{(Pythagoras, radius perp. tangent)} \\
\therefore e^2 &= 6{,}25 && \\
\therefore e &= 2{,}5 \text{ cm} &&
\end{aligned}
$$

3.



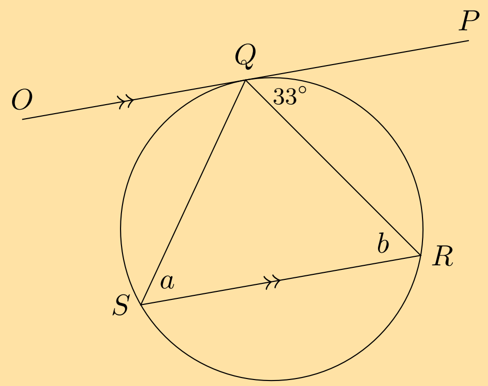
Solution:

$f = 3$ cm

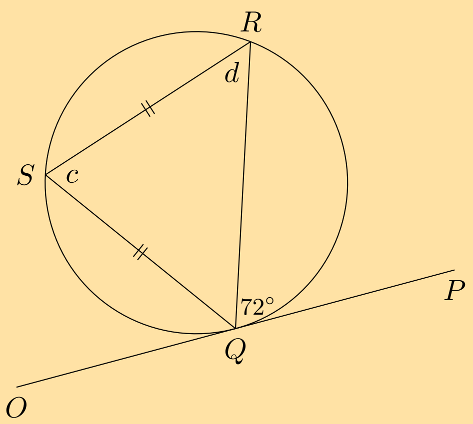
Exercise 8 – 6: Tangent-chord theorem

1. Find the values of the unknown letters, stating reasons.

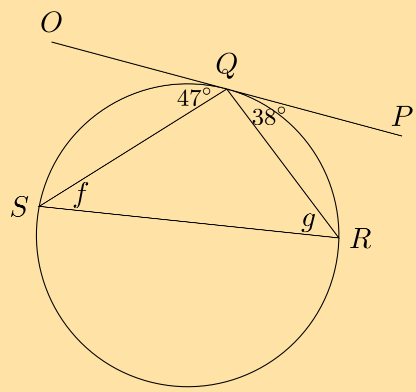
a)



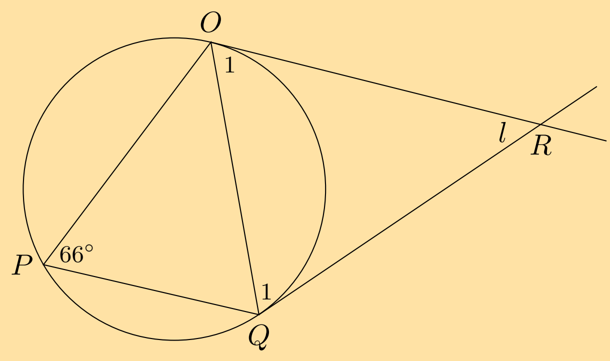
b)



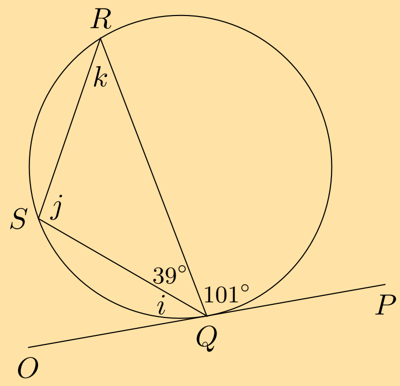
c)



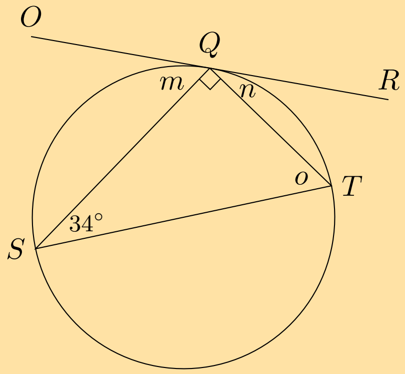
d)



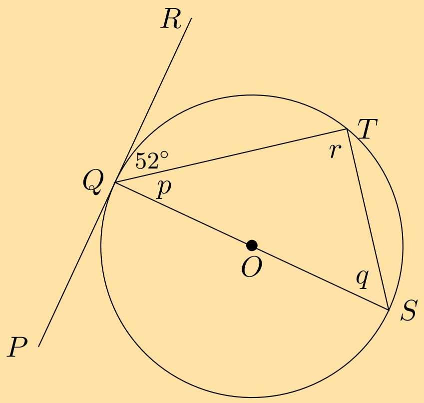
e)



f)



g)



Solution:

a)

$$a = 33° \quad \text{(tangent-chord)}$$
$$b = 33° \quad \text{(alt. angles, } OP \parallel SR\text{)}$$

b)

$$c = 72° \quad \text{(tangent-chord)}$$
$$d = \frac{180° - 72°}{2} \quad \text{(isosceles triangle)}$$
$$= 54° \quad \text{(alt. angles, } OP \parallel SR\text{)}$$

c)

$$f = 38° \quad \text{(tangent-chord)}$$
$$g = 47° \quad \text{(tangent-chord)}$$

d)

$$\hat{O}_1 = \hat{Q}_1 = 66° \quad \text{(isosceles, tangent-chord)}$$
$$\therefore l = 180° - 2 \times 66° \quad \text{(angles sum } \triangle\text{)}$$
$$= 48°$$

e)

$$i = 180° - 101° - 39° \quad (\angle\text{s on str. line })$$
$$\therefore i = 40°$$
$$j = 101° \quad \text{(tangent-chord)}$$
$$k = i = 40° \quad \text{(tangent-chord)}$$

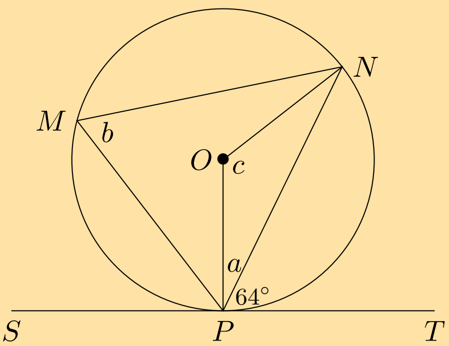
f)

$$n = 34° \quad \text{(tangent-chord)}$$
$$o = 180° - 90° - 34° \quad \text{(angles sum } \triangle\text{)}$$
$$\therefore o = 56°$$
$$m = 56° \quad \text{(tangent-chord)}$$

g)

$$q = 52° \quad \text{(tangent-chord)}$$
$$p = 90° - 52° \quad \text{(tangent perp. radius)}$$
$$\therefore p = 38°$$
$$r = 90° \quad (\angle \text{ in semi-circle})$$

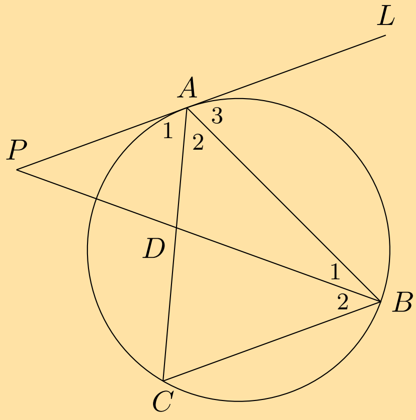
2. O is the centre of the circle and SPT is a tangent, with $OP \perp ST$. Determine a, b and c, giving reasons.



Solution:

$$a = 90° - 64° \quad \text{(tangent perp. radius)}$$
$$= 26°$$
$$b = 64° \quad \text{(tangent chord)}$$
$$c = 2 \times 64° \quad (\angle \text{ at centre} = 2\angle \text{ at circum.})$$
$$= 128°$$

3.



Given $AB = AC$, $AP \parallel BC$ and $\hat{A}_2 = \hat{B}_2$. Prove:

a) PAL is a tangent to the circle ABC.

b) AB is a tangent to the circle ADP.

Solution:

a)

$$\begin{aligned} \hat{A}_1 &= A\hat{C}B && \text{(alt. angles, } AP \parallel BC) \\ A\hat{C}B &= A\hat{B}C && (\angle\text{s opp. equal sides , } AB = AC) \\ \therefore \hat{A}_1 &= A\hat{B}C \\ \therefore PAL \text{ tangent to circle } ABC & && (\angle\text{btn line chord} = \angle \text{ in alt. seg.}) \end{aligned}$$

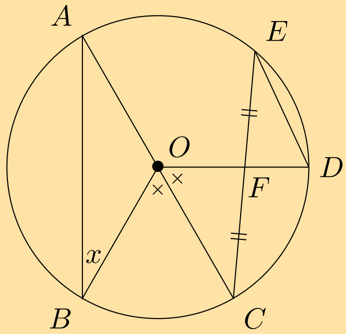
b)

$$\begin{aligned} \hat{A}_2 &= \hat{B}_2 && \text{(given)} \\ \text{And } A\hat{P}B &= \hat{B}_2 && \text{(alt. angles, } AP \parallel BC) \\ \text{Therefore } A\hat{P}B &= \hat{A}_2 \\ \therefore AB \text{ tangent to circle } ADP & && (\angle\text{btn line chord} = \angle \text{ in alt. seg.}) \end{aligned}$$

8.2 Summary

Exercise 8 – 7: End of chapter exercises

1.



AOC is a diameter of the circle with centre O. F is the mid-point of chord EC. $B\hat{O}C = C\hat{O}D$ and $\hat{B} = x$. Express the following angles in terms of x, stating reasons:

a) $\hat{A}$

b) $C\hat{O}D$

c) $\hat{D}$

Solution:

a)

$$\hat{A} = x \qquad \text{(radius } OA = OB\text{)}$$

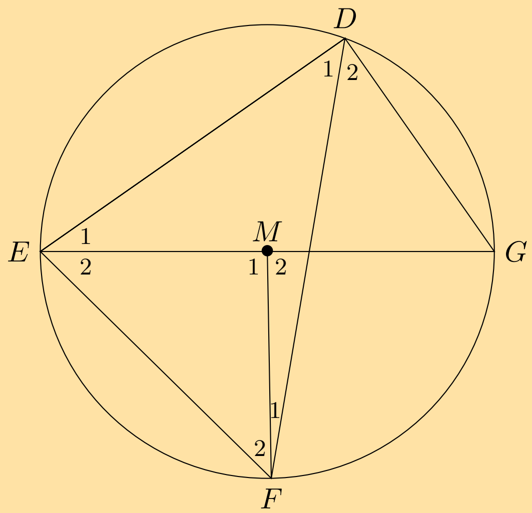
b)

$$\begin{aligned} C\hat{O}B &= 2x && \text{(angles at centre and on circumference)} \\ &= C\hat{O}D && \text{(given)} \\ &= 2x \end{aligned}$$

c)

$$\begin{aligned} \hat{E} &= \tfrac{1}{2}C\hat{O}D \\ &= x && \text{(}\angle\text{'s at centre and circumference)} \\ \therefore \hat{D} &= 90° - x && \text{(sum } \angle s\triangle = 180°\text{)} \end{aligned}$$

2.



D, E, F and G are points on circle with centre M. $\hat{F}_1 = 7°$ and $\hat{D}_2 = 51°$. Determine the sizes of the following angles, stating reasons:

a) $\hat{M}_1$

b) $\hat{D}_1$

c) $\hat{F}_2$

d) $\hat{G}$

e) $\hat{E}_1$

Solution:

a)

$$\begin{aligned} \hat{M}_2 &= 2 \times 51° && \text{(}\angle\text{ at centre } = 2 \times \text{circumference)} \\ \therefore \hat{M}_2 &= 102° \\ \hat{M}_1 &= 180° - 102° && \text{(}\angle\text{ on str.line)} \\ \therefore \hat{M}_1 &= 78° \end{aligned}$$

b)

$$\hat{D}_1 = \frac{1}{2} \times 78° \qquad (\angle \text{ at centre } = 2 \times \text{ circumference})$$
$$\therefore \hat{D}_1 = 39°$$

c)

$$\hat{F}_2 + \hat{E}_2 = 102° \qquad (\text{exterior } \angle\triangle)$$
$$\hat{F}_2 = \hat{E}_2 \qquad (ME = MF)$$
$$= 51°$$

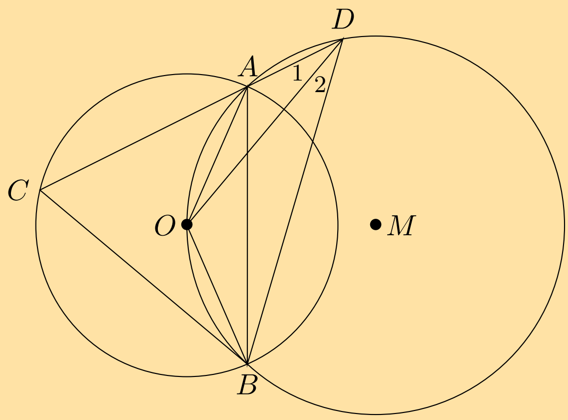
d)

$$\hat{G} = E\hat{F}D \qquad (\angle\text{s on same chord})$$
$$= 58°$$

e)

$$\hat{E}_1 = 180° - 90° - \hat{G}$$
$$= 180° - 90° - 58°$$
$$\therefore \hat{E}_1 = 32°$$

3.



O is a point on the circle with centre M. O is also the centre of a second circle. DA cuts the smaller circle at C and $\hat{D}_1 = x$. Express the following angles in terms of x, stating reasons:

a) $\hat{D}_2$
b) $O\hat{A}B$
c) $O\hat{B}A$
d) $A\hat{O}B$
e) $\hat{C}$

Solution:

a)

$$\hat{D}_2 = \hat{D}_1 \qquad (\angle\text{s on same chord} OA = OB)$$
$$= x$$

b)

$$O\hat{A}B = x \qquad (\angle\text{'s on same chord})$$

c)

$$O\hat{B}A = x \qquad (\text{equal radii} OA = OB)$$

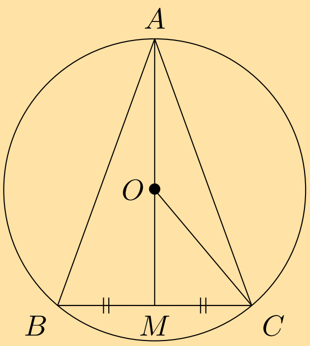
d)

$$A\hat{O}B = 180° - 2x \qquad (\text{sum } \angle\text{s } \triangle = 180°)$$

e)

$$\hat{C} = 90° - x \qquad (\angle\text{'s on same chord})$$

4.



O is the centre of the circle with radius 5 cm and chord $BC = 8$ cm. Calculate the lengths of:

a) OM

b) AM

c) AB

Solution:

a)

$$\begin{aligned} \text{In } OMC, \quad OC^2 &= OM^2 + MC^2 \qquad (Pythagoras) \\ 5^2 &= OM^2 + 4^2 \\ \therefore OM &= 3 \text{ cm} \end{aligned}$$

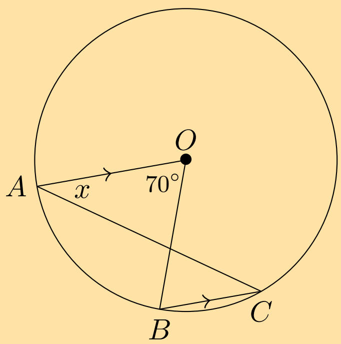
b)

$$\begin{aligned} AM &= 5 + 3 \\ \therefore AM &= 8 \text{ cm} \end{aligned}$$

c)

$$\begin{aligned} \text{In } ABM, \quad AB^2 &= BM^2 + AM^2 \qquad (Pythagoras) \\ AB^2 &= 8^2 + 4^2 \\ AB &= \sqrt{80} \\ \therefore AB &= 4\sqrt{5}\text{cm} \end{aligned}$$

5.

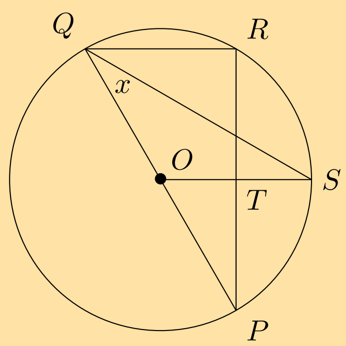


$AO \parallel CB$ in circle with centre O. $A\hat{O}B = 70°$ and $O\hat{A}C = x$. Calculate the value of x, giving reasons.

Solution:

$$\begin{aligned} \hat{C} &= \tfrac{1}{2} A\hat{O}B \qquad (\angle\text{s at centre = twice circumference}) \\ &= 35° \\ \therefore x &= 35° \qquad (\text{alt. } \angle\text{'s, } AO \parallel BC) \end{aligned}$$

6.



PQ is a diameter of the circle with centre O. SQ bisects $P\hat{Q}R$ and $P\hat{Q}S = x$.

a) Write down two other angles that are also equal to x.

b) Calculate $P\hat{O}S$ in terms of x, giving reasons.

c) Prove that OS is a perpendicular bisector of PR.

Solution:

a)

$$R\hat{Q}S = x$$
$$Q\hat{S}O = x$$

b)

$$P\hat{O}S = 2 \times P\hat{Q}S \qquad (\angle\text{'s at centre and circumference on same chord})$$
$$= 2x$$

c)

$x_1 = x_2$ (proven)

$\therefore QR \parallel OS$

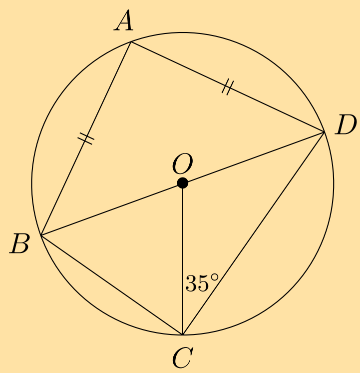
$\therefore \hat{R} = R\hat{T}S$ (alt. $\angle$'s, $QR \parallel OS$)

$= 90°$ ($\hat{R} = \angle$ in semi-circle)

$\therefore PT = TR$

$\therefore OS$ is perpendicular bisector of PR

7.



$B\hat{O}D$ is a diameter of the circle with centre O. $AB = AD$ and $O\hat{C}D = 35°$. Calculate the value of the following angles, giving reasons:

a) $O\hat{D}C$

b) $C\hat{O}D$

c) $C\hat{B}D$

d) $B\hat{A}D$

e) $A\hat{D}B$

Solution:

a)

$$O\hat{D}C = 35° \qquad \text{(radii } OC = OD\text{)}$$

b)

$$\begin{aligned} C\hat{O}D &= 180° - 70° \qquad (\text{sum}\angle\text{'s } \triangle = 180°) \\ &= 110° \end{aligned}$$

c)

$$\begin{aligned} C\hat{B}D &= \tfrac{1}{2} C\hat{O}D \qquad (\angle\text{'s on } CD) \\ &= 55° \end{aligned}$$

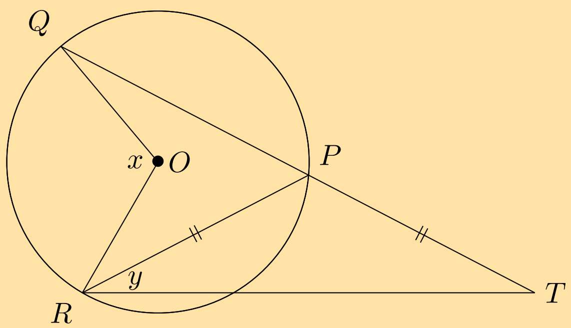
d)

$$B\hat{A}D = 90° \qquad (\angle \text{ in semi-circle})$$

e)

$$\begin{aligned} A\hat{D}B &= \tfrac{180° - 90°}{2} \qquad (\text{sum } \angle\text{'s in } \triangle = 180°) \\ &= 45° \end{aligned}$$

8.

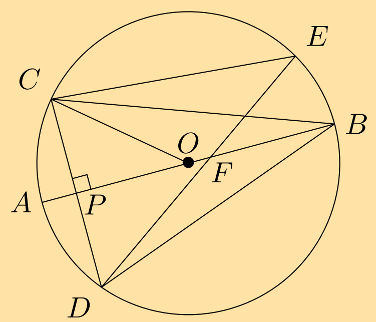


QP in the circle with centre O is protracted to T so that $PR = PT$. Express y in terms of x.

Solution:

$$\begin{aligned} T &= y \qquad (PT = PR) \\ \therefore Q\hat{P}R &= 2y \qquad (\text{ext. } \angle\triangle = \text{ sum interior } \angle\text{'s}) \\ \therefore x &= 2(2y) \qquad (\angle\text{'s at centre and circumference on } QR) \\ x &= 4y \end{aligned}$$

9.



O is the centre of the circle with diameter AB. $CD \perp AB$ at P and chord DE cuts AB at F. Prove that:

a) $C\hat{B}P = D\hat{P}B$

b) $C\hat{E}D = 2C\hat{B}A$

c) $A\hat{B}D = \frac{1}{2}C\hat{O}A$

Solution:

a)

In $\triangle CBP$ and $\triangle DBP$:

$CP = DP$	$(OP \perp CD)$
$C\hat{P}B = D\hat{P}B = 90^\circ$	(given)
$BP = BP$	(common)
$\therefore \triangle CBP \equiv DBP$	(SAS)

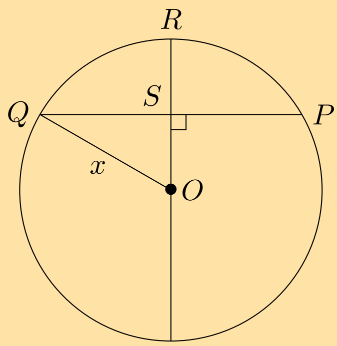
b)

$C\hat{E}D = C\hat{B}D$	($\angle$'s on chord CD)
But $C\hat{B}A = D\hat{B}A$	$(\triangle CBP \equiv \triangle DBP)$
$\therefore C\hat{E}D = 2C\hat{B}A$	

c)

$D\hat{B}A = C\hat{B}A$	$(\triangle CBP \equiv \triangle DBP)$
$C\hat{B}A = \frac{1}{2}C\hat{O}A$	($\angle$'s at centre and circumference on arc AC)
$\therefore A\hat{B}D = \frac{1}{2}C\hat{O}A$	

10.



In the circle with centre O, $OR \perp QP$, $PQ = 30$ mm and $RS = 9$ mm. Determine the length of OQ.

Solution:

In $\triangle QOS$,

$QO^2 = OS^2 + QS^2$ (Pythagoras)

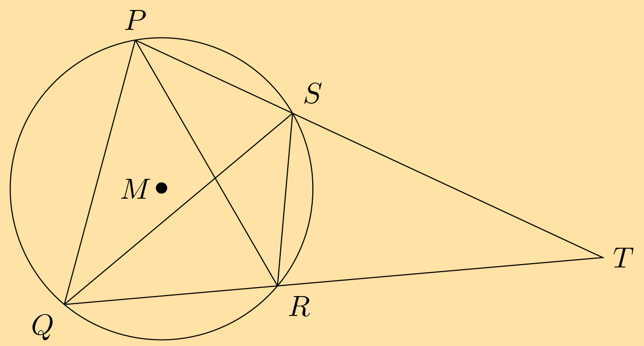
$x^2 = (x-9)^2 + 15^2$

$x^2 = x^2 - 18x + 81 + 225$

$\therefore 18x = 306$

$\therefore x = 17$

11.



P, Q, R and S are points on the circle with centre M. PS and QR are extended and meet at T. $PQ = PR$ and $P\hat{Q}R = 70°$.

a) Determine, stating reasons, three more angles equal to $70°$.

b) If $Q\hat{P}S = 80°$, calculate $S\hat{R}T$, $S\hat{T}R$ and $P\hat{Q}S$.

c) Explain why PQ is a tangent to the circle QST at point Q.

d) Determine $P\hat{M}Q$.

Solution:

a) $Q\hat{R}P, Q\hat{S}P, R\hat{S}T$

b)

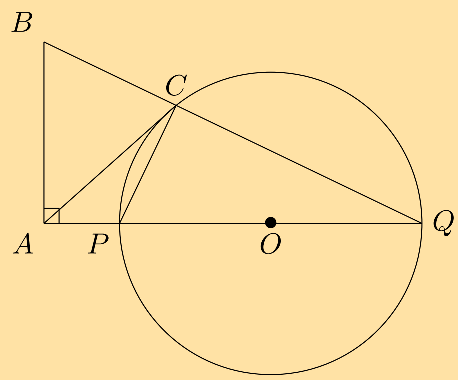
$$\begin{aligned} &\text{In } \triangle QOS, && \\ &S\hat{R}T = 80° && \text{(ext. angle cyclic quad.)} \\ &S\hat{T}R = 180° - 80° - 30° && (\angle\text{s sum of } \triangle) \\ &\therefore S\hat{T}R = 30° && \\ In\triangle PQS, \quad &P\hat{Q}S = 180° - 80° - 30° && (\angle\text{s sum of } \triangle) \\ &\therefore P\hat{Q}S = 30° && \end{aligned}$$

c) $P\hat{Q}S = Q\hat{T}S = 30°$, therefore PQ is a tangent to the circle through QST, (angle between line and chord equals angle in alt. seg.)

d)

$$\begin{aligned} In\triangle PMQ, \quad &P\hat{M}Q = 180° - 30° - 40° && (\angle\text{s sum of } \triangle) \\ &\therefore P\hat{M}Q = 110° && \end{aligned}$$

12.



POQ is a diameter of the circle with centre O. QP is protruded to A and AC is a tangent to the circle. $BA \perp AQ$ and BCQ is a straight line. Prove:

a) $P\hat{C}Q = B\hat{A}P$

b) $BAPC$ is a cyclic quadrilateral

c) $AB = AC$

Solution:

a)

$$\begin{aligned} P\hat{C}Q &= 90° && (\angle \text{ in semi-circle }) \\ B\hat{A}Q &= 90° && (\text{ given } BA \perp AQ) \\ \therefore P\hat{C}Q &= B\hat{A}Q \end{aligned}$$

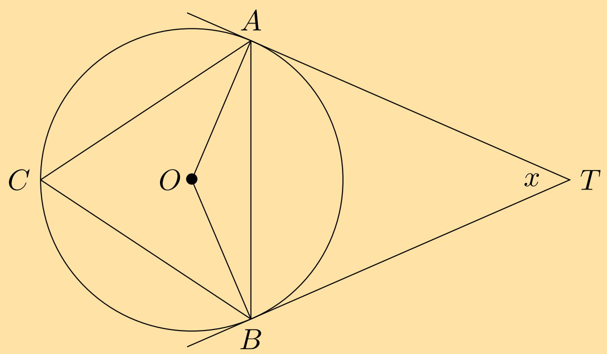
b)

$$\begin{aligned} P\hat{C}Q &= B\hat{A}Q && (\text{ proven }) \\ \therefore BAPC & \text{ is a cyclic quad.} && (\text{ ext. angle = int. opp. angle }) \end{aligned}$$

c)

$$\begin{aligned} C\hat{P}Q &= A\hat{B}C && (\text{ ext. angle of cyclic quad. }) \\ B\hat{C}P &= C\hat{P}Q + C\hat{Q}P && (\text{ ext. angle of } \triangle) \\ A\hat{C}P &= C\hat{Q}P && (\text{ tangent-chord }) \\ \therefore B\hat{C}A &= C\hat{P}Q \\ \therefore &= A\hat{B}C \\ \therefore AB &= AC && \text{angles opp. equal sides} \end{aligned}$$

13.



TA and TB are tangents to the circle with centre O. C is a point on the circumference and $A\hat{T}B = x$. Express the following in terms of x, giving reasons:

a) $A\hat{B}T$

b) $O\hat{B}A$

c) $\hat{C}$

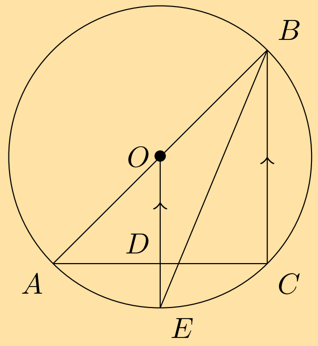
Solution:

a) $A\hat{B}T = 90° - \frac{x}{2}$

b) $O\hat{B}A = \frac{x}{2}$

c) $\hat{C} = 90° - \frac{x}{2}$

14.



AOB is a diameter of the circle $AECB$ with centre O. $OE \parallel BC$ and cuts AC at D.

a) Prove $AD = DC$

b) Show that $A\hat{B}C$ is bisected by EB

c) If $O\hat{E}B = x$, express $B\hat{A}C$ in terms of x

d) Calculate the radius of the circle if $AC = 10$ cm and $DE = 1$ cm

Solution:

a)

$$\begin{aligned} BC &\parallel OE && \text{(given)} \\ A\hat{C}B &= 90^\circ && (\angle \text{ in semi-circle)} \\ \therefore O\hat{D}C &= 90^\circ && \text{(corresp. } \angle, BC \parallel OE) \\ \therefore AD &= DC && \text{(perp. from centre to mid-point.)} \end{aligned}$$

b)

$$\begin{aligned} O\hat{E}B &= E\hat{B}C && \text{(alt. } \angle, OE \parallel BC) \\ OE &= OB && (\angle \text{ equal radii)} \\ \therefore O\hat{E}B &= O\hat{B}E && (\angle \text{ opp. equal sides)} \\ \therefore O\hat{B}E &= E\hat{B}C \\ \therefore A\hat{B}C & \text{ is bisected} \end{aligned}$$

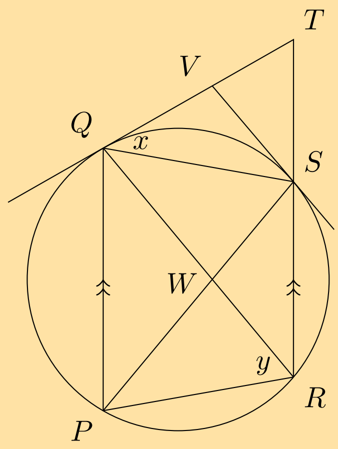
c)

$$\begin{aligned} \text{In } & \triangle BAC, \\ B\hat{A}C &= 180^\circ - 90^\circ - 2x && (\angle \text{ sum of } \triangle) \\ \therefore B\hat{A}C &= 90^\circ - 2x \end{aligned}$$

d)

$$\begin{aligned} &\text{In } \triangle AOD, \\ &\text{Let } AO = r \\ &AO^2 = 5^2 + (r-1)^2 && \text{(Pythagoras)} \\ &r^2 = 25 + r^2 - 2r + 1 \\ &2r = 26 \\ &\therefore r = 13 \text{ cm} \end{aligned}$$

15.



PQ and RS are chords of the circle and $PQ \parallel RS$. The tangent to the circle at Q meets RS protruded at T. The tangent at S meets QT at V. QS and PR are drawn.

Let $T\hat{Q}S = x$ and $Q\hat{R}P = y$. Prove that:

a) $T\hat{V}S = 2Q\hat{R}S$
b) $QVSW$ is a cyclic quadrilateral
c) $Q\hat{P}S + \hat{T} = P\hat{R}T$
d) W is the centre of the circle

Solution:

a)

$$\begin{aligned} \text{In } &\triangle VQS, \\ VQ &= VS && (\text{ tangents from same pt. }) \\ \therefore V\hat{S}Q &= V\hat{Q}S = x && (\angle\text{s opp. equal sides }) \\ \therefore T\hat{V}S &= x + x && (\text{ext. angle of } \triangle) \\ &= 2x \\ \text{And } Q\hat{R}S &= Q\hat{P}S = x && (\angle \text{ same seg. }) \\ \therefore T\hat{V}S &= 2Q\hat{R}S \end{aligned}$$

b)

$$\begin{aligned} T\hat{V}S &= 2Q\hat{R}S && (\text{ proven }) \\ V\hat{Q}S &= Q\hat{P}S && (\text{ tangent-chord }) \\ Q\hat{P}S &= P\hat{S}R = x && (\text{ alt. } \angle, PQ \parallel RT) \\ \text{And } Q\hat{W}S &= 2Q\hat{R}S && (\angle \text{ ext. angle of } \triangle WSR) \\ \therefore T\hat{V}S &= Q\hat{W}S && = 2x \\ \therefore QVSW &= \text{ is cyclic quad.} && \text{ext. angle } = \text{ int. opp. angle} \end{aligned}$$

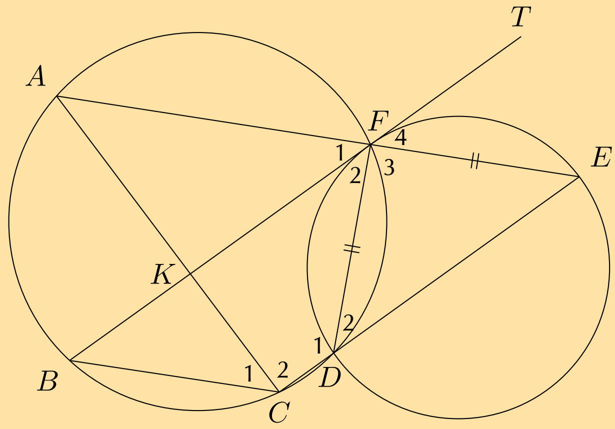
c)

$$\begin{aligned} \hat{Q} &= Q\hat{R}P = y && (\text{ tangent-chord }) \\ \hat{Q} &= \hat{T} = y && (\text{ alt. } \angle, PQ \parallel RT) \\ P\hat{R}T &= x + y \\ \text{And } Q\hat{P}S &= x && (\angle \text{ proven }) \\ \therefore Q\hat{P}S + \hat{T} &= x + y && = P\hat{R}T \end{aligned}$$

d)

$$Q\hat{R}S = x \quad \text{(proven)}$$
$$Q\hat{W}S = 2x \quad \text{(proven)}$$
$$\therefore W = \text{ is circle centre.} \quad (\angle \text{ centre } = 2 \text{ angle at circum.}$$

16.



The two circles shown intersect at points F and D. BFT is a tangent to the smaller circle at F. Straight line AFE is drawn such that $DF = EF$. CDE is a straight line and chord AC and BF cut at K. Prove that:

a) $BT \parallel CE$

b) $BCEF$ is a parallelogram

c) $AC = BF$

Solution:

a)

$$\hat{E} = \hat{D}_2 \quad (\angle\text{s opp. equal sides)}$$
$$\hat{F}_4 = \hat{F}_1 \quad \text{(vert. opp angles)}$$
$$\text{and } \hat{F}_4 = \hat{D}_2 \quad \text{(tangent-chord)}$$
$$\therefore \hat{F}_1 = \hat{D}_2$$
$$\therefore BT \parallel CE \quad (\angle \text{ corresp. angles}$$

b)

$$\therefore BT \parallel CE \quad \text{(proven)}$$
$$\therefore \hat{F}_1 = \hat{C}_1 \quad \text{(angles same seg.)}$$
$$\therefore AE \parallel BC \quad (\angle \text{ alt. angles}$$
$$\therefore BCEF \text{ is parallelogram} \quad (\angle \text{ both opp. sides } \parallel$$

c)

$$\hat{F}_1 = \hat{C}_1 \quad \text{(proven)}$$
$$\hat{F}_1 = \hat{E} \quad \text{(proven)}$$
$$\hat{A} = \hat{C}_1 \quad \text{(alt. angles , } AE \parallel BC\text{)}$$
$$\therefore \hat{A} = \hat{E}$$
$$\therefore AC = CE \quad (\angle \text{ angles opp. equal sides}$$
$$BF = CE \quad (\angle \text{ opp. sides parm. } =$$
$$\therefore AC = BF$$

CHAPTER 9

Finance, growth and decay

- Discuss terminology.
- Very important to emphasize not rounding off calculations until final answer.
- Learners should do calculation in one step using the memory function on their calculators.
- Draw timelines showing the different time periods, interest rates and any deposits/withdrawals.
- Discuss real-life financial issues; savings, budgets, tax, retirement etc.

9.1 Revision

Exercise 9 – 1: Revision

1. Determine the value of an investment of R 10 000 at $12{,}1\%$ p.a. simple interest for 3 years.

   Solution:

$$\begin{aligned} A &= P(1 + in) \\ &= 10\ 000(1 + 0{,}121 \times 3) \\ &= \text{R } 13\ 630 \end{aligned}$$

2. Calculate the value of R 8000 invested at $8{,}6\%$ p.a. compound interest for 4 years.

   Solution:

$$\begin{aligned} A &= P(1 + i)^n \\ &= 8000(1 + 0{,}086)^3 \\ &= \text{R } 10\ 246{,}59 \end{aligned}$$

3. Calculate how much interest John will earn if he invests R 2000 for 4 years at:

   a) $6{,}7\%$ p.a. simple interest
   b) $5{,}4\%$ p.a. compound interest

   Solution:

   a)

$$\begin{aligned} A &= P(1 + in) \\ &= 2000(1 + 0{,}067 \times 4) \\ &= \text{R } 2536 \end{aligned}$$

b)

$$\begin{aligned} A &= P(1+i)^n \\ &= 2000(1+0{,}054)^4 \\ &= \text{R } 2468{,}27 \end{aligned}$$

4. The value of an investment grows from R 2200 to R 3850 in 8 years. Determine the simple interest rate at which it was invested.

Solution:

$$\begin{aligned} A &= P(1+in) \\ 3850 &= 2200(1+8i) \\ \frac{3850}{2200} - 1 &= 8i \\ \therefore \frac{1}{8}\left(\frac{3850}{2200} - 1\right) &= i \\ \therefore i &= 9{,}38\% \end{aligned}$$

5. James had R 12 000 and invested it for 5 years. If the value of his investment is R 15 600, what compound interest rate did it earn?

Solution:

$$\begin{aligned} A &= P(1+i)^n \\ 15\ 000 &= 12\ 000(1+i)^5 \\ \frac{15\ 000}{12\ 000} &= (1+i)^5 \\ \sqrt[5]{\frac{15\ 000}{12\ 000}} - 1 &= i \\ \therefore i &= 4{,}56\% \end{aligned}$$

9.2 Simple and compound depreciation

Simple depreciation

Exercise 9 – 2: Simple decay

1. A business buys a truck for R 560 000. Over a period of 10 years the value of the truck depreciates to R 0 using the straight-line method. What is the value of the truck after 8 years?

Solution:

$$\begin{aligned}
\text{Depreciation} &= \frac{560\ 000}{10} \\
&= \text{R } 56\ 000 \text{ per year} \\
\text{For } n &= 8 \\
A &= 560\ 000 - 8(56\ 000) \\
&= \text{R } 112\ 000
\end{aligned}$$

2. Harry wants to buy his grandpa's donkey for R 800. His grandpa is quite pleased with the offer, seeing that it only depreciated at a rate of 3% per year using the straight-line method. Grandpa bought the donkey 5 years ago. What did grandpa pay for the donkey then?

   Solution:

$$\begin{aligned}
A &= P(1 - in) \\
800 &= P(1 - (0{,}03 \times 5)) \\
\therefore \frac{800}{0{,}85} &= P \\
\therefore P &= \text{R } 941{,}18
\end{aligned}$$

3. Seven years ago, Rocco's drum kit cost him R 12 500. It has now been valued at R 2300. What rate of simple depreciation does this represent?

   Solution:

$$\begin{aligned}
A &= P(1 - in) \\
2300 &= 12\ 500(1 - (i \times 7)) \\
\therefore \frac{2300}{12\ 500} &= 1 - 7i \\
0{,}184 - 1 &= -7i \\
\frac{-0{,}816}{-7} &= i \\
\therefore i &= 11{,}66\%
\end{aligned}$$

4. Fiona buys a DStv satellite dish for R 3000. Due to weathering, its value depreciates simply at 15% per annum. After how long will the satellite dish have a book value of zero?

   Solution:

$$\begin{aligned}
\text{Depreciation} &= 3000 \times \frac{15}{100} \\
&= \text{R } 450 \text{ per year} \\
\therefore n &= \frac{3000}{450} \\
&= 6{,}666\ldots \\
\therefore n &= 7 \text{ years}
\end{aligned}$$

   Or

$$
\begin{aligned}
A &= P(1 - in) \\
0 &= 3000(1 - 0{,}15 \times n) \\
\therefore 0 &= 1 - 0{,}15n \\
0{,}15n &= 1 \\
n &= \frac{1}{0{,}15} \\
&= 6{,}666\ldots \\
\therefore n &= 7 \text{ years}
\end{aligned}
$$

Compound depreciation

Exercise 9 – 3: Compound depreciation

1. Jwayelani buys a truck for R 89 000 and depreciates it by 9% p.a. using the compound depreciation method. What is the value of the truck after 14 years?

   Solution:

$$
\begin{aligned}
A &= P(1 - i)^n \\
&= 89\ 000(1 - 0{,}09)^{14} \\
&= 89\ 000(0{,}91)^{14} \\
\therefore A &= \text{R } 23\ 766{,}73
\end{aligned}
$$

2. The number of cormorants at the Amanzimtoti river mouth is decreasing at a compound rate of 8% p.a. If there are now 10 000 cormorants, how many will there be in 18 years' time?

   Solution:

$$
\begin{aligned}
A &= P(1 - i)^n \\
&= 10\ 000(1 - 0{,}08)^{18} \\
&= 10\ 000(0{,}92)^{18} \\
&= 2229{,}36\ldots \\
\therefore A &= 2229 \text{ cormorants}
\end{aligned}
$$

3. On January 1, 2008 the value of my Kia Sorento is R 320 000. Each year after that, the car's value will decrease 20% of the previous year's value. What is the value of the car on January 1, 2012?

   Solution:

$$
\begin{aligned}
A &= P(1-i)^n \\
&= 320\ 000(1-0{,}2)^4 \\
&= 320\ 000(0{,}8)^4 \\
\therefore A &= \text{R } 131\ 072
\end{aligned}
$$

4. The population of Bonduel decreases at a reducing-balance rate of 9,5% per annum as people migrate to the cities. Calculate the decrease in population over a period of 5 years if the initial population was 2 178 000.

   Solution:

$$
\begin{aligned}
A &= P(1-i)^n \\
&= 2\ 178\ 000(1-0{,}095)^5 \\
&= 2\ 178\ 000(0{,}905)^5 \\
\therefore A &= 132\ 221
\end{aligned}
$$

5. A 20 kg watermelon consists of 98% water. If it is left outside in the sun it loses 3% of its water each day. How much does it weigh after a month of 31 days?

   Solution:

$$
\begin{aligned}
A &= P(1-i)^n \\
A &= \left(20 \times \frac{98}{100}\right)(1-0{,}03)^{31} \\
&= 19{,}6(0{,}97)^{31} \\
\therefore A &= 7{,}62 \text{ kg}
\end{aligned}
$$

6. Richard bought a car 15 years ago and it depreciated by 17% p.a. on a compound depreciation basis. How much did he pay for the car if it is now worth R 5256?

   Solution:

$$
\begin{aligned}
A &= P(1-i)^n \\
5256 &= P(1-0{,}17)^{15} \\
\frac{5256}{0{,}83^{15}} &= P \\
\therefore P &= \text{R } 85\ 997{,}13
\end{aligned}
$$

Exercise 9 – 4: Finding i

1. A machine costs R 45 000 and has a scrap value of R 9000 after 10 years. Determine the annual rate of depreciation if it is calculated on the reducing balance method.

   Solution:

$$\begin{aligned} A &= P(1-i)^n \\ 9000 &= 45\ 000(1-i)^{10} \\ \frac{9000}{45\ 000} &= (1-i)^{10} \\ \sqrt[10]{\frac{9000}{45\ 000}} - 1 &= -i \\ \therefore i &= 14{,}9\% \end{aligned}$$

2. After 15 years, an aeroplane is worth $\frac{1}{6}$ of its original value. At what annual rate was depreciation compounded?

   Solution:

$$\begin{aligned} A &= P(1-i)^n \\ \frac{1}{6}P &= P(1-i)^{15} \\ \frac{1}{6} &= (1-i)^{15} \\ \sqrt[10]{\frac{1}{6}} - 1 &= -i \\ \therefore i &= 16{,}4\% \end{aligned}$$

3. Mr. Mabula buys furniture for R 20 000. After 6 years he sells the furniture for R 9300. Calculate the annual compound rate of depreciation of the furniture.

   Solution:

$$\begin{aligned} A &= P(1-i)^n \\ 9300 &= 20\ 000(1-i)^{6} \\ \frac{9300}{20\ 000} &= (1-i)^{6} \\ \sqrt[6]{\frac{9300}{20\ 000}} - 1 &= -i \\ \therefore i &= 12{,}0\% \end{aligned}$$

4. Ayanda bought a new car 7 years ago for double what it is worth today. At what yearly compound rate did her car depreciate?

Solution:

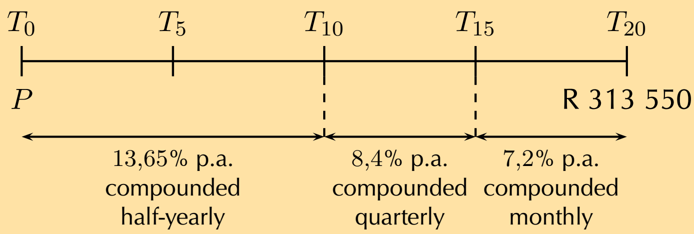
$$
\begin{aligned}
A &= P(1-i)^n \\
\frac{1}{2}P &= P(1-i)^7 \\
\frac{1}{2} &= (1-i)^7 \\
\frac{1}{2} &= (1-i)^7 \\
\sqrt[7]{\frac{1}{2}} - 1 &= -i \\
\therefore i &= 9{,}4\%
\end{aligned}
$$

9.3 Timelines

Exercise 9 – 5: Timelines

1. After a 20-year period Josh's lump sum investment matures to an amount of R 313 550. How much did he invest if his money earned interest at a rate of 13,65% p.a. compounded half yearly for the first 10 years, 8,4% p.a. compounded quarterly for the next five years and 7,2% p.a. compounded monthly for the remaining period?

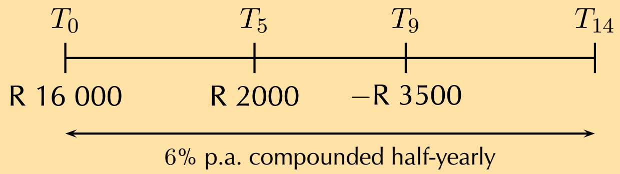
   Solution:

$$
\begin{aligned}
A &= P(1+i)^n \\
313\ 550 &= P\left(1+\frac{0{,}1365}{2}\right)^{10\times 2}\left(1+\frac{0{,}084}{4}\right)^{5\times 4}\left(1+\frac{0{,}072}{12}\right)^{5\times 12} \\
&= P\,(1{,}06825)^{20}\,(1{,}021)^{20}\,(1{,}006)^{60} \\
&= \text{R } 38\ 588{,}25
\end{aligned}
$$

2. Sindisiwe wants to buy a motorcycle. The cost of the motorcycle is R 55 000. In 1998 Sindisiwe opened an account at Sutherland Bank with R 16 000. Then in 2003 she added R 2000 more into the account. In 2007 Sindisiwe made another change: she took R 3500 from the account. If the account pays 6% p.a. compounded half-yearly, will Sindisiwe have enough money in the account at the end of 2012 to buy the motorcycle?

Solution:



$$\begin{aligned} A &= P(1+i)^n \\ &= 16\ 000\left(1+\frac{0{,}06}{2}\right)^{14\times 2} + 2000\left(1+\frac{0{,}06}{2}\right)^{9\times 2} - 3500\left(1+\frac{0{,}06}{2}\right)^{5\times 2} \\ &= 16\ 000\,(1{,}03)^{28} + 2000\,(1{,}03)^{18} - 3500\,(1{,}03)^{10} \\ &= \text{R } 35\ 308{,}00 \end{aligned}$$

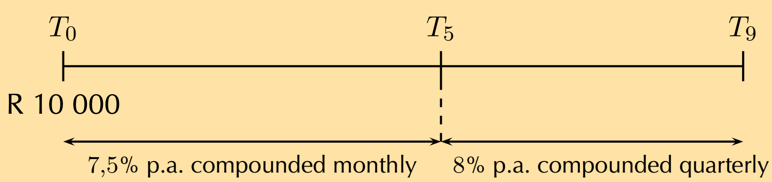
3. A loan has to be returned in two equal semi-annual instalments. If the rate of interest is 16% per annum, compounded semi-annually and each instalment is R 1458, find the sum borrowed.

   Solution:

$$\begin{aligned} A &= P(1+i)^n \\ &= P_1(1+\frac{0{,}16}{2})^1 \\ &= P_1(1{,}08) \\ \text{Therefore } P_1(1{,}08) &= 1458 + P_2 \\ \text{and } P_2(1{,}08) &= 1458 \\ \therefore P_2 &= \frac{1458}{1{,}08} \\ \therefore P_1(1{,}08) &= 1458 + P_2 \\ \therefore P_1 &= \frac{1458}{1{,}08} + \frac{1458}{(1{,}08)^2} \\ &= \text{R } 2600 \end{aligned}$$

4. A man named Phillip invests R 10 000 into an account at North Bank at an interest rate of $7{,}5\%$ p.a. compounded monthly. After 5 years the bank changes the interest rate to 8% p.a. compounded quarterly. How much money will Phillip have in his account 9 years after the original deposit?

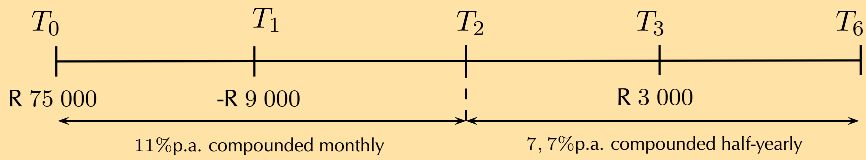
   Solution:



$$
\begin{aligned}
A &= P(1+i)^n \\
&= 10\ 000\left(1+\frac{0{,}075}{12}\right)^{5\times 12}\left(1+\frac{0{,}08}{4}\right)^{4\times 4} \\
&= 10\ 000\,(1{,}00625)^{60}\,(1{,}02)^{16} \\
&= \text{R } 19\ 950{,}62
\end{aligned}
$$

5. R 75 000 is invested in an account which offers interest at 11% p.a. compounded monthly for the first 24 months. Then the interest rate changes to $7{,}7\%$ p.a. compounded half-yearly. If R 9000 is withdrawn from the account after one year and then a deposit of R 3000 is made three years after the initial investment, how much will be in the account at the end of 6 years?

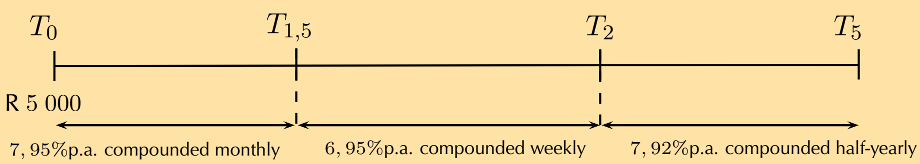
Solution:



$$
\begin{aligned}
A &= P(1+i)^n \\
&= 75\ 000\left(1+\frac{0{,}11}{12}\right)^{2\times 12}\left(1+\frac{0{,}077}{2}\right)^{4\times 2} \\
&\quad - 9000\left(1+\frac{0{,}11}{12}\right)^{1\times 12}\left(1+\frac{0{,}077}{2}\right)^{4\times 2} + 3000\left(1+\frac{0{,}077}{2}\right)^{3\times 2} \\
&= \text{R } 1\ 149\ 283{,}50
\end{aligned}
$$

6. Christopher wants to buy a computer, but right now he doesn't have enough money. A friend told Christopher that in 5 years the computer will cost R 9150. He decides to start saving money today at Durban United Bank. Christopher deposits R 5000 into a savings account with an interest rate of $7{,}95\%$ p.a. compounded monthly. Then after 18 months the bank changes the interest rate to $6{,}95\%$ p.a. compounded weekly. After another 6 months, the interest rate changes again to $7{,}92\%$ p.a. compounded two times per year. How much money will Christopher have in the account after 5 years, and will he then have enough money to buy the computer?

Solution:



$$
\begin{aligned}
A &= P(1+i)^n \\
&= 5000\left(1+\frac{0,0795}{12}\right)^{1,5\times 12}\left(1+\frac{0,0695}{52}\right)^{0,5\times 52}\left(1+\frac{0,0792}{2}\right)^{3\times 2} \\
&= 5000\left(1+\frac{0,0795}{12}\right)^{18}\left(1+\frac{0,0695}{52}\right)^{26}\left(1+\frac{0,0792}{2}\right)^{6} \\
&= \text{R } 7359,83
\end{aligned}
$$

9.4 Nominal and effective interest rates

Exercise 9 – 6: Nominal and effect interest rates

1. Determine the effective annual interest rate if the nominal interest rate is:
   a) 12% p.a. compounded quarterly.
   b) $14,5\%$ p.a. compounded weekly.
   c) 20% p.a. compounded daily.

Solution:

a)

$$
\begin{aligned}
1+i &= \left(1+\frac{i^{(m)}}{m}\right)^m \\
i &= \left(1+\frac{0,12}{4}\right)^4 - 1 \\
&= (1,03)^4 - 1 \\
&= 0,121255\ldots \\
\therefore i &= 12,6\%
\end{aligned}
$$

b)

$$
\begin{aligned}
1+i &= \left(1+\frac{i^{(m)}}{m}\right)^m \\
i &= \left(1+\frac{0,145}{12}\right)^{12} - 1 \\
&= 0,155035\ldots \\
\therefore i &= 15,5\%
\end{aligned}
$$

c)

$$
\begin{aligned}
1+i &= \left(1+\frac{i^{(m)}}{m}\right)^m \\
i &= \left(1+\frac{0,2}{365}\right)^{365} - 1 \\
&= 0,221335\ldots \\
\therefore i &= 22,1\%
\end{aligned}
$$

2. Consider the following:

   - 16,8% p.a. compounded annually.
   - 16,4% p.a. compounded monthly.
   - 16,5% p.a. compounded quarterly.

   a) Determine the effective annual interest rate of each of the nominal rates listed above.

   b) Which is the best interest rate for an investment?

   c) Which is the best interest rate for a loan?

   Solution:

   a)

$$\begin{aligned} 1 + i &= \left(1 + \frac{i^{(m)}}{m}\right)^{m} \\ i &= \left(1 + \frac{0{,}164}{12}\right)^{12} - 1 \\ &= 0{,}176906\ldots \\ \therefore i &= 17{,}7\% \end{aligned}$$

$$\begin{aligned} 1 + i &= \left(1 + \frac{i^{(m)}}{m}\right)^{m} \\ i &= \left(1 + \frac{0{,}165}{4}\right)^{4} - 1 \\ &= 0{,}175493\ldots \\ \therefore i &= 17{,}5\% \end{aligned}$$

   b) 17,7%

   c) 16,8%

3. Calculate the effective annual interest rate equivalent to a nominal interest rate of 8,75% p.a. compounded monthly.

   Solution:

$$\begin{aligned} 1 + i &= \left(1 + \frac{i^{(m)}}{m}\right)^{m} \\ i &= \left(1 + \frac{0{,}0875}{12}\right)^{12} - 1 \\ &= 0{,}091095\ldots \\ \therefore i &= 9{,}1\% \end{aligned}$$

4. Cebela is quoted a nominal interest rate of 9,15% per annum compounded every four months on her investment of R 85 000. Calculate the effective rate per annum.

Solution:

$$\begin{aligned}1+i &= \left(1+\frac{i^{(m)}}{m}\right)^{m}\\ i &= \left(1+\frac{0{,}0915}{3}\right)^{3}-1\\ &= 0{,}094319\ldots\\ \therefore i &= 9{,}4\%\end{aligned}$$

5. Determine which of the following would be the better agreement for paying back a student loan:

   a) $9{,}1\%$ p.a. compounded quarterly.
   b) 9% p.a. compounded monthly.
   c) $9{,}3\%$ p.a. compounded half-yearly.

Solution:

a)

$$\begin{aligned}1+i &= \left(1+\frac{i^{(m)}}{m}\right)^{m}\\ i &= \left(1+\frac{0{,}091}{4}\right)^{4}-1\\ &= 0{,}094152\ldots\\ \therefore i &= 9{,}42\%\end{aligned}$$

b)

$$\begin{aligned}1+i &= \left(1+\frac{i^{(m)}}{m}\right)^{m}\\ i &= \left(1+\frac{0{,}09}{12}\right)^{12}-1\\ &= 0{,}093806\ldots\\ \therefore i &= 9{,}38\%\end{aligned}$$

c)

$$\begin{aligned}1+i &= \left(1+\frac{i^{(m)}}{m}\right)^{m}\\ i &= \left(1+\frac{0{,}093}{2}\right)^{2}-1\\ &= 0{,}095162\ldots\\ \therefore i &= 9{,}52\%\end{aligned}$$

6. Miranda invests R 8000 for 5 years for her son's study fund. Determine how much money she will have at the end of the period and the effective annual interest rate if the nominal interest of 6% is compounded:

	Calculation	Accumulated amount	Effective annual interest rate
yearly			
half-yearly			
quarterly			
monthly			

Solution:

	Calculation	Accumulated amount	Effective annual interest rate
yearly	$8000\left(1+\frac{0{,}06}{1}\right)^{5}$	R 10 705,80	6%
half-yearly	$8000\left(1+\frac{0{,}06}{2}\right)^{10}$	R 10 751,33	$\left(1+\frac{0{,}06}{2}\right)^{2}-1=6{,}09\%$
quarterly	$8000\left(1+\frac{0{,}06}{4}\right)^{20}$	R 10 774,84	$\left(1+\frac{0{,}06}{4}\right)^{4}-1=6{,}14\%$
monthly	$8000\left(1+\frac{0{,}06}{12}\right)^{60}$	R 10 790,80	$\left(1+\frac{0{,}06}{12}\right)^{12}-1=6{,}17\%$

9.5 Summary

Exercise 9 – 7: End of chapter exercises

1. Thabang buys a Mercedes worth R 385 000 in 2007. What will the value of the Mercedes be at the end of 2013 if:

   a) the car depreciates at 6% p.a. straight-line depreciation.

   b) the car depreciates at 6% p.a. reducing-balance depreciation.

   Solution:

   a)

$$\begin{aligned} A &= P(1-in) \\ &= 385\ 000(1-0{,}06\times 6) \\ &= 385\ 000(0{,}64) \\ \therefore i &= \text{R } 246\ 400 \end{aligned}$$

   b)

$$\begin{aligned} A &= P(1-in) \\ &= 385\ 000(1-0{,}06)^{6} \\ &= 385\ 000(0{,}94)^{6} \\ \therefore i &= \text{R } 265\ 599{,}87 \end{aligned}$$

2. Greg enters into a 5-year hire-purchase agreement to buy a computer for R 8900. The interest rate is quoted as 11% per annum based on simple interest. Calculate the required monthly payment for this contract.

Solution:

$$\begin{aligned} A &= P(1 + in) \\ &= 8900(1 + 0{,}11 \times 5) \\ &= 8900(1{,}55) \\ &= \text{R } 13\ 795 \\ \therefore \text{ monthly repayment} &= \frac{13\ 795}{5 \times 12} \\ &= \text{R } 229{,}92 \end{aligned}$$

3. A computer is purchased for R 16 000. It depreciates at 15% per annum.

   a) Determine the book value of the computer after 3 years if depreciation is calculated according to the straight-line method.

   b) Find the rate according to the reducing-balance method that would yield, after 3 years, the same book value as calculated in the previous question.

Solution:

a)

$$\begin{aligned} A &= P(1 - in) \\ &= 16\ 000(1 - 0{,}15 \times 3) \\ &= 16\ 000(0{,}55) \\ &= \text{R } 8800 \end{aligned}$$

b)

$$\begin{aligned} A &= P(1 - i)^n \\ 8800 &= 16\ 000(1 - i)^3 \\ \frac{8800}{16\ 000} &= (1 - i)^3 \\ \sqrt[3]{\frac{8800}{16\ 000}} &= 1 - i \\ \sqrt[3]{\frac{8800}{16\ 000}} - 1 &= -i \\ \therefore i &= 0{,}180678\ldots \\ \therefore i &= 18{,}1\% \end{aligned}$$

4. Maggie invests R 12 500 for 5 years at 12% per annum compounded monthly for the first 2 years and 14% per annum compounded semi-annually for the next 3 years. How much will Maggie receive in total after 5 years?

Solution:

$$\begin{aligned} A &= P(1+i)^n \\ &= 125\ 000 \left(1+\frac{0{,}12}{12}\right)^{2\times 12} \left(1+\frac{0{,}14}{2}\right)^{3\times 2} \\ &= 125\ 000\,(1{,}01)^{24}\,(1{,}07)^6 \\ \therefore A &= \text{R } 238\ 191{,}17 \end{aligned}$$

5. Tintin invests R 120 000. He is quoted a nominal interest rate of 7,2% per annum compounded monthly.

   a) Calculate the effective rate per annum (correct to two decimal places).

   b) Use the effective rate to calculate the value of Tintin's investment if he invested the money for 3 years.

   c) Suppose Tintin invests his money for a total period of 4 years, but after 18 months makes a withdrawal of R 20 000, how much will he receive at the end of the 4 years?

   Solution:

   a)

$$\begin{aligned} 1+i &= \left(1+\frac{i^{(m)}}{m}\right)^m \\ i &= \left(1+\frac{0{,}072}{12}\right)^{12} - 1 \\ &= 0{,}074424\ldots \\ \therefore i &= 7{,}44\% \end{aligned}$$

   b)

$$\begin{aligned} A &= P(1+i)^n \\ &= 120\ 000\,(1+0{,}0744)^3 \\ &= 120\ 000\,(1{,}0744)^3 \\ \therefore A &= \text{R } 148\ 826{,}15 \end{aligned}$$

   c)

$$\begin{aligned} A &= P(1+i)^n \\ &= 120\ 000\,(1+0{,}0744)^4 - 20\ 000\,(1+0{,}0744)^{2{,}5} \\ &= 120\ 000\,(1{,}0744)^4 - 20\ 000\,(1{,}0744)^{2{,}5} \\ \therefore A &= \text{R } 135\ 968{,}69 \end{aligned}$$

6. Ntombi opens accounts at a number of clothing stores and spends freely. She gets herself into terrible debt and she cannot pay off her accounts. She owes Fashion World R 5000 and the shop agrees to let her pay the bill at a nominal interest rate of 24% compounded monthly.

   a) How much money will she owe Fashion World after two years?

   b) What is the effective rate of interest that Fashion World is charging her?

   Solution:

a)

$$\begin{aligned} A &= P(1+i)^n \\ &= 5000\left(1+\frac{0{,}24}{12}\right)^{2\times 12} \\ &= 5000\,(1{,}02)^{24} \\ \therefore A &= \text{R } 8042{,}19 \end{aligned}$$

b)

$$\begin{aligned} 1+i &= \left(1+\frac{i^{(m)}}{m}\right)^{m} \\ i &= \left(1+\frac{0{,}24}{12}\right)^{12}-1 \\ &= 0{,}268241\ldots \\ \therefore i &= 26{,}82\% \end{aligned}$$

7. John invests R 30 000 in the bank for a period of 18 months. Calculate how much money he will have at the end of the period and the effective annual interest rate if the nominal interest of 8% is compounded:

	Calculation	Accumulated amount	Effective annual interest rate
yearly			
half-yearly			
quarterly			
monthly			
daily			

Solution:

	Calculation	Accumulated amount	Effective annual interest rate
yearly	$30\ 000\,(1+0{,}08)^{1}$	R 33 671,07	
half-yearly	$30\ 000\left(1+\frac{0{,}08}{2}\right)^{1{,}5\times 2}$	R 33 745,92	$\left(1+\frac{0{,}08}{2}\right)^{2}-1 =$ 8,16%
quarterly	$30\ 000\left(1+\frac{0{,}08}{4}\right)^{1{,}5\times 4}$	R 33 784,87	$\left(1+\frac{0{,}08}{4}\right)^{4}-1 =$ 8,24%
monthly	$30\ 000\left(1+\frac{0{,}08}{12}\right)^{1{,}5\times 12}$	R 33 811,44	$\left(1+\frac{0{,}08}{12}\right)^{12}-1 =$ 8,30%
daily	$30\ 000\left(1+\frac{0{,}08}{365}\right)^{1{,}5\times 365}$	R 33 828,17	$\left(1+\frac{0{,}08}{365}\right)^{365}-1 =$ 8,33%

8. Convert an effective annual interest rate of 11,6% p.a. to a nominal interest rate compounded:

a) half-yearly

b) quarterly

c) monthly

Solution:

a)

$$\begin{aligned}1+i &= \left(1+\frac{i^{(m)}}{m}\right)^m\\ 1+0{,}116 &= \left(1+\frac{i^{(2)}}{2}\right)^2\\ \sqrt[2]{1{,}116}-1 &= \frac{i^{(2)}}{2}\\ 2\left(\sqrt[2]{1{,}116}-1\right) &= i^{(2)}\\ \therefore i^{(2)} &= 11{,}3\%\end{aligned}$$

b)

$$\begin{aligned}1+i &= \left(1+\frac{i^{(m)}}{m}\right)^m\\ 1+0{,}116 &= \left(1+\frac{i^{(4)}}{4}\right)^4\\ \sqrt[4]{1{,}116}-1 &= \frac{i^{(4)}}{4}\\ 4\left(\sqrt[4]{1{,}116}-1\right) &= i^{(4)}\\ \therefore i^{(4)} &= 11{,}1\%\end{aligned}$$

c)

$$\begin{aligned}1+i &= \left(1+\frac{i^{(m)}}{m}\right)^m\\ 1+0{,}116 &= \left(1+\frac{i^{(12)}}{12}\right)^{12}\\ \sqrt[12]{1{,}116}-1 &= \frac{i^{(12)}}{12}\\ 12\left(\sqrt[12]{1{,}116}-1\right) &= i^{(12)}\\ \therefore i^{(12)} &= 11{,}0\%\end{aligned}$$

9. Joseph must sell his plot on the West Coast and he needs to get R 300 000 on the sale of the land. If the estate agent charges him 7% commission on the selling price, what must the buyer pay for the plot?

Solution:

$$\text{Let the selling price} = k$$
$$300\ 000 + \frac{7}{100} \times k = k$$
$$300\ 000 = k - 0{,}07k$$
$$300\ 000 = 0{,}93k$$
$$\frac{300\ 000}{0{,}93} = k$$
$$\therefore k = \text{R } 322\ 580{,}65$$

10. Mrs. Brown retired and received a lump sum of R 200 000. She deposited the money in a fixed deposit savings account for 6 years. At the end of the 6 years the value of the investment was R 265 000. If the interest on her investment was compounded monthly, determine:

   a) the nominal interest rate per annum
   b) the effective annual interest rate

Solution:

a)

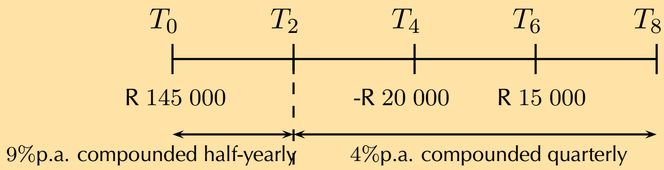
$$\text{Let the selling price} = k$$
$$265\ 000 = 200\ 000\left(1 + \frac{i}{12}\right)^{6\times 12}$$
$$\frac{265\ 000}{200\ 000} = \left(1 + \frac{i}{12}\right)^{72}$$
$$\sqrt[72]{\frac{265\ 000}{200\ 000}} - 1 = \frac{i}{12}$$
$$\therefore i = 12\left(\sqrt[72]{\frac{265\ 000}{200\ 000}} - 1\right)$$
$$= 0{,}046993\ldots$$
$$\therefore i = 4{,}7\%$$

b)

$$1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^{m}$$
$$1 + i = \left(1 + \frac{0{,}047}{12}\right)^{12}$$
$$i = \left(1 + \frac{0{,}047}{12}\right)^{12} - 1$$
$$\therefore i = 4{,}8\%$$

11. R 145 000 is invested in an account which offers interest at 9% p.a. compounded half-yearly for the first 2 years. Then the interest rate changes to 4% p.a. compounded quarterly. Four years after the initial investment, R 20 000 is withdrawn. 6 years after the initial investment, a deposit of R 15 000 is made. Determine the balance of the account at the end of 8 years.

Solution:



$$
\begin{aligned}
A &= P(1+i)^n \\
&= 145\ 000\left(1+\frac{0{,}09}{2}\right)^{2\times 2}\left(1+\frac{0{,}04}{4}\right)^{6\times 4} \\
&- 20\ 000\left(1+\frac{0{,}04}{4}\right)^{4\times 4} + 15\ 000\left(1+\frac{0{,}04}{4}\right)^{2\times 4} \\
&= 145\ 000\,(1{,}045)^4\,(1{,}01)^{24} - 20\ 000\,(1{,}01)^{16} + 15\ 000\,(1{,}01)^8 \\
\therefore A &= \text{R } 212\ 347{,}69
\end{aligned}
$$

CHAPTER 10

Probability

- Discuss terminology. This chapter has many words that can be confusing for learners.
- This chapter provides good opportunity for experiments and activities in classroom.
- Union and intersection symbols have been included, but "and" and "or" is the preferred notation.
- The prime symbol has been included, but "not" is the preferred notation.
- It is very important to define the events: fair dice, full deck of cards etc.

10.1 Revision

Exercise 10 – 1: Revision

1. A bag contains r red balls, b blue balls and y yellow balls. What is the probability that a ball drawn from the bag at random is yellow?

   Solution:

   To get the probability of drawing a yellow ball, we need to count the number of outcomes that result in a yellow ball and divide by the total number of possible outcomes. The number of yellow balls is y and the total number of balls is $r + b + y$. So, the probability of drawing a yellow ball is

   $$P(\text{yellow}) = \frac{y}{r + b + y}$$

2. A packet has yellow and pink sweets. The probability of taking out a pink sweet is $\frac{7}{12}$. What is the probability of taking out a yellow sweet?

   Solution:

   Since there are only yellow and pink sweets in the packet the event of getting a yellow sweet is the complement of the event of getting a pink sweet. So,

   $$\begin{aligned} P(\text{yellow}) &= 1 - P(\text{pink}) \\ &= 1 - \frac{7}{12} \\ &= \frac{5}{12} \end{aligned}$$

3. You flip a coin 4 times. What is the probability that you get 2 heads and 2 tails? Write down the sample space and the event set to determine the probability of this event.

Solution:

The sample space is the collection of all the ways in which a coin can land when we flip it 4 times. We use H to represent heads and T to represent tails. The sample space is

$$S = \left\{ \begin{array}{cccc} (H;H;H;H) & (H;H;H;T) & (H;H;T;H) & (H;H;T;T) \\ (H;T;H;H) & (H;T;H;T) & (H;T;T;H) & (H;T;T;T) \\ (T;H;H;H) & (T;H;H;T) & (T;H;T;H) & (T;H;T;T) \\ (T;T;H;H) & (T;T;H;T) & (T;T;T;H) & (T;T;T;T) \end{array} \right\}$$

We want to compute the probability of the event of getting 2 heads and 2 tails. From the sample space, we find all the outcomes for which this is true:

$$\{(H;H;T;T);(H;T;H;T);(H;T;T;H);(T;H;H;T);(T;H;T;H);(T;T;H;H)\}$$

Since there are 6 outcomes in the event E and 16 outcomes in the sample space S, the probability of the event is

$$P(E) = \frac{6}{16} = \frac{3}{8}$$

4. In a class of 37 children, 15 children walk to school, 20 children have pets at home and 12 children who have a pet at home also walk to school. How many children walk to school and do not have a pet at home?

   Solution:

   Let E be the event that a child walks to school; and let F be the event that a child has a pet at home. Then, from the information in the problem statement

   $$\begin{aligned} n(E) &= 15 \\ n(F) &= 20 \\ n(E \text{ and } F) &= 12 \end{aligned}$$

   We are asked to compute $n(E \text{ and } (\text{not } F))$.

   $$\begin{aligned} n(E \text{ and } (\text{not } F)) &= n(E) - n(E \text{ and } F) \\ &= 15 - 12 \\ &= 3 \end{aligned}$$

5. You roll two 6-sided dice and are interested in the following two events:

   - A: the sum of the dice equals 8
   - B: at least one of the dice shows a 1

   Show that these events are mutually exclusive.

   Solution:

   The event A has the following elements:

   $$\{(2;6);(3;5);(4;4);(5;3);(6;2)\}$$

   Since A does not include any outcomes where a die shows a 1 and since B requires that a die shows at least one 1, the two events can have no outcomes in common: $(A \text{ and } B) = \emptyset$. Therefore the events are, by definition, mutually exclusive.

6. You ask a friend to think of a number from 1 to 100. You then ask her the following questions:

   - Is the number even?
   - Is the number divisible by 7?

   How many possible numbers are less than 80 if she answered "yes" to both questions?

   Solution:

   The first question requires the number to be divisible by 2. The second question requires the number to be divisible by 7. Therefore we should count the multiples of 14 that are less than 80. These are $\{14; 28; 42; 56; 70\}$, giving a total of 5 numbers.

7. In a group of 42 pupils, all but 3 had a packet of chips or a Fanta or both. If 23 had a packet of chips and 7 of these also had a Fanta, what is the probability that one pupil chosen at random has:

   a) both chips and Fanta
   b) only Fanta

   Solution:

   a) $\frac{7}{42} = \frac{1}{6}$
   b) Since $42 - 3 = 39$ had at least one of chips and a Fanta, and 23 had a packet of chips, then $39 - 23 = 16$ had only Fanta.

   $$\frac{16}{42} = \frac{8}{21}$$

8. Tamara has 18 loose socks in a drawer. Eight of these are orange and two are pink. Calculate the probability that the first sock taken out at random is:

   a) orange
   b) not orange
   c) pink
   d) not pink
   e) orange or pink
   f) neither orange nor pink

   Solution:

   a) $\frac{8}{18} = \frac{4}{9}$
   b) $1 - \frac{4}{9} = \frac{5}{9}$
   c) $\frac{2}{18} = \frac{1}{9}$
   d) $1 - \frac{1}{9} = \frac{8}{9}$
   e) $\frac{1}{9} + \frac{4}{9} = \frac{5}{9}$
   f) $1 - \frac{5}{9} = \frac{4}{9}$

9. A box contains coloured blocks. The number of blocks of each colour is given in the following table.

Colour	Purple	Orange	White	Pink
Number of blocks	24	32	41	19

A block is selected randomly. What is the probability that the block will be:

a) purple
b) purple or white
c) pink and orange
d) not orange?

Solution:

a) Before we answer the questions we first work out how many blocks there are in total. This gives us the size of the sample space as

$$n(S) = 24 + 32 + 41 + 19 = 116$$

The probability that a block is purple is:

$$\begin{aligned} P(\text{purple}) &= \frac{n(E)}{n(S)} \\ &= \frac{24}{116} \\ &= 0{,}21 \end{aligned}$$

b) The probability that a block is either purple or white is:

$$\begin{aligned} P(\text{purple or white}) &= P(\text{purple}) + P(\text{white}) \\ &= \frac{24}{116} + \frac{41}{116} \\ &= 0{,}56 \end{aligned}$$

c) Since one block cannot be two colours the probability of this event is 0.

d) We first work out the probability that a block is orange:

$$\begin{aligned} P(\text{orange}) &= \frac{32}{116} \\ &= 0{,}28 \end{aligned}$$

The probability that a block is not orange is:

$$\begin{aligned} P(\text{not orange}) &= 1 - 0{,}28 \\ &= 0{,}72 \end{aligned}$$

10. The surface of a soccer ball is made up of 32 faces. 12 faces are regular pentagons, each with a surface area of about 37 cm^2. The other 20 faces are regular hexagons, each with a surface area of about 56 cm^2.

You roll the soccer ball. What is the probability that it stops with a pentagon touching the ground?

Solution:

Since a soccer ball is round, the probability of stopping on a face is proportional to the area of the face. There are 12 pentagons each with an area of 37 cm^2, for a total area of $12 \times 37 = 444$ cm^2. There are 20 hexagons each with an area of 56 cm^2, for a total area of $20 \times 56 = 1120$ cm^2. So the probability of stopping on a pentagon is

$$\begin{aligned} \frac{\text{area of pentagons}}{\text{total area}} &= \frac{444}{444 + 1120} \\ &= 0{,}28 \end{aligned}$$

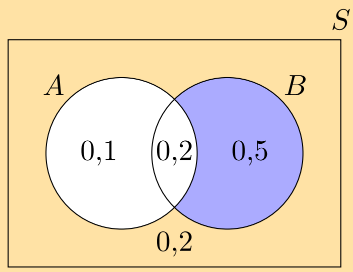
to 2 decimal places.

Exercise 10 – 2: Venn diagram revision

1. Given the following information:
   - $P(A) = 0{,}3$
   - $P(B \text{ and } A) = 0{,}2$
   - $P(B) = 0{,}7$

   First draw a Venn diagram to represent this information. Then compute the value of $P(B \text{ and } (\text{not } A))$.

   Solution:

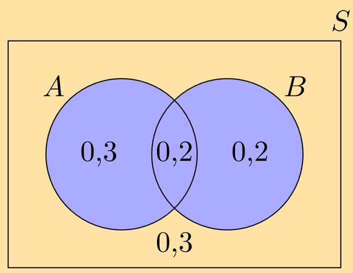
   

   $P(B \text{ and } (\text{not } A)) = 0{,}5$ is marked in blue on the Venn diagram.

2. You are given the following information:
   - $P(A) = 0{,}5$
   - $P(A \text{ and } B) = 0{,}2$
   - $P(\text{not } B) = 0{,}6$

   Draw a Venn diagram to represent this information and determine $P(A \text{ or } B)$.

   Solution:

   $P(A \text{ or } B) = 0{,}7$ is marked in blue on the Venn diagram.

3. A study was undertaken to see how many people in Port Elizabeth owned either a Volkswagen or a Toyota. 3% owned both, 25% owned a Toyota and 60% owned a Volkswagen. What percentage of people owned neither car?

Solution:

Let T be the event that a person owns a Toyota; and V be the event that a person owns a Volkswagen. According to the information in the problem:

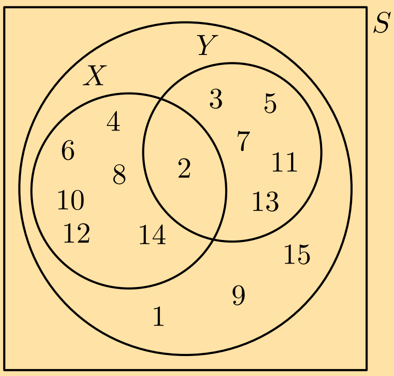
$$P(T \text{ and } V) = 0{,}03$$
$$P(T) = 0{,}25$$
$$P(V) = 0{,}6$$

We are asked to compute $P(\text{not } (T \text{ or } V))$.

$$\begin{aligned}
&P(\text{not } (T \text{ or } V)) \\
&= 1 - P(T \text{ or } V) && \text{(complementary rule)} \\
&= 1 - (P(T) + P(V) - P(T \text{ and } V)) && \text{(sum rule)} \\
&= 1 - (0{,}25 + 0{,}6 - 0{,}03) \\
&= 0{,}18
\end{aligned}$$

4. Let S denote the set of whole numbers from 1 to 15, X denote the set of even numbers from 1 to 15 and Y denote the set of prime numbers from 1 to 15. Draw a Venn diagram depicting S, X and Y.

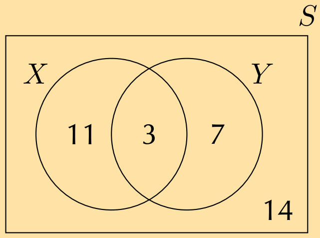
Solution:



10.2 Dependent and independent events

Exercise 10 – 3: Dependent and independent events

1. Use the following Venn diagram to determine whether events X and Y are
   a) mutually exclusive or not mutually exclusive;
   b) dependent or independent.



Solution:

a) The events are mutually exclusive if and only if $(X \text{ and } Y) = \emptyset$. From the Venn diagram we see that $(X \text{ and } Y)$ contains 3 elements. Therefore X and Y are not mutually exclusive.

b) The events are independent if and only if $P(X \text{ and } Y) = P(X) \times P(Y)$. In this case,

$$\begin{aligned} P(X \text{ and } Y) &= \frac{3}{35} \\ P(X) &= \frac{14}{35} \\ P(Y) &= \frac{10}{35} \end{aligned}$$

So $P(X) \times P(Y) = \frac{4}{35} \neq P(X \text{ and } Y)$. Therefore X and Y are dependent.

2. Of the 30 learners in a class 17 have black hair, 11 have brown hair and 2 have red hair. A learner is selected from the class at random.

a) What is the probability that the learner has black hair?

b) What is the probability that the learner has brown hair?

c) Are these two events mutually exclusive?

d) Are these two events independent?

Solution:

a) $\frac{17}{30}$

b) $\frac{11}{30}$

c) Yes, since each learner has only one hair colour, so $P(\text{black and brown}) = 0$.

d) No, since $P(\text{black and brown}) = 0 \neq P(\text{black}) \times P(\text{brown})$.

3. $P(M) = 0{,}45$; $P(N) = 0{,}3$ and $P(M \text{ or } N) = 0{,}615$. Are the events M and N mutually exclusive, independent or neither mutually exclusive nor independent?

Solution:

From the sum rule,

$$\begin{aligned} P(M \text{ and } N) &= P(M) + P(N) - P(M \text{ or } N) \\ &= 0{,}45 + 0{,}3 - 0{,}615 \\ &= 0{,}135 \end{aligned}$$

$P(M \text{ and } N) \neq 0$, therefore the events are not mutually exclusive.

Therefore the events are independent since

$$P(M) \times P(N) = 0{,}135 = P(M \text{ and } N)$$

4. (For enrichment)

   Prove that if event A and event B are mutually exclusive with $P(A) \neq 0$ and $P(B) \neq 0$, then A and B are always dependent.

   Solution:

   Proof by contradiction.

   Assume that A and B are independent events. From the definition of independence, we then have $P(A \text{ and } B) = P(A) \times P(B)$. Since A and B are mutually exclusive, we know that $P(A \text{ and } B) = 0$. Therefore $P(A) \times P(B) = 0$.

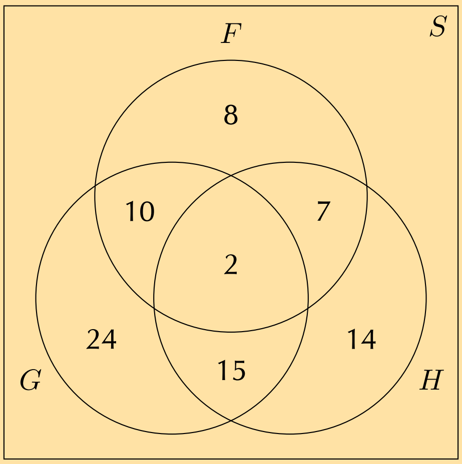
   For the product of two numbers to be zero, at least one of the two numbers must be zero. But in the problem statement, we were told $P(A) \neq 0$ and $P(B) \neq 0$. Therefore we have a contradiction and our original assumption, that A and B are independent events, must be false.

   Conclusion: A and B are dependent.

10.3 More Venn diagrams

Exercise 10 – 4: Venn diagrams

1. Use the Venn diagram below to answer the following questions. Also given: $n(S) = 120$.



   a) Compute $P(F)$.

   b) Compute $P(G \text{ or } H)$.

   c) Compute $P(F \text{ and } G)$.

   d) Are F and G dependent or independent?

   Solution:

   a) $\frac{27}{120} = \frac{9}{40}$

   b) $\frac{72}{120} = \frac{3}{5}$

   c) $\frac{12}{120} = \frac{1}{10}$

d)

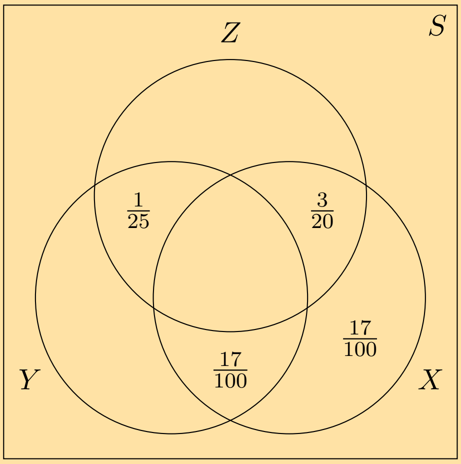
$$P(F) \times P(G) = \frac{9}{40} \times \frac{51}{120}$$
$$= \frac{153}{1600}$$

and

$$P(F \text{ and } G) = \frac{1}{10}$$

Therefore $P(F \text{ and } G) \neq P(F) \times P(G)$ and the events are dependent.

2. The Venn diagram below shows the probabilities of 3 events. Complete the Venn diagram using the additional information provided.



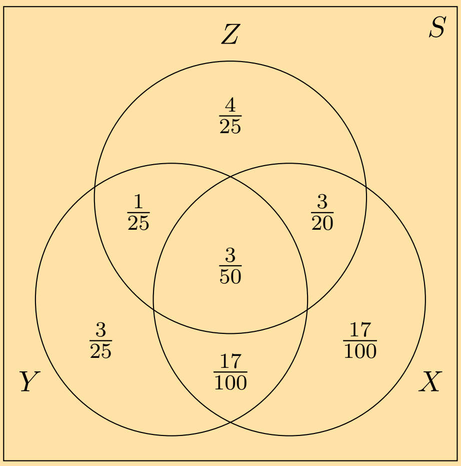
- $P(Z \text{ and } (\text{not } Y)) = \frac{31}{100}$
- $P(Y \text{ and } X) = \frac{23}{100}$
- $P(Y) = \frac{39}{100}$

After completing the Venn diagram, compute the following:

$$P(Z \text{ and not } (X \text{ or } Y))$$

Solution:

The completed Venn diagram is below.



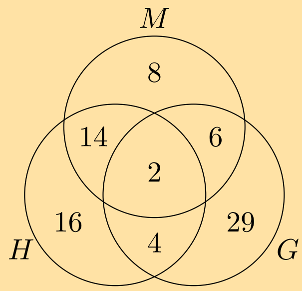
$(Z \text{ and not } (X \text{ or } Y))$ is the top part of Z, which excludes all of X and Y. Therefore $P(Z \text{ and not } (X \text{ or } Y)) = \frac{4}{25}$.

3. There are 79 Grade 10 learners at school. All of these take some combination of Maths, Geography and History. The number who take Geography is 41; those who take History is 36; and 30 take Maths. The number who take Maths and History is 16; the number who take Geography and History is 6, and there are 8 who take Maths only and 16 who take History only.

   a) Draw a Venn diagram to illustrate all this information.
   b) How many learners take Maths and Geography but not History?
   c) How many learners take Geography only?
   d) How many learners take all three subjects?

   Solution:

   a)



   b) Each student must do exactly one of the following:
      - take only Geography;
      - only take Maths and/or History.

      There are $30 + 36 - 16 = 50$ learners taking Maths and/or History, therefore there must be $79 - 50 = 29$ learners doing only Geography.

      Each learner must do exactly one of the following:
      - take only Geography (29 learners);
      - take only Maths (8 learners);
      - take History (36 learners);
      - take Geography and Maths, but not History.

      Given the number of learners list for each of the first three items above, the final answer is that $79 - 29 - 8 - 36 = 6$ learners take Geography and Maths, but not History.

   c) Calculated already: 29 learners.

   d) Each learner must take exactly one of:
      - Geography;
      - only Maths;
      - only History;
      - Maths and History but not Geography.

      Using the same method as before, the number of learners in the last group is $79 - 41 - 8 - 16 = 14$. But, 16 learners do Maths and History, so there must be $16 - 14 = 2$ learners who do all three.

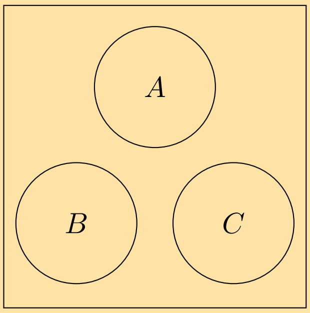
4. Draw a Venn diagram with 3 mutually exclusive events. Use the diagram to show that for 3 mutually exclusive events, A, B and C, the following is true:

$$P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C)$$

   This is the addition rule for 3 mutually exclusive events.

Solution:

Recall for events to be mutually exclusive, they must have no elements in common. This mean that in the Venn diagram there must be no overlap between any of A, B and C.



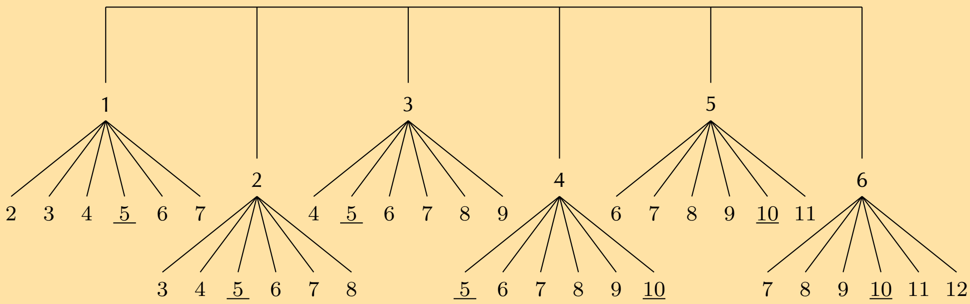
From the Venn diagram we can see that the probability of the three events together is simply the sum of their individual probabilities. So $P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C)$. This will be true for any mutually exclusive events since they never overlap in the Venn diagram.

10.4 Tree diagrams

Exercise 10 – 5: Tree diagrams

1. You roll a die twice and add up the dots to get a score. Draw a tree diagram to represent this experiment. What is the probability that your score is a multiple of 5?

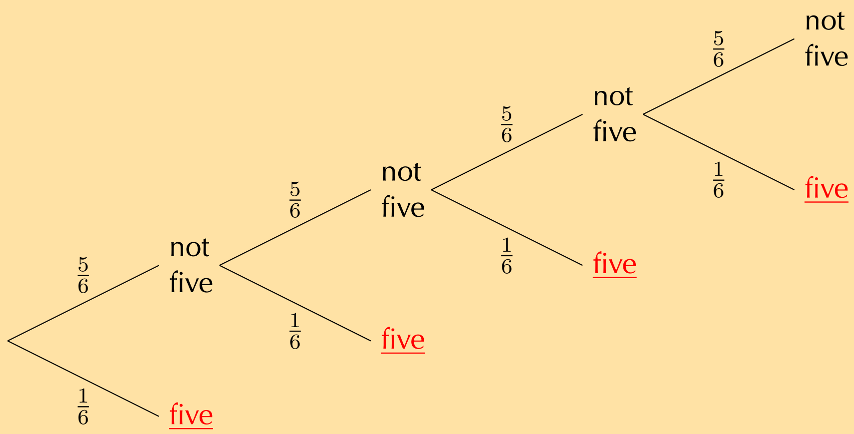
   Solution:

   The tree diagram for the experiment is shown above. To save space, probabilities were not indicated on the branches of the tree, but every branch has a probability of $\frac{1}{6}$. The multiples of 5 are underlined. Since the probability of each of the underlined outcomes in $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$ and since there are 7 outcomes that are multiples of 5, the probability of getting a multiple of 5 is $\frac{7}{36}$.

2. What is the probability of throwing at least one five in four rolls of a regular 6-sided die? Hint: do not show all possible outcomes of each roll of the die. We are interested in whether the outcome is 5 or not 5 only.

Solution:



The outcomes that lead to at least one 5 in four rolls of the die are marked on the tree diagram above. Summing the probabilities along all the branches gives

$$\left(\frac{1}{6}\right)+\left(\frac{5}{6}\times\frac{1}{6}\right)+\left(\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}\right)+\left(\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}\right)=\frac{671}{1296}$$

3. You flip one coin 4 times.

   a) What is the probability of getting exactly 3 heads?

   b) What is the probability of getting at least 3 heads?

   Solution:

   a) There are $2^4 = 16$ possible outcomes for 4 coin tosses. There are 4 outcomes that contain exactly 3 heads, namely $\{H; H; H; T\}$, $\{H; H; T; H\}$, $\{H; T; H; H\}$, $\{T; H; H; H\}$. Therefore the probability of getting exactly 3 heads is $\frac{4}{16} = \frac{1}{4}$.

   b) Getting **at least** 3 heads is the same as getting either exactly 3 heads or exactly 4 heads. We have already seen that there are 4 ways to get exactly 3 heads. There is 1 way of getting exactly 4 heads, namely the outcome $\{H; H; H; H\}$. Hence there are 5 ways of getting at least 3 heads and the probability of this event is $\frac{5}{16}$.

4. You flip 4 different coins at the same time.

   a) What is the probability of getting exactly 3 heads?

   b) What is the probability of getting at least 3 heads?

   Solution:

   a) The mathematics of this problem is exactly the same as the previous problem since it does not matter whether we flip 4 different coins at the same time or the same coin 4 different times. The correct answer is $\frac{1}{4}$.

   b) $\frac{5}{16}$

Exercise 10 – 6: Contingency tables

1. Use the contingency table below to answer the following questions.

	Brown eyes	Not brown eyes	Totals
Black hair	50	30	80
Red hair	70	80	150
Totals	120	110	230

a) What is the probability that someone with black hair has brown eyes?

b) What is the probability that someone has black hair?

c) What is the probability that someone has brown eyes?

d) Are having black hair and having brown eyes dependent or independent events?

Solution:

a) 80 people have black hair and of those, 50 people also have brown eyes. Therefore the probability that someone with black hair has brown eyes is $\frac{50}{80} = \frac{5}{8}$.
Note: this is different from asking for the probability of having black hair **and** brown eyes. (This probability is computed in part (d) below.) The question was phrased to ask for the probability of having brown eyes **given** that a person has black hair.

b) Out of a total of 230, 80 have black hair. Therefore the probability that someone has black hair is $\frac{80}{230} = \frac{8}{23}$.

c) Out of a total of 230, 120 have brown eyes. Therefore the probability that someone has brown eyes is $\frac{120}{230} = \frac{12}{23}$.

d) We already computed that the probability of having

- black hair is $\frac{8}{23}$; and
- brown eyes is $\frac{12}{23}$.

Since 50 out of 230 people have black hair and brown eyes, the probability of having black hair and brown eyes is $\frac{5}{23}$.
We conclude that having black hair and brown eyes are dependent events since $\frac{5}{23} \neq \frac{8}{23} \times \frac{12}{23}$.

2. Given the following contingency table, identify the events and determine whether they are dependent or independent.

	Location A	Location B	Totals
Buses left late	15	40	55
Buses left on time	25	20	45
Totals	40	60	100

Solution:

The events are whether a bus leaves from Location A or not and whether a bus left late or not.

We test whether the Location A and the left late events are independent. The total number of buses in the contingency table is 100. We determine the probabilities of the different events from the values in the table —

- leaving from Location A: $\frac{40}{100} = 0{,}4$;
- leaving late: $\frac{55}{100} = 0{,}55$;
- leaving from Location A and leaving late: $\frac{15}{100} = 0{,}15$.

Since $0{,}4 \times 0{,}55 = 0{,}22 \neq 0{,}15$, the events are dependent.

3. You are given the following information.

   - Events A and B are independent.
   - $P(\text{not } A) = 0{,}3$.
   - $P(B) = 0{,}4$.

   Complete the contingency table below.

	A	not A	Totals
B			
not B			
Totals			50

Solution:

From the given table, we see that the total number of outcomes is 50. Since $P(\text{not } A) = 0{,}3$ we have $n(\text{not } A) = 0{,}3 \times 50 = 15$ and $n(A) = 50 - 15 = 35$. Since $P(B) = 0{,}4$ we have $n(B) = 0{,}4 \times 50 = 20$ and $n(\text{not } B) = 50 - 20 = 30$. From this we can partially complete the table:

	A	not A	Totals
B			20
not B			30
Totals	35	15	50

Next, we use the fact that A and B are independent. From the definition of independence

$$P((\text{not } A) \text{ and } B) = P(\text{not } A) \times P(B) = 0{,}3 \times 0{,}4 = 0{,}12$$

Therefore $n((\text{not } A) \text{ and } B) = 0{,}12 \times 50 = 6$. We find the rest of the values in the table by making sure that each row and column sums to its total.

	A	not A	Totals
B	14	6	20
not B	21	9	30
Totals	35	15	50

Exercise 10 – 7: End of chapter exercises

1. Jane invested in the stock market. The probability that she will not lose all her money is $0{,}32$. What is the probability that she will lose all her money? Explain.

   Solution:

   Since the events "Jane will lose all her money" and "Jane will not lose all her money" are complementary events, their probabilities sum to 1. Therefore the probability that Jane will lose all her money is $1 - 0{,}32 = 0{,}68$.

2. If D and F are mutually exclusive events, with $P(\text{not } D) = 0{,}3$ and $P(D \text{ or } F) = 0{,}94$, find $P(F)$.

   Solution:

   Since D and F are mutually exclusive, the sum rule for mutually exclusive events applies:

   $$P(D \text{ or } F) = P(D) + P(F)$$

   We are given $P(\text{not } D) = 0{,}3$, therefore $P(D) = 1 - 0{,}3 = 0{,}7$. From the sum rule we have

   $$\begin{aligned} P(F) &= P(D \text{ or } F) - P(D) \\ &= 0{,}94 - 0{,}7 \\ &= 0{,}24 \end{aligned}$$

3. A car sales person has pink, lime-green and purple models of car A and purple, orange and multicolour models of car B. One dark night a thief steals a car.

   a) What is the experiment and sample space?
   b) What is the probability of stealing either a model of A or a model of B?
   c) What is the probability of stealing both a model of A and a model of B?

   Solution:

   a) The experiment is the outcome of selecting a particular model and colour car from the sample space of available cars.

      The sample space is {pink model A; lime-green model A; purple model A; purple model B; orange model B; multicolour model B}.
   b) Since there are only two models, namely A and B, the probability of stealing one of the two models is 1.
   c) Since there is no overlap between the two models and since the thief steals only one car, the probability of stealing both of the models is 0.

4. The probability of event X is $0{,}43$ and the probability of event Y is $0{,}24$. The probability of both occurring together is $0{,}10$. What is the probability that X or Y will occur?

Solution:

From the addition rule

$$\begin{aligned} P(X \text{ or } Y) &= P(X) + P(Y) - P(X \text{ and } Y) \\ &= 0{,}43 + 0{,}24 - 0{,}10 \\ &= 0{,}57 \end{aligned}$$

5. $P(H) = 0{,}62$; $P(J) = 0{,}39$ and $P(H \text{ and } J) = 0{,}31$. Calculate:

   a) $P(H')$
   b) $P(H \text{ or } J)$
   c) $P(H' \text{ or } J')$
   d) $P(H' \text{ or } J)$
   e) $P(H' \text{ and } J')$

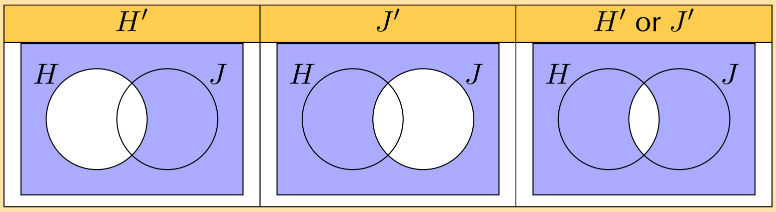
Solution:

a) $P(H') = 1 - P(H) = 1 - 0{,}62 = 0{,}38$

b) From the addition rule:

$$\begin{aligned} P(H \text{ or } J) &= P(H) + P(J) - P(H \text{ and } J) \\ &= 0{,}62 + 0{,}39 - 0{,}31 \\ &= 0{,}7 \end{aligned}$$

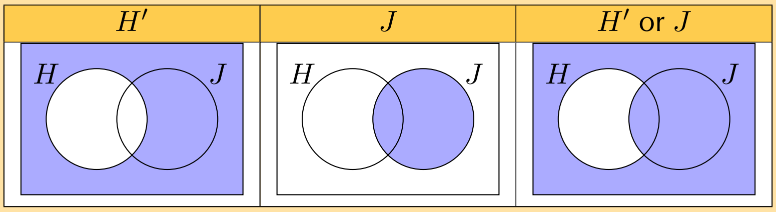
c) We draw a Venn diagram to get an idea of what the event $(H' \text{ or } J')$ looks like.



From the third diagram above we can see that

$$\begin{aligned} P(H' \text{ or } J') &= 1 - P(H \text{ and } J) \\ &= 0{,}69 \end{aligned}$$

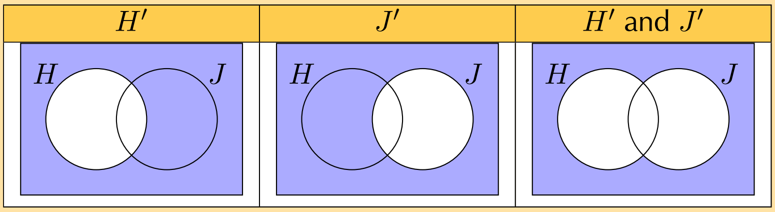
d) We draw a Venn diagram to get an idea of what the event $(H' \text{ or } J)$ looks like.



From the third diagram above we can see that

$$\begin{aligned} P(H' \text{ or } J) &= P(H') + P(H \text{ and } J) \\ &= 0{,}38 + 0{,}31 \\ &= 0{,}69 \end{aligned}$$

e) We draw a Venn diagram to get an idea of what the event $(H'$ and $J')$ looks like.



From the third diagram above we can see that

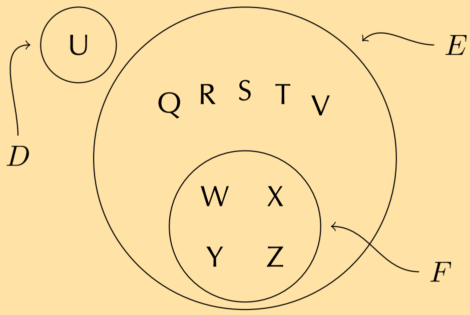
$$\begin{aligned} P(H' \text{ and } J') &= 1 - P(H \text{ or } J) \\ &= 0{,}3 \end{aligned}$$

6. The last ten letters of the alphabet are placed in a hat and people are asked to pick one of them. Event D is picking a vowel, event E is picking a consonant and event F is picking one of the last four letters. Draw a Venn diagram showing the outcomes in the sample space and the different events. Then calculate the following probabilities:

   a) $P(\text{not } F)$
   b) $P(F \text{ or } D)$
   c) $P(\text{neither } E \text{ nor } F)$
   d) $P(D \text{ and } E)$
   e) $P(E \text{ and } F)$
   f) $P(E \text{ and } D')$

   Solution:

   a) The Venn diagram is shown below. (Note that Y is not a vowel.)



   We can then read off all the required probabilities.
   $P(\text{not } F) = 1 - \frac{4}{10} = \frac{3}{5}$

   b) $\frac{5}{10} = \frac{1}{2}$
   c) This leaves only the outcomes in D, so $P(\text{neither } E \text{ nor } F) = \frac{1}{10}$.
   d) 0 since D and E are mutually exclusive.
   e) $\frac{4}{10} = \frac{2}{5}$
   f) Since D and E are complementary (a letter is either a vowel or a consonant), $D' = E$, so $P(E \text{ and } D') = P(E \text{ and } E) = P(E) = \frac{9}{10}$.

7. Thobeka compares three neighbourhoods (we'll call them A, B and C) to see where the best place is to live. She interviews 80 people and asks them whether they like each of the neighbourhoods, or not.

- 40 people like neighbourhood A.
- 35 people like neighbourhood B.
- 40 people like neighbourhood C.
- 21 people like both neighbourhoods A and C.
- 18 people like both neighbourhoods B and C.
- 68 people like at least one neighbourhood.
- 7 people like all three neighbourhoods.

a) Use this information to draw a Venn diagram.

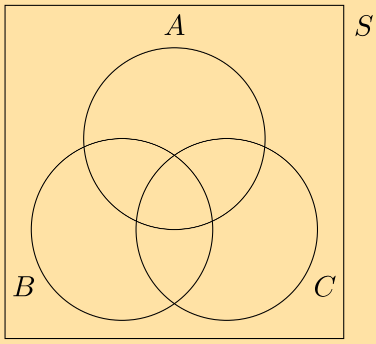
b) How many people like none of the neighbourhoods?

c) How many people like neighbourhoods A and B, but not C?

d) What is the probability that a randomly chosen person from the survey likes at least one of the neighbourhoods?

Solution:

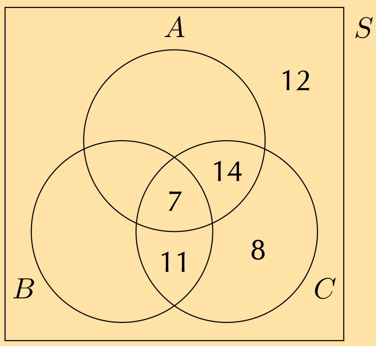
a) We first draw the outline of the Venn diagram.



Next, we determine the counts of the different regions in the diagram, using the information provided.

- There are 7 people who like all three neighbourhoods, so $n(A \text{ and } B \text{ and } C) = 7$.
- There are 21 people who like A and C, so $n(A \text{ and } C) = 21$.
- There are 18 people who like B and C, so $n(B \text{ and } C) = 18$.
- There are 40 people who like C, so $n(C) = 40$.
- Since 68 people like at least one neighbourhood and since there are 80 people in total, 12 people like none of the neighbourhoods.

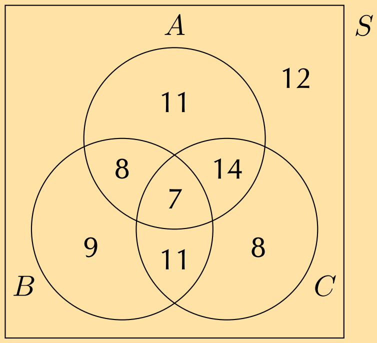
Using the above information we can complete part of the Venn diagram.



Since $n(A \text{ or } B \text{ or } C) = 68$, we can see from the diagram above that $n(A \text{ or } B) = 60$. From the sum rule

$$n(A \text{ and } B) = n(A) + n(B) - n(A \text{ or } B) = 40 + 35 - 60 = 15$$

This allows us to complete the diagram:



b) 12

c) 8

d) $P(A \text{ or } B \text{ or } C) = \frac{68}{80} = \frac{17}{20}$

8. Let G and H be two events in a sample space. Suppose that $P(G) = 0{,}4$; $P(H) = h$; and $P(G \text{ or } H) = 0{,}7$.

   a) For what value of h are G and H mutually exclusive?

   b) For what value of h are G and H independent?

   Solution:

   a) G and H are mutually exclusive when $P(G \text{ and } H) = 0$.
   From the sum rule we have

   $$P(G \text{ or } H) = P(G) + P(H) - P(G \text{ and } H)$$

   By substituting in the given values and $P(G \text{ and } H) = 0$, we get

   $$0{,}7 = 0{,}4 + h - 0$$

   Therefore $h = 0{,}3$.

   b) G and H are independent when $P(G \text{ and } H) = P(G) \times P(H)$.
   From the sum rule we have

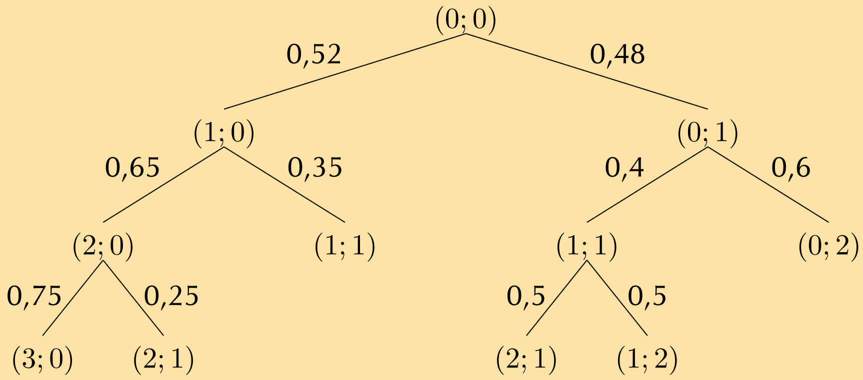
   $$P(G \text{ or } H) = P(G) + P(H) - P(G \text{ and } H)$$

   By substituting in the given values and $P(G \text{ and } H) = P(G) \times P(H)$, we get

   $$0{,}7 = 0{,}4 + h - 0{,}4 \times h$$
   $$0{,}3 = 0{,}6 \times h$$

   Therefore $h = 0{,}5$.

9. The following tree diagram represents points scored by two teams in a soccer game. At each level in the tree, the points are shown as (points for Team 1; points for Team 2).



Use this diagram to determine the probability that:

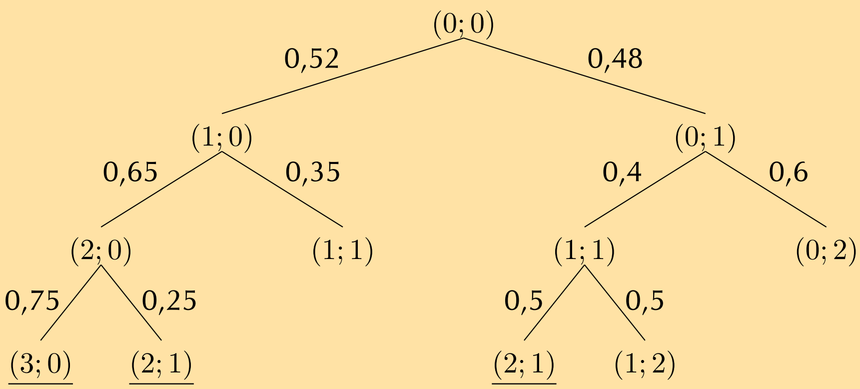
a) Team 1 will win

b) The game will be a draw

c) The game will end with an even number of total points

Solution:

a) The outcomes where Team 1 wins (when the first score is greater than the second score) are underlined in the tree diagram below.

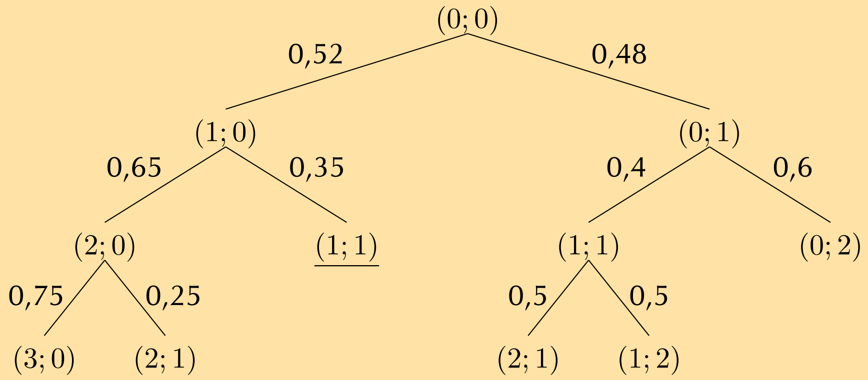


We compute the probability of Team 1 winning by multiplying the probabilities along each path and adding them up.

$$0{,}52 \times 0{,}65 \times 0{,}75 = 0{,}2535$$
$$0{,}52 \times 0{,}65 \times 0{,}25 = 0{,}0845$$
$$0{,}48 \times 0{,}4 \times 0{,}5 = 0{,}096$$

$$0{,}2535 + 0{,}0845 + 0{,}096 = \underline{0{,}434}$$

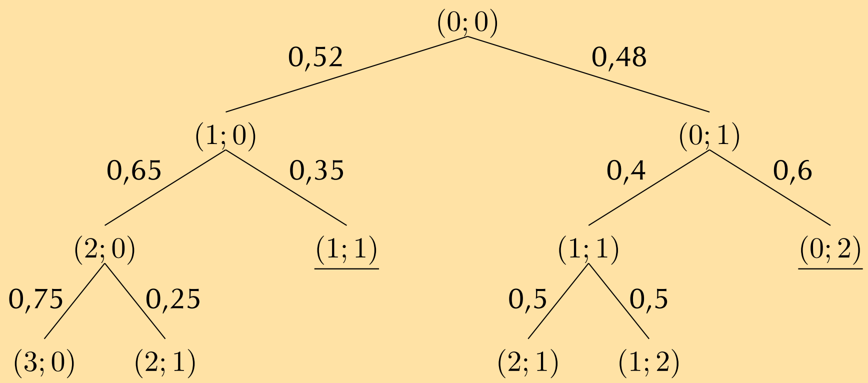
b) The outcomes where the game is a draw (when the first score equals the second score) are underlined in the tree diagram below.



There is only one such outcome and so the probability of a draw is the product of the probabilities along the path.

$$0{,}52 \times 0{,}35 = 0{,}182$$

c) The outcomes where the sum of the points is even are underlined in the tree diagram below.



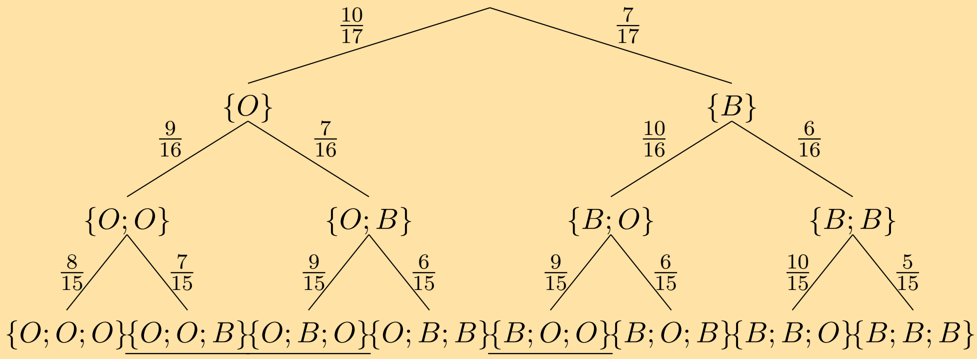
$$0{,}52 \times 0{,}35 = 0{,}182$$
$$0{,}48 \times 0{,}6 = 0{,}288$$

$$0{,}182 + 0{,}288 = \underline{0{,}47}$$

10. A bag contains 10 orange balls and 7 black balls. You draw 3 balls from the bag **without replacement**. What is the probability that you will end up with exactly 2 orange balls? Represent this experiment using a tree diagram.

Solution:

The tree diagram for the experiment is shown below. Since balls are drawn without replacement, the total number of balls (which appears in the denominators of the fractions) decreases by 1 at each step. Depending on whether the ball drawn was orange or black, the numerator either decreases by 1 or stays the same. The outcomes containing exactly 2 orange balls are underlined.



$$
\begin{aligned}
P(\text{two orange balls}) &= \left(\frac{10}{17} \times \frac{9}{16} \times \frac{7}{15}\right) + \left(\frac{10}{17} \times \frac{7}{16} \times \frac{9}{15}\right) + \left(\frac{7}{17} \times \frac{10}{16} \times \frac{9}{15}\right) \\
&= \frac{63}{136}
\end{aligned}
$$

11. Complete the following contingency table and determine whether the events are dependent or independent.

	Durban	Bloemfontein	Totals
Liked living there	130	30	
Did not like living there	140		340
Totals		230	500

Solution:

We complete the contingency table by making sure that all the rows and columns sum to the correct totals.

	Durban	Bloemfontein	Totals
Liked living there	130	30	160
Did not like living there	140	200	340
Totals	270	230	500

Since

- $P(\text{Durban}) = \frac{270}{500} = 0{,}54$;
- $P(\text{liked living there}) = \frac{160}{500} = 0{,}32$;
- $P(\text{Durban and liked living there}) = \frac{130}{500} = 0{,}26$;

and since $0{,}54 \times 0{,}32 = 0{,}1728 \neq 0{,}26$ the events are dependent.

12. Summarise the following information about a medical trial with 2 types of multivitamin in a contingency table and determine whether the events are dependent or independent.

- 960 people took part in the medical trial.
- 540 people used multivitamin A for a month and 400 of those people showed an improvement in their health.
- 300 people showed an improvement in health when using multivitamin B for a month.

If the events are independent, it means that the two multivitamins have the same effect on people. If the events are dependent, it means that one multivitamin is better than the other. Which multivitamin is better than the other, or are the both equally effective?

Solution:

From the information in the problem, we can draw up the following partially completed contingency table.

	Multivitamin A	Multivitamin B	Totals
Improvement in health	400	300	
No improvement in health			
Totals	540		960

We complete the table by ensuring that all rows and columns add up to the correct totals.

	Multivitamin A	Multivitamin B	Totals
Improvement in health	400	300	700
No improvement in health	140	120	260
Totals	540	420	960

Since

- $P(\text{Multivitamin A}) = \frac{540}{960} = \frac{9}{16}$;
- $P(\text{Improvement in health}) = \frac{700}{960} = \frac{35}{48}$;
- $P(\text{Multivitamin A and Improvement in health}) = \frac{400}{960} = \frac{5}{12}$;

and since $\frac{9}{16} \times \frac{35}{48} = \frac{105}{256} \neq \frac{5}{12}$ the events are dependent.

With Multivitamin A, $\frac{400}{540} = 74{,}1\%$ of the people showed an improvement in health. With Multivitamin B, $\frac{300}{420} = 71{,}4\%$ of the people showed an improvement in health. Therefore Multivitamin A is more effective than Multivitamin B.

CHAPTER 11

Statistics

- Ogives are not always grounded at $(0;0)$.
- Histograms must have bars of equal widths.
- The formula for population variance is used and not sample variance.
- Encourage learners to use the STATS functions on their calculators.
- Learners do not need to draw scatter plots, they need only identify outliers.
- Discuss the misuse of statistics in the real world and encourage awareness.

11.1 Revision

Exercise 11 – 1: Revision

1. For each of the following data sets, compute the mean and all the quartiles. Round your answers to one decimal place.

   a) $-3,4\ ;\ -3,1\ ;\ -6,1\ ;\ -1,5\ ;\ -7,8\ ;\ -3,4\ ;\ -2,7\ ;\ -6,2$

   b) $-6\ ;\ -99\ ;\ 90\ ;\ 81\ ;\ 13\ ;\ -85\ ;\ -60\ ;\ 65\ ;\ -49$

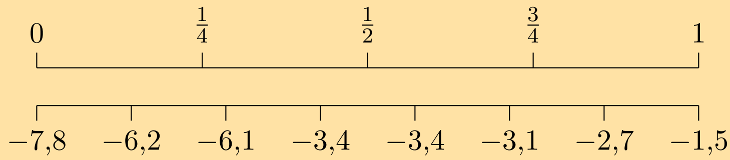
   c) $7\ ;\ 45\ ;\ 11\ ;\ 3\ ;\ 9\ ;\ 35\ ;\ 31\ ;\ 7\ ;\ 16\ ;\ 40\ ;\ 12\ ;\ 6$

   Solution:

   a) Mean:

$$\begin{aligned}\overline{x} &= \frac{(-3,4)+(-3,1)+(-6,1)+(-1,5)+(-7,8)+(-3,4)+(-2,7)+(-6,2)}{8} \\ &\approx -4,3\end{aligned}$$

   To compute the quartiles, we order the data:
   $-7,8\ ;\ -6,2\ ;\ -6,1\ ;\ -3,4\ ;\ -3,4\ ;\ -3,1\ ;\ -2,7\ ;\ -1,5$
   We use the diagram below to find at or between which values the quartiles lie.

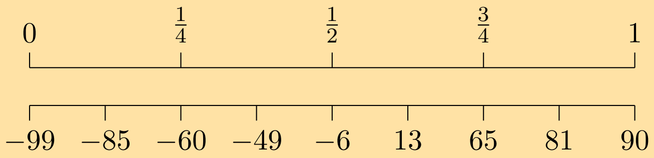


   For the first quartile the position is between the second and third values. The second value is $-6,2$ and the third value is $-6,1$, which means that the first quartile is $\frac{-6,2-6,1}{2} = -6,15$.
   For the median (second quartile) the position is halfway between the fourth and fifth values. Since both these values are $-3,4$, the median is $-3,4$.
   For the third quartile the position is between the sixth and seventh values. Therefore the third quartile is $-2,9$.

b) Mean:

$$\overline{x} = \frac{(-6)+(-99)+(90)+(81)+(13)+(-85)+(-60)+(65)+(-49)}{9}$$
$$\approx -5{,}6$$

To compute the quartiles, we order the data:
$-99\ ;\ -85\ ;\ -60\ ;\ -49\ ;\ -6\ ;\ 13\ ;\ 65\ ;\ 81\ ;\ 90$
We use the diagram below to find at or between which values the quartiles lie.

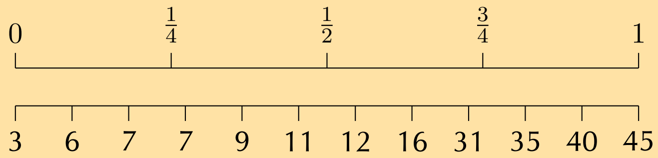


We see that the quartiles are at -60; -6; and 65.

c) The mean is $\overline{x} = 18{,}5$.
To compute the quartiles, we order the data:

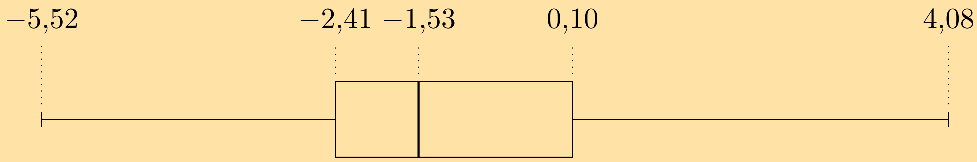
$$3\ ;\ 6\ ;\ 7\ ;\ 7\ ;\ 9\ ;\ 11\ ;\ 12\ ;\ 16\ ;\ 31\ ;\ 35\ ;\ 40\ ;\ 45$$

We use the diagram below to find at or between which values the quartiles lie.



For the first quartile the position is between the third and fourth values. Since both these values are equal to 7, the first quartile is 7.
For the median (second quartile) the position is halfway between the sixth and seventh values. The sixth value is 11 and the seventh value is 12, which means that the median is $\frac{11+12}{2} = 11{,}5$.
For the third quartile the position is between the ninth and tenth values. Therefore the third quartile is $\frac{31+35}{2} = 33$.

2. Use the following box and whisker diagram to determine the range and inter-quartile range of the data.



Solution:

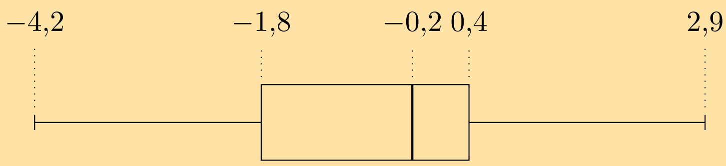
The range is the difference between the minimum and maximum values. From the box-and-whisker diagram, the minimum is $-5{,}52$ and the maximum is $4{,}08$. Therefore the range is $4{,}08 - (-5{,}52) = 9{,}6$.

The inter-quartile range is the difference between the first and third quartiles. From the box-and-whisker diagram, the first quartile is $-2{,}41$ and the third quartile is $0{,}10$. Therefore the range is $0{,}10 - (-2{,}41) = 2{,}51$.

3. Draw the box and whisker diagram for the following data.

$0{,}2\ ;\ -0{,}2\ ;\ -2{,}7\ ;\ 2{,}9\ ;\ -0{,}2\ ;\ -4{,}2\ ;\ -1{,}8\ ;\ 0{,}4\ ;\ -1{,}7\ ;\ -2{,}5\ ;\ 2{,}7\ ;\ 0{,}8\ ;\ -0{,}5$

Solution:

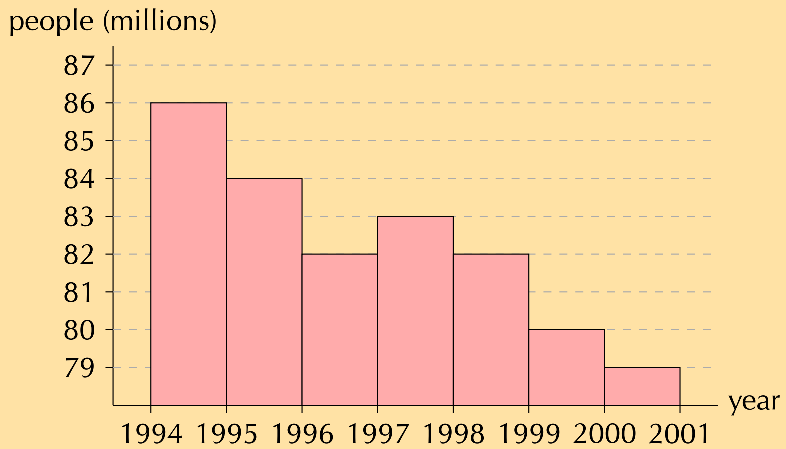


11.2 Histograms

Frequency polygons

Exercise 11 – 2: Histograms

1. Use the histogram below to answer the following questions. The histogram shows the number of people born around the world each year. The ticks on the x-axis are located at the start of each year.



   a) How many people were born between the beginning of 1994 and the beginning of 1996?
   b) Is the number people in the world population increasing or decreasing? (Ignore the rate at which people are dying for this question.)
   c) How many more people were born in 1994 than in 1997?

Solution:

   a) $86 + 84 = 170$ million

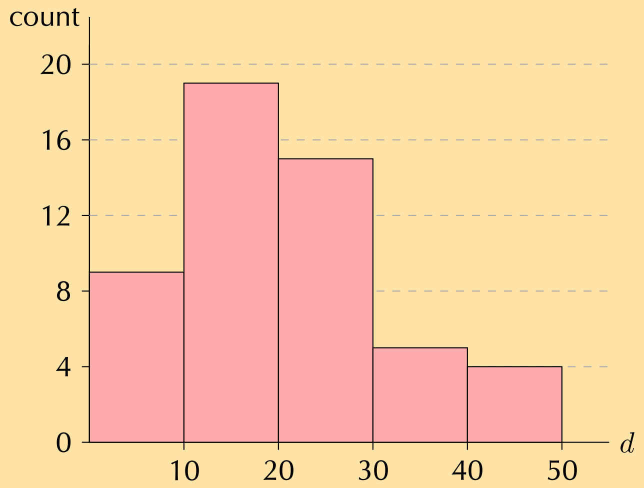
b) Even though the rate at which people are born seems to be decreasing, there are still new people born every year and so the world population is increasing.

c) $86 - 83 = 3$ million

2. In a traffic survey, a random sample of 50 motorists were asked the distance they drove to work daily. The results of the survey are shown in the table below. Draw a histogram to represent the data.

d (km)	$0 < d \leq 10$	$10 < d \leq 20$	$20 < d \leq 30$	$30 < d \leq 40$	$40 < d \leq 50$
f	9	19	15	5	4

Solution:

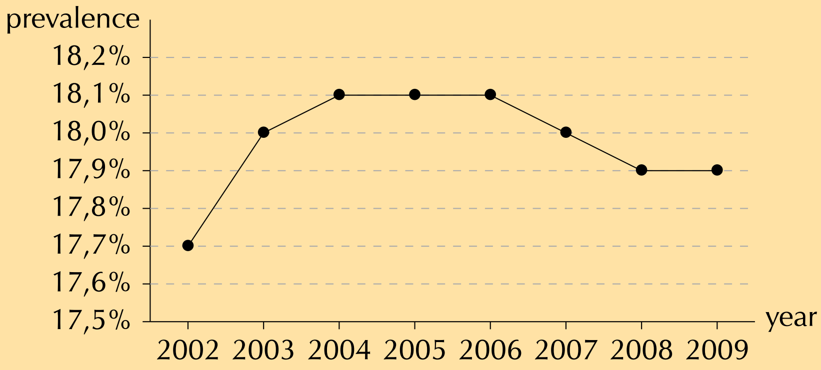


3. Below is data for the prevalence of HIV in South Africa. HIV prevalence refers to the percentage of people between the ages of 15 and 49 who are infected with HIV.

year	2002	2003	2004	2005	2006	2007	2008	2009
prevalence (%)	17,7	18,0	18,1	18,1	18,1	18,0	17,9	17,9

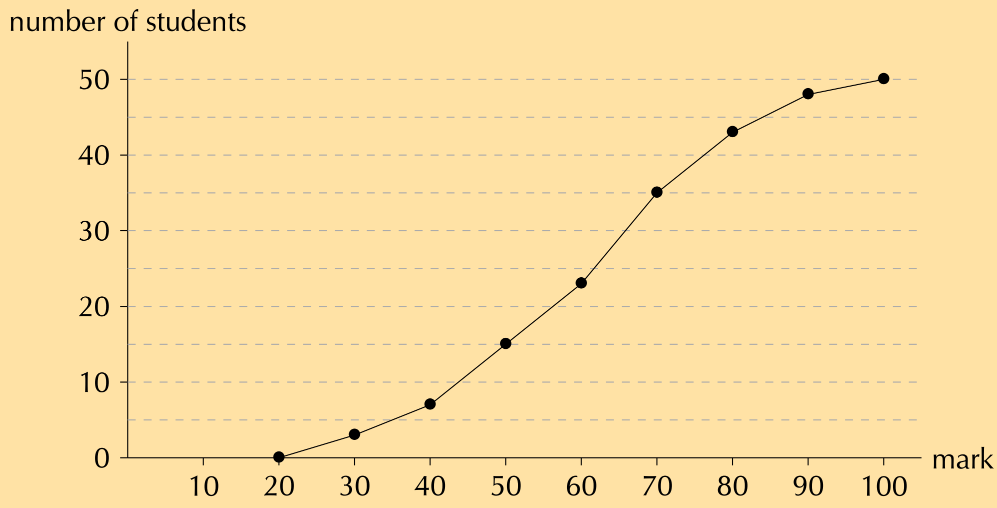
Draw a frequency polygon of this data set.

Solution:



Exercise 11 – 3: Ogives

1. Use the ogive to answer the questions below. Marks give as a percentage (%).



a) How many students got between 50% and 70%?

b) How many students got at least 70%?

c) Compute the average mark for this class, rounded to the nearest integer.

Solution:

a) The cumulative plot shows that 15 students got below 50% and 35 students got below 70%. Therefore $35 - 15 = 20$ students got between 50% and 70%.

b) The cumulative plot shows that 35 students got below 70% and that there are 50 students in total. Therefore $50 - 35 = 15$ students got at least (greater than or equal to) 70%.

c) To compute the average, we first need to use the ogive to determine the frequency of each interval. The frequency of an interval is the difference between the cumulative counts at the top and bottom of the interval on the ogive. It might be difficult to read the exact cumulative count for some of the points on the ogive. But since the final answer will be rounded to the nearest integer, small errors in the counts will not make a difference. The table below summarises the counts.

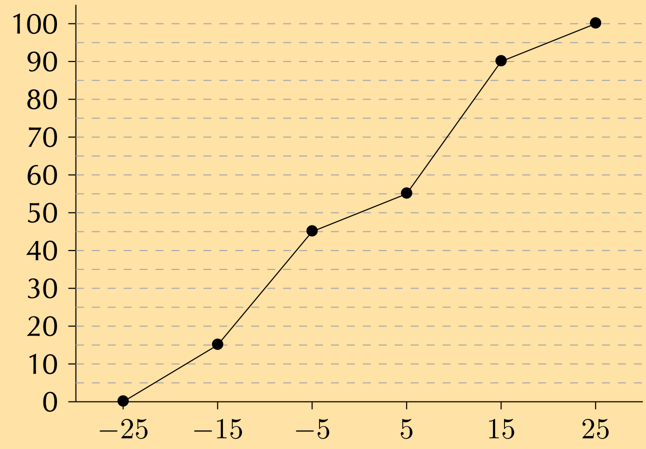
Interval	$[20, 30)$	$[30, 40)$	$[40, 50)$	$[50, 60)$
f	3	4	8	8
Interval	$[60, 70)$	$[70, 80)$	$[80, 90)$	$[90, 100)$
f	12	8	5	2

The average is then the centre of each interval, weighted by the count in that interval.

$$\frac{3 \times 25 + 4 \times 35 + 8 \times 45 + 8 \times 55 + 12 \times 65 + 8 \times 75 + 5 \times 85 + 2 \times 95}{3 + 4 + 8 + 8 + 12 + 8 + 5 + 2}$$
$$= 60{,}2$$

The average mark, rounded to the nearest integer, is 60%.

2. Draw the histogram corresponding to this ogive.



Solution:

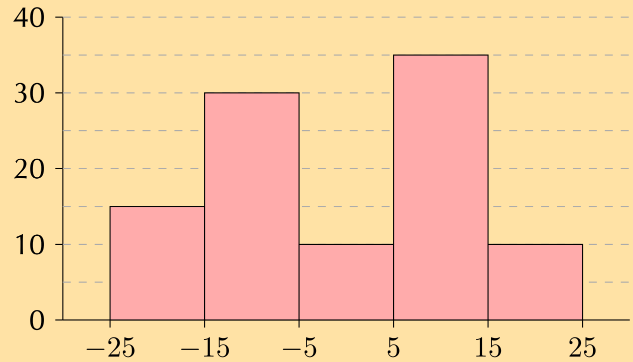
To draw the histogram we need to determine the count in each interval.

Firstly, we can find the intervals by looking where the points are plotted on the ogive. Since the points are at x-coordinates of -25; -15; -5; 5; 15 and 25, it means that the intervals are $[-25;-15)$, etc.

To get the count in each interval we subtract the cumulative count at the start of the interval from the cumulative count at the end of the interval.

Interval	$[-25;-15)$	$[-15;-5)$	$[-5;5)$	$[5;15)$	$[15;25)$
Count	15	30	10	35	10

From these counts we can draw the following histogram:



3. The following data set lists the ages of 24 people.

   $2;\ 5;\ 1;\ 76;\ 34;\ 23;\ 65;\ 22;\ 63;\ 45;\ 53;\ 38$

   $4;\ 28;\ 5;\ 73;\ 79;\ 17;\ 15;\ 5;\ 34;\ 37;\ 45;\ 56$

   Use the data to answer the following questions.

   a) Using an interval width of 8 construct a cumulative frequency plot.
   b) How many are below 30?
   c) How many are below 60?
   d) Giving an explanation, state below what value the bottom 50% of the ages fall.

e) Below what value do the bottom 40% fall?

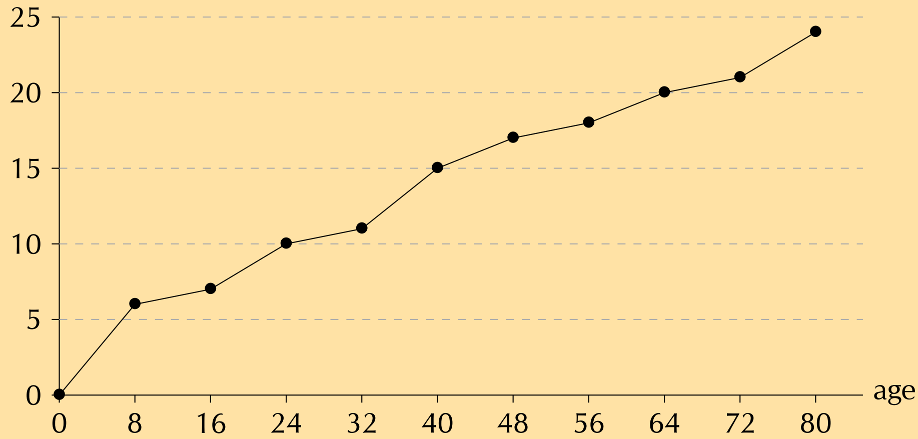
f) Construct a frequency polygon.

Solution:

a) The table below shows the number of people in each age bracket of width 8.

Interval	$[0;8)$	$[8;16)$	$[16;24)$	$[24;32)$	$[32;40)$
Count	6	1	3	1	4
Cumulative	6	7	10	11	15
Interval	$[40;48)$	$[48;56)$	$[56;64)$	$[64;72)$	$[72;80)$
Count	2	1	2	1	3
Cumulative	17	18	20	21	24

From this table we can draw the cumulative frequency plot:



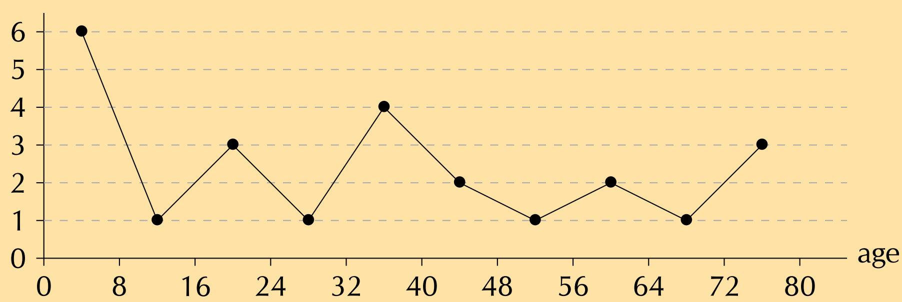
b) 11 people

c) 19 people

d) This question is asking for the median of the data set. The median is, by definition, the value below which 50% of the data lie. Since there are 24 values, the median lies between the middle two values, giving 34.

e) There are 24 values. By drawing a number line, as we do for determining quartiles, we can see that the 40% point is between the tenth and eleventh values. The tenth value is 23 and the eleventh value is 28. Therefore 40% of the values lie below $\frac{23+28}{2} = 25{,}5$.

f) We already have all the values needed to construct the frequency polygon in the table of values above.



4. The weights of bags of sand in grams is given below (rounded to the nearest tenth):

$50{,}1$; $40{,}4$; $48{,}5$; $29{,}4$; $50{,}2$; $55{,}3$; $58{,}1$; $35{,}3$; $54{,}2$; $43{,}5$

$60{,}1$; $43{,}9$; $45{,}3$; $49{,}2$; $36{,}6$; $31{,}5$; $63{,}1$; $49{,}3$; $43{,}4$; $54{,}1$

a) Decide on an interval width and state what you observe about your choice.
b) Give your lowest interval.
c) Give your highest interval.
d) Construct a cumulative frequency graph and a frequency polygon.
e) Below what value do 53% of the cases fall?
f) Below what value of 60% of the cases fall?

Solution:

a) Learner-dependent answer.
b) Learner-dependent answer.
c) Learner-dependent answer.
d) Learner-dependent answer.
e) $49{,}25$
f) $49{,}7$

11.4 Variance and standard deviation

Exercise 11 – 4: Variance and standard deviation

1. Bridget surveyed the price of petrol at petrol stations in Cape Town and Durban. The data, in rands per litre, are given below.

Cape Town	3,96	3,76	4,00	3,91	3,69	3,72
Durban	3,97	3,81	3,52	4,08	3,88	3,68

a) Find the mean price in each city and then state which city has the lower mean.
b) Find the standard deviation of each city's prices.
c) Which city has the more consistently priced petrol? Give reasons for your answer.

Solution:

a) Cape Town: $3{,}84$. Durban: $3{,}82$. Durban has the lower mean.
b) Standard deviation:

$$\sigma = \sqrt{\frac{\sum (x - \overline{x})^2}{n}}$$

For Cape Town:

$$\begin{aligned}\sigma &= \sqrt{\frac{\sum (x - (3{,}84))^2}{6}} \\ &= \sqrt{\frac{0{,}0882}{6}} \\ &= \sqrt{0{,}0147} \\ &\approx 0{,}121\end{aligned}$$

For Durban:

$$\begin{aligned}\sigma &= \sqrt{\frac{\sum \left(x - (3{,}82\dot{3})\right)^2}{6}} \\ &= \sqrt{\frac{0{,}20\dot{3}}{6}} \\ &= \sqrt{0{,}033\dot{8}} \\ &\approx 0{,}184\end{aligned}$$

c) The standard deviation of Cape Town's prices is lower than that of Durban's. That means that Cape Town has more consistent (less variable) prices than Durban.

2. Compute the mean and variance of the following set of values.

   $150 ; 300 ; 250 ; 270 ; 130 ; 80 ; 700 ; 500 ; 200 ; 220 ; 110 ; 320 ; 420 ; 140$

   Solution:

   Mean $= 270{,}7$. Variance $= 27\ 435{,}2$.

3. Compute the mean and variance of the following set of values.

   $-6{,}9 ; -17{,}3 ; 18{,}1 ; 1{,}5 ; 8{,}1 ; 9{,}6 ; -13{,}1 ; -14{,}0 ; 10{,}5 ; -14{,}8 ; -6{,}5 ; 1{,}4$

   Solution:

   Mean $= -1{,}95$. Variance $= 127{,}5$.

4. The times for 8 athletes who ran a 100 m sprint on the same track are shown below. All times are in seconds.

   $10{,}2 ; 10{,}8 ; 10{,}9 ; 10{,}3 ; 10{,}2 ; 10{,}4 ; 10{,}1 ; 10{,}4$

   a) Calculate the mean time.

   b) Calculate the standard deviation for the data.

   c) How many of the athletes' times are more than one standard deviation away from the mean?

   Solution:

   a) $\overline{x} = 10{,}4$

   b) $\sigma = 0{,}27$

   c) The mean is $10{,}4$ and the standard deviation is $0{,}27$. Therefore the interval containing all values that are one standard deviation from the mean is $[10{,}4 - 0{,}27; 10{,}4 + 0{,}27] = [10{,}13; 10{,}67]$. We are asked how many values are **further** than one standard deviation from the mean, meaning **outside** the interval. There are 3 values from the data set outside the interval.

5. The following data set has a mean of $14{,}7$ and a variance of $10{,}01$.

$$18\ ;\ 11\ ;\ 12\ ;\ a\ ;\ 16\ ;\ 11\ ;\ 19\ ;\ 14\ ;\ b\ ;\ 13$$

Compute the values of a and b.

Solution:

From the formula of the mean we have

$$\begin{aligned} 14{,}7 &= \frac{114 + a + b}{10} \\ \therefore a + b &= 147 - 114 \\ \therefore a &= 33 - b \end{aligned}$$

From the formula of the variance we have

$$\begin{aligned} \sigma^2 &= \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n} \\ \therefore 10{,}01 &= \frac{69{,}12 + (a - 14{,}7)^2 + (b - 14{,}7)^2}{10} \end{aligned}$$

Substitute $a = 33 - b$ into this equation to get

$$\begin{aligned} 10{,}01 &= \frac{69{,}12 + (18{,}3 - b)^2 + (b - 14{,}7)^2}{10} \\ \therefore 100{,}1 &= 2b^2 - 66b + 620{,}1 \\ \therefore 0 &= b^2 - 33b + 260 \\ &= (b - 13)(b - 20) \end{aligned}$$

Therefore $b = 13$ or $b = 20$.

Since $a = 33 - b$ we have $a = 20$ or $a = 13$. So, the two unknown values in the data set are 13 and 20.

We do not know which of these is a and which is b since the mean and variance tell us nothing about the order of the data.

11.5 Symmetric and skewed data

Exercise 11 – 5: Symmetric and skewed data

1. Is the following data set symmetric, skewed right or skewed left? Motivate your answer.

   $27\ ;\ 28\ ;\ 30\ ;\ 32\ ;\ 34\ ;\ 38\ ;\ 41\ ;\ 42\ ;\ 43\ ;\ 44\ ;\ 46\ ;\ 53\ ;\ 56\ ;\ 62$

   Solution:

   The statistics of the data set are

   - mean: $41{,}1$;
   - first quartile: 33;

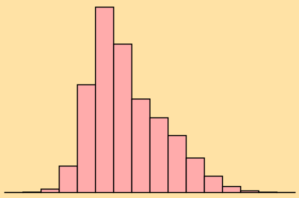
- median: $41{,}5$;
- third quartile: 45.

We can conclude that the data set is skewed left for two reasons.

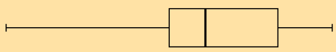
- The mean is less than the median. There is only a very small difference between the mean and median, so this is not a very strong reason.
- A better reason is that the median is closer to the third quartile than the first quartile.

2. State whether each of the following data sets are symmetric, skewed right or skewed left.

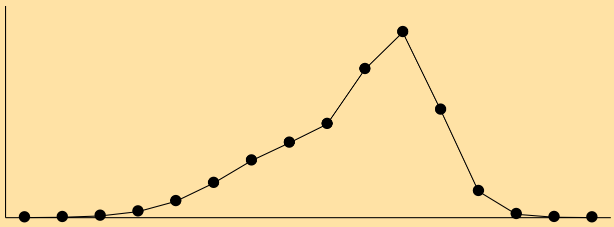
   a) A data set with this histogram:



   b) A data set with this box and whisker plot:



   c) A data set with this frequency polygon:



   d) The following data set:
   $11{,}2$; 5 ; $9{,}4$; $14{,}9$; $4{,}4$; $18{,}8$; $-0{,}4$; $10{,}5$; $8{,}3$; $17{,}8$

Solution:

a) skewed right

b) skewed right

c) skewed left

d) The statistics of the data set are

- mean: $9{,}99$;
- first quartile: $6{,}65$;
- median: $9{,}95$;
- third quartile: $13{,}05$.

Note that we get contradicting indications from the different ways of determining whether the data is skewed right or left.

- The mean is slightly greater than the median. This would indicate that the data set is skewed right.
- The median is slightly closer to the third quartile than the first quartile. This would indicate that the data set is skewed left.

Since these differences are so small and since they contradict each other, we conclude that the data set is symmetric.

3. Two data sets have the same range and interquartile range, but one is skewed right and the other is skewed left. Sketch the box and whisker plot for each of these data sets. Then, invent data (6 points in each data set) that matches the descriptions of the two data sets.

Solution:

Learner-dependent answer.

11.6 Identification of outliers

Exercise 11 – 6: Outliers

1. For each of the following data sets, draw a box and whisker diagram and determine whether there are any outliers in the data.

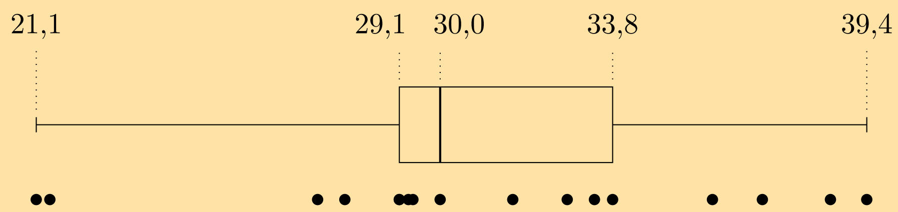
   a) 30 ; 21,4 ; 39,4 ; 33,4 ; 21,1 ; 29,3 ; 32,8 ; 31,6 ; 36 ; 27,9 ; 27,3 ; 29,4 ; 29,1 ; 38,6 ; 33,8 ; 29,1 ; 37,1

   b) 198 ; 166 ; 175 ; 147 ; 125 ; 194 ; 119 ; 170 ; 142 ; 148

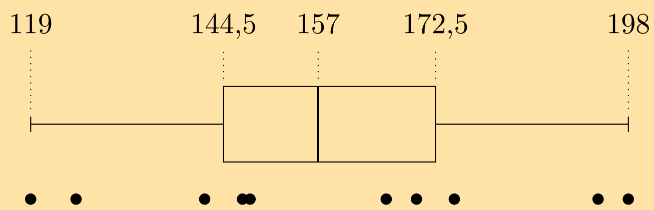
   c) 7,1 ; 9,6 ; 6,3 ; −5,9 ; 0,7 ; −0,1 ; 4,4 ; −11,7 ; 10 ; 2,3 ; −3,7 ; 5,8 ; −1,4 ; 1,7 ; −0,7

Solution:

a) Below is the box-and-whisker diagram of the data as well as dots representing the data themselves. Note that learners do not neeed to draw the dots, but this helps us to see that there are two outliers on the left.

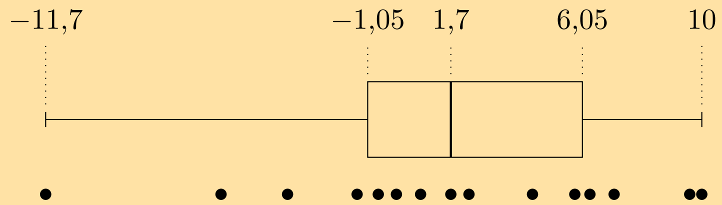


b)



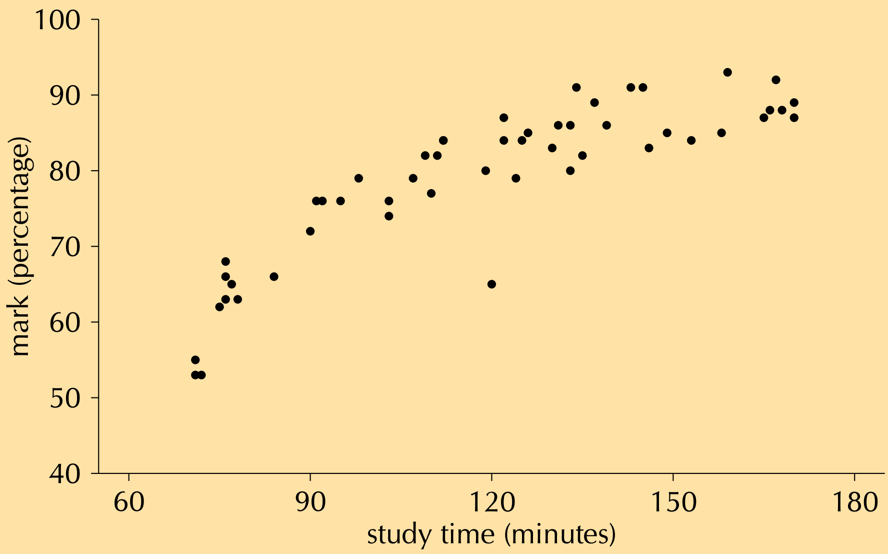
There are no outliers.

c)



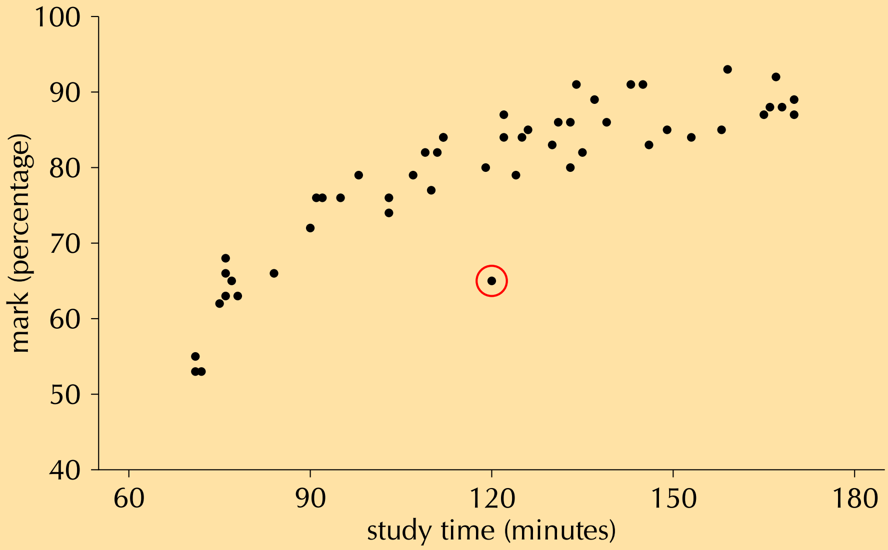
There is one outlier on the left.

2. A class's results for a test were recorded along with the amount of time spent studying for it. The results are given below. Identify any outliers in the data.



Solution:

There is one outlier, marked in red below.



Exercise 11 – 7: End of chapter exercises

1. Draw a histogram, frequency polygon and ogive of the following data set. To count the data, use intervals with a width of 1, starting from 0.

   0,4 ; 3,1 ; 1,1 ; 2,8 ; 1,5 ; 1,3 ; 2,8 ; 3,1 ; 1,8 ; 1,3 ;

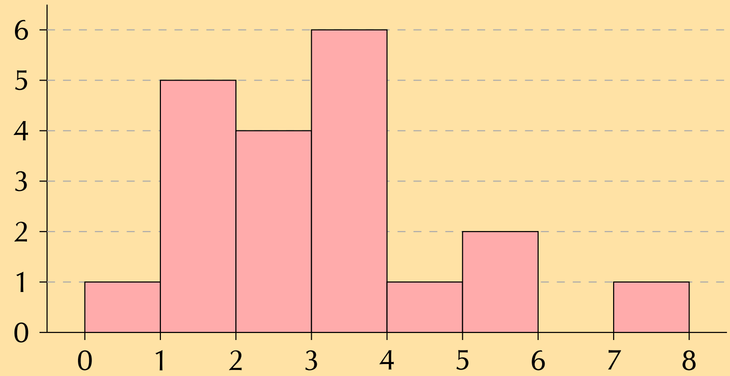
   2,6 ; 3,7 ; 3,3 ; 5,7 ; 3,7 ; 7,4 ; 4,6 ; 2,4 ; 3,5 ; 5,3

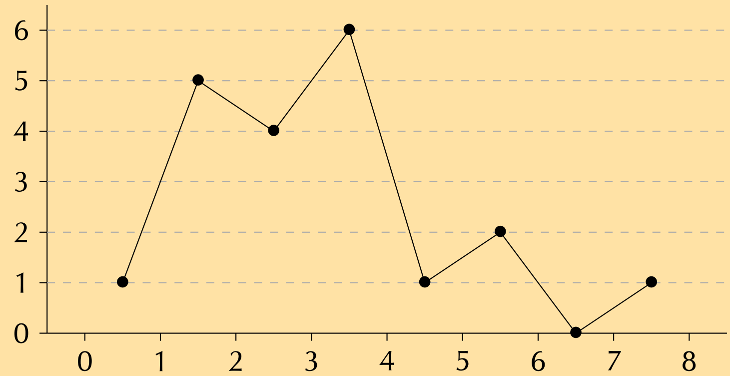
   Solution:

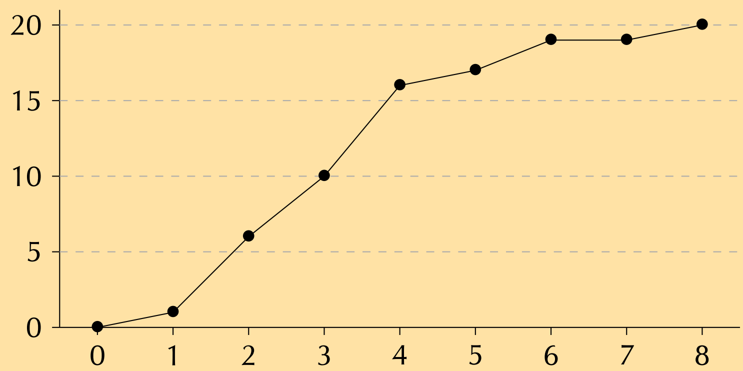
   We first organise the data into a table using an interval width of 1, showing the count in each interval as well as the cumulative count across intervals.

Interval	$[0;1)$	$[1;2)$	$[2;3)$	$[3;4)$	$[4;5)$	$[5;6)$	$[6;7)$	$[7;8)$
Count	1	5	4	6	1	2	0	1
Cumulative	1	6	10	16	17	19	19	20

   From the table above we can draw the histogram, frequency polygon and ogive.







2. Draw a box and whisker diagram of the following data set and explain whether it is symmetric, skewed right or skewed left.

   $-4{,}1$; $-1{,}1$; -1 ; $-1{,}2$; $-1{,}5$; $-3{,}2$; -4 ; $-1{,}9$; -4 ;

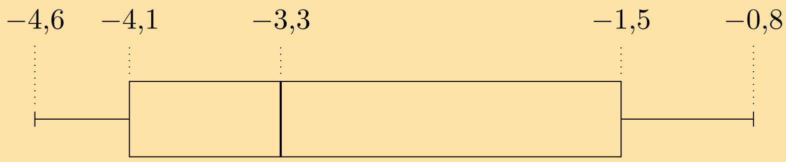
   $-0{,}8$; $-3{,}3$; $-4{,}5$; $-2{,}5$; $-4{,}4$; $-4{,}6$; $-4{,}4$; $-3{,}3$

   Solution:

   The statistics of the data set are

   - minimum: $-4{,}6$;
   - first quartile: $-4{,}1$;
   - median: $-3{,}3$;
   - third quartile: $-1{,}5$;
   - maximum: $-0{,}8$.

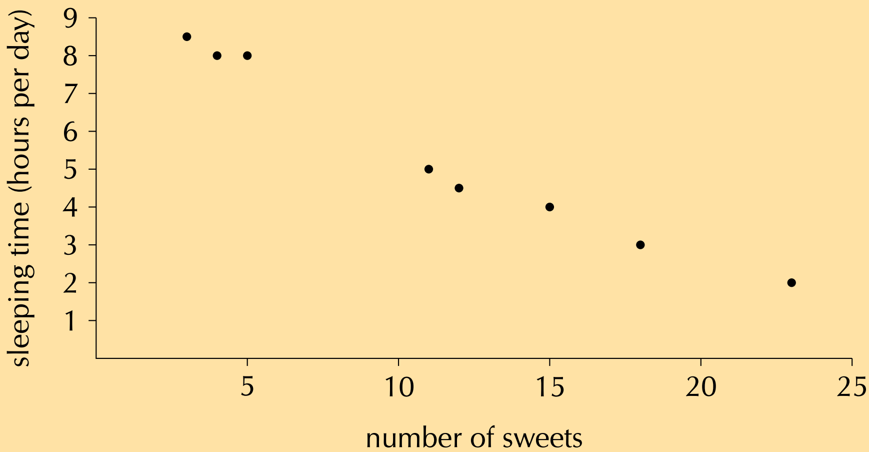
   From this we can draw the box-and-whisker plot as follows.



   Since the median is closer to the first quartile than the third quartile, the data set is skewed right.

3. Eight children's sweet consumption and sleeping habits were recorded. The data are given in the following table and scatter plot.

Number of sweets per week	15	12	5	3	18	23	11	4
Average sleeping time (hours per day)	4	4,5	8	8,5	3	2	5	8



   a) What is the mean and standard deviation of the number of sweets eaten per day?
   b) What is the mean and standard deviation of the number of hours slept per day?
   c) Make a list of all the outliers in the data set.

Solution:

a) Mean $= 11\frac{3}{8}$. Standard deviation $= 6{,}69$.

b) Mean $= 5\frac{3}{8}$. Standard deviation $= 2{,}33$.

c) There are no outliers.

4. The monthly incomes of eight teachers are as follows:

R 10 050; R 14 300; R 9800; R 15 000; R 12 140; R 13 800; R 11 990; R 12 900.

a) What is the mean and standard deviation of their incomes?

b) How many of the salaries are less than one standard deviation away from the mean?

c) If each teacher gets a bonus of R 500 added to their pay what is the new mean and standard deviation?

d) If each teacher gets a bonus of 10% on their salary what is the new mean and standard deviation?

e) Determine for both of the above, how many salaries are less than one standard deviation away from the mean.

f) Using the above information work out which bonus is more beneficial financially for the teachers.

Solution:

a) Mean = R 12 497,50. Standard deviation = R 1768,55.

b) All salaries within the range $(10\ 728{,}95\ ;\ 14\ 266{,}05)$ are less than one standard deviation away from the mean. There are 4 salaries inside this range.

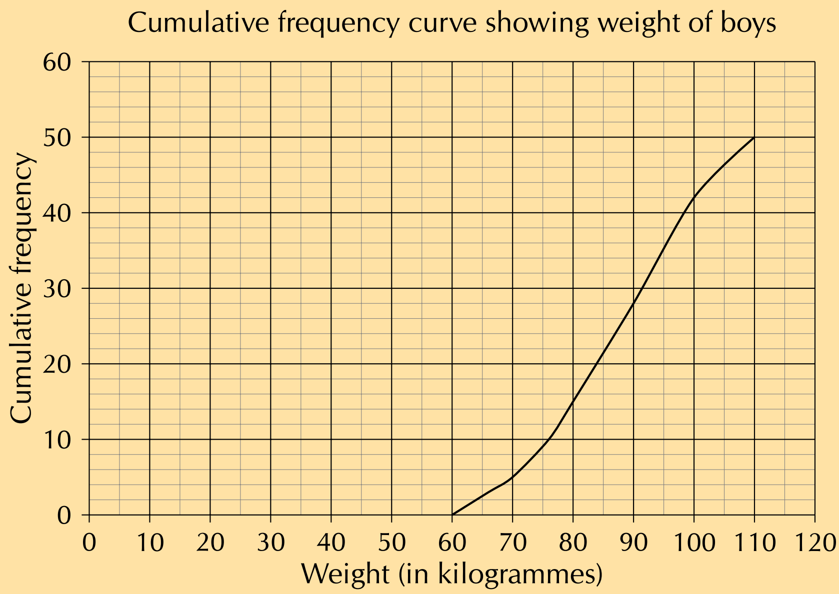
c) Since the increase in each salary is the same absolute amount, the mean simply increases by the bonus. The standard deviation does not change since every value is increased by exactly the same amount. Mean = R 12 997,50. Standard deviation = R 1768,55.

d) With a relative increase, the mean and standard deviation are both multiplied by the same factor. With an increase of 10% the factor is $1{,}1$. Mean = R 13 747,25. Standard deviation = R 1945,41.

e) Adding a constant amount or multiplying by a constant factor (that is, applying a linear transformation) does not change the number of values that lie within one standard deviation from the mean. Therefore the answer is still 4.

f) Since the mean is greater in the second case it means that, on average, the teachers are getting better salaries when the increase is 10%.

5. The weights of a random sample of boys in Grade 11 were recorded. The cumulative frequency graph (ogive) below represents the recorded weights.



a) How many of the boys weighed between 90 and 100 kilogrammes?

b) Estimate the median weight of the boys.

c) If there were 250 boys in Grade 11, estimate how many of them would weigh less than 80 kilogrammes?

Solution:

a) $42 - 28 = 14$

b) There are 50 boys in total, so the median weight is that of the 25^{th} boy. The weight corresponding to a cumulative frequency of 25 is approximately 88 kg.

Note: Accept a range from 86 to 89 kg.

c) 15 boys in the sample have a weight of less than 80 kg. One would expect $\frac{15}{50} \times 250 = 75$ boys in the grade to have a weight of less than 80 kg.

6. Three sets of 12 learners each had their test scores recorded. The test was out of 50. Use the given data to answer the following questions.

Set A	Set B	Set C
25	32	43
47	34	47
15	35	16
17	32	43
16	25	38
26	16	44
24	38	42
27	47	50
22	43	50
24	29	44
12	18	43
31	25	42

a) For each of the sets calculate the mean and the five number summary.

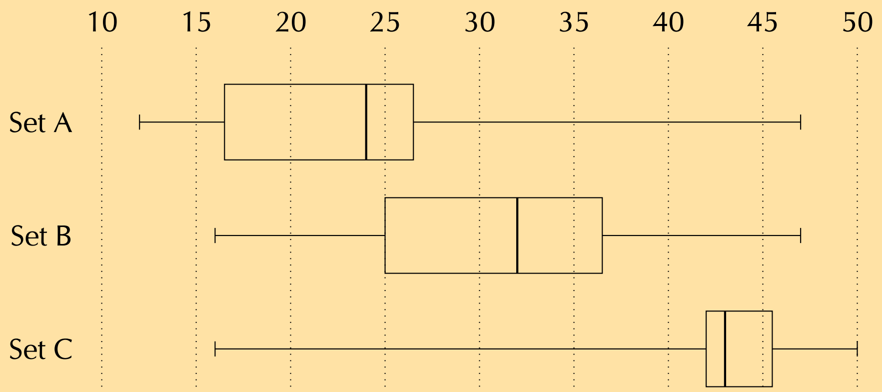
b) Make box and whisker plots of the three data sets on the same set of axes.

c) State, with reasons, whether each of the three data sets are symmetric or skewed (either right or left).

Solution:

a) A. Mean $= 23{,}83$. Five number summary $= [\,12 ; 16{,}5 ; 24 ; 26{,}5 ; 47\,]$.
B. Mean $= 31{,}17$. Five number summary $= [\,16 ; 25 ; 32 ; 36{,}5 ; 47\,]$.
C. Mean $= 41{,}83$. Five number summary $= [\,16 ; 42 ; 43 ; 45{,}5 ; 50\,]$.

b)



c) Set A: skewed left. Set B: slightly skewed left. Set C: skewed right.

CHAPTER 12

Linear programming

12.1 Introduction

- This chapter has been included for enrichment/projects.

Exercise 12 – 1: Optimisation

1. Furniture store opening special:

   As part of their opening special, a furniture store has promised to give away at least 40 prizes with a total value of at least R 4000. They intend to give away kettles and toasters. They decide there will be at least 10 units of each prize. A kettle costs the company R 120 and a toaster costs R 100.

   Determine how many of each prize will represent the cheapest option for the company. Calculate how much this combination of kettles and toasters will cost.

   Use a suitable strategy to organise the information and solve the problem.

   Solution:

   In this situation there are two variables that we need to consider: let the number of kettles produced be K and let the number of toasters produced be T.

   Write down a summary of the information given in the problem so that we consider all the different components in the situation.

minimum number for K	$= 10$
minimum number for T	$= 10$
cost of K	= R 120
cost of T	= R 100
minimum number of prizes	$= 40$
minimum total value of prizes	= R 4000

   Use the summary to draw up a table of the number of kettles and toasters that are needed. We will consider multiples of 5 to simplify the calculations:

K	T				
	10	15	20	25	30
10	$(10;10)$	$(10;15)$	$(10;20)$	$(10;25)$	$(10;30)$
15	$(15;10)$	$(15;15)$	$(15;20)$	$(15;25)$	$(15;30)$
20	$(20;10)$	$(20;15)$	$(20;20)$	$(20;25)$	$(20;30)$
25	$(25;10)$	$(25;15)$	$(25;20)$	$(25;25)$	$(25;30)$
30	$(30;10)$	$(30;15)$	$(30;20)$	$(30;25)$	$(30;30)$

   We need to have at least 40 kettles and toasters together.

   With this limitation, we are able to eliminate some of the combinations in the table where $K + T < 40$:

K	T				
	10	15	20	25	30
10	~~$(10; 10)$~~	~~$(10; 15)$~~	~~$(10; 20)$~~	~~$(10; 25)$~~	$(10; 30)$
15	~~$(15; 10)$~~	~~$(15; 15)$~~	~~$(15; 20)$~~	$(15; 25)$	$(15; 30)$
20	~~$(20; 10)$~~	~~$(20; 15)$~~	$(20; 20)$	$(20; 25)$	$(20; 30)$
25	~~$(25; 10)$~~	$(25; 15)$	$(25; 20)$	$(25; 25)$	$(25; 30)$
30	$(30; 10)$	$(30; 15)$	$(30; 20)$	$(30; 25)$	$(30; 30)$

These combinations have been excluded as possible answers.

We can express the minimum cost as: $C = 120(K) + 100(T)$.

By substituting the different combinations for K and T, we can find the value that gives the minimum cost.

K	T				
	10	15	20	25	30
10	~~$(10; 10)$~~	~~$(10; 15)$~~	~~$(10; 20)$~~	~~$(10; 25)$~~	$(15; 30) \Rightarrow$ R 4200
15	~~$(15; 10)$~~	~~$(15; 15)$~~	~~$(15; 20)$~~	$(15; 25) \Rightarrow$ R 4300	$(15; 30) \Rightarrow$ R 4800
20	~~$(20; 10)$~~	~~$(20; 15)$~~	$(20; 20) \Rightarrow$ R 4400	$(20; 25) \Rightarrow$ R 4900	$(20; 30) \Rightarrow$ R 5400
25	~~$(25; 10)$~~	$(25; 15) \Rightarrow$ R 4500	$(25; 20) \Rightarrow$ R 5000	$(25; 25) \Rightarrow$ R 5500	$(25; 30) \Rightarrow$ R 6000
30	$(30; 10) \Rightarrow$ R 4600	$(30; 15) \Rightarrow$ R 5100	$(30; 20) \Rightarrow$ R 5600	$(30; 25) \Rightarrow$ R 6100	$(30; 30) \Rightarrow$ R 6600

We are looking for a pair of values that give the minimum cost. We can easily check the non-multiples of 5 for kettles and toasters to confirm that 10 kettles and 30 toasters meet the minimum cost.

For example if 11 kettles and 29 toasters are given away, the cost will be R 4220. And if 12 kettles and 28 toasters are given away, the cost will be R 4340.

The minimum cost of R 4000 can be obtained if 10 kettles and 30 toasters are given away. This will result in a cost of R 4200.

Exercise 12 – 2: Optimisation

1. You are given a test consisting of two sections. The first section is on algebra and the second section is on geometry. You are not allowed to answer more than 10 questions from any section, but you have to answer at least 4 algebra questions. The time allowed is not more than 30 minutes. An algebra problem will take 2 minutes and a geometry problem will take 3 minutes to solve.

   Let x be the number of algebra questions and y be the number of geometry questions.

   a) Formulate the equations and inequalities that satisfy the above constraints.
   b) The algebra questions carry 5 marks each and the geometry questions carry 10 marks each. If T is the total marks, write down an expression for T.

Solution:

a) You are not allowed to answer more than 10 questions from any section:

$$x \leq 10$$
$$y \leq 10$$

You have to answer at least 4 algebra questions:

$$x \geq 4$$

The time allowed is not more than 30 minutes. An algebra problem will take 2 minutes and a geometry problem will take 3 minutes to solve:

$$2x + 3y \leq 30$$

b)

$$\begin{aligned} T =& (5 \text{ marks}) \times (\text{algebra questions answered}) + \\ & (10 \text{ marks}) \times (\text{geometry questions answered}) \\ T =& 5x + 10y \end{aligned}$$

2. A local clinic wants to produce a guide to healthy living. The clinic intends to produce the guide in two formats: a short video and a printed book. The clinic needs to decide how many of each format to produce for sale. Estimates show that no more than 10 000 copies of both items together will be sold. At least 4000 copies of the video and at least 2000 copies of the book could be sold, although sales of the book are not expected to exceed 4000 copies. Let x be the number of videos sold, and y the number of printed books sold.

   a) Write down the constraint inequalities that can be deduced from the given information.
   b) Represent these inequalities graphically and indicate the feasible region clearly.
   c) The clinic is seeking to maximise the income, I, earned from the sales of the two products. Each video will sell for R 50 and each book for R 30. Write down the objective function for the income.
   d) What maximum income will be generated by the two guides?

Solution:

a) Estimates show that no more than 10 000 copies of both items together will be sold:

$$x + y \leq 10\ 000$$

At least 4000 copies of the video could be sold:

$$x \geq 4000$$

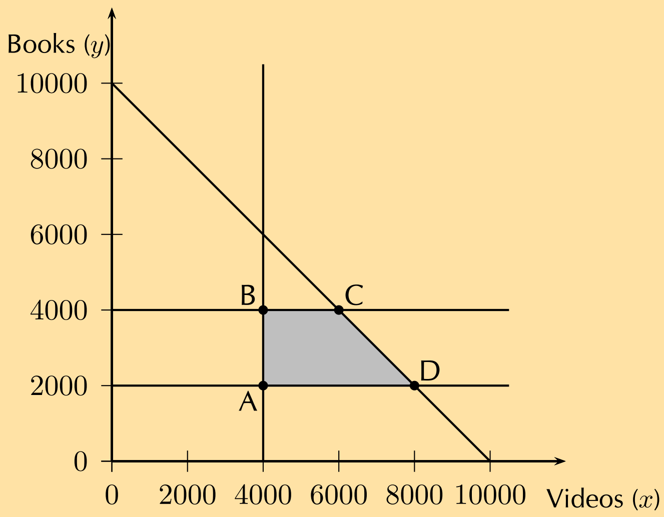
At least 2000 copies of the book could be sold:

$$y \geq 2000$$

Sales of the book are not expected to exceed 4000 copies:

$$y \leq 4000$$

b)



c)

$$\begin{aligned} I =& (\text{R } 50) \times (\text{videos sold}) + \\ & (\text{R } 30) \times (\text{printed books sold}) \\ I =& 50x + 30y \end{aligned}$$

d) The vertices of the feasible region are as follows:

$$\begin{aligned} \text{Point A: } & (4000; 2000) \\ \text{Point B: } & (4000; 4000) \\ \text{Point C: } & (6000; 4000) \\ \text{Point D: } & (8000; 2000) \end{aligned}$$

The cost at each vertex is as follows:

$$\begin{aligned} \text{Point A: } I &= 50(4000) + 30(2000) = 260\ 000 \\ \text{Point B: } I &= 50(4000) + 30(4000) = 320\ 000 \\ \text{Point C: } I &= 50(6000) + 30(4000) = 420\ 000 \\ \text{Point D: } I &= 50(8000) + 30(2000) = 460\ 000 \end{aligned}$$

The maximum profit of R 460 000 can be made if 8000 videos and 2000 books are sold.

3. A certain motorcycle manufacturer produces two basic models, the Super X and the Super Y. These motorcycles are sold to dealers at a profit of R 20 000 per Super X and R 10 000 per Super Y. A Super X requires 150 hours for assembly, 50 hours for painting and finishing and 10 hours for checking and testing. The Super Y requires 60 hours for assembly, 40 hours for painting and finishing and 20 hours for checking and testing. The total number of hours available per month is: $30\ 000$ in the assembly department, $13\ 000$ in the painting and finishing department and 5000 in the checking and testing department.

The above information is summarised by the following table:

Department	Hours for Super X	Hours for Super Y	Hours available per month
Assembly	150	60	$30\ 000$
Painting and finishing	50	40	$13\ 000$
Checking and testing	10	20	5000

Let x be the number of Super X and y be the number of Super Y models manufactured per month.

a) Write down the set of constraint inequalities.

b) Use graph paper to represent the set of constraint inequalities.

c) Shade the feasible region on the graph paper.

d) Write down the profit generated in terms of x and y.

e) How many motorcycles of each model must be produced in order to maximise the monthly profit?

f) What is the maximum monthly profit?

Solution:

a) Adding the assembly hours:

$$(150)x + (60)y \geq 30\ 000$$

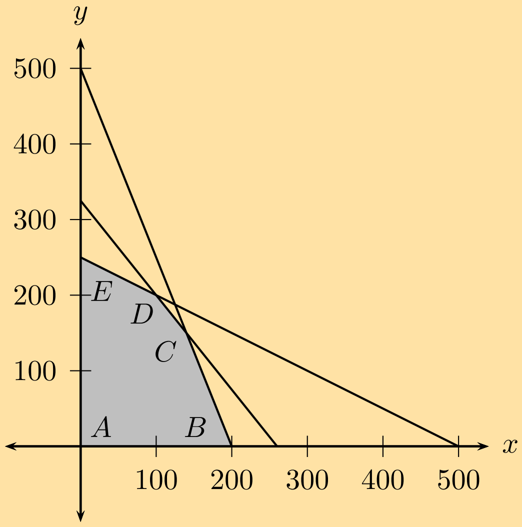
Adding the painting and finishing:

$$(50)x + (40)y \geq 13\ 000$$

Adding the checking and testing:

$$(10)x + (20)y \geq 5000$$

b)



c) See above

d)

$$E = (\text{profit per Super X}) \times (\text{number of Super X bikes sold}) + (\text{profit per Super Y}) \times (\text{number of Super Y bikes sold})$$
$$E = (20\ 000)x + (10\ 000)y$$

e) The vertices of the feasible region are as follows:

Point A: $(0; 0)$
Point B: $(200; 0)$
Point C: $(140; 150)$
Point D: $(100; 200)$
Point E: $(0; 250)$

The cost at each vertex is as follows:

Point A: $E = 20\ 000(0) + 10\ 000(0) = 0$
Point B: $E = 20\ 000(200) + 10\ 000(0) = 4\ 000\ 000$
Point C: $E = 20\ 000(140) + 10\ 000(150) = 4\ 300\ 000$
Point D: $E = 20\ 000(100) + 10\ 000(200) = 4\ 000\ 000$
Point E: $E = 20\ 000(0) + 10\ 000(250) = 2\ 500\ 000$

To maximize the objective function select **Point C**.
The maximum profit can be made if **140** Super X motorbikes and **150** Super Y motorbikes are sold.

f) The maximum possible profit is R 4 300 000.

4. A group of students plan to sell x hamburgers and y chicken burgers at a rugby match. They have meat for at most 300 hamburgers and at most 400 chicken burgers. Each burger of both types is sold in a packet. There are 500 packets available. The demand is likely to be such that the number of chicken burgers sold is at least half the number of hamburgers sold.

   a) Write the constraint inequalities and draw a graph of the feasible region.

b) A profit of R 3 is made on each hamburger sold and R 2 on each chicken burger sold. Write the equation which represents the total profit P in terms of x and y.

c) The objective is to maximise profit. How many of each type of burger should be sold?

Solution:

a) They have meat for at most 300 hamburgers:

$$x \leq 300$$

They have meat for at most 400 chicken burgers:

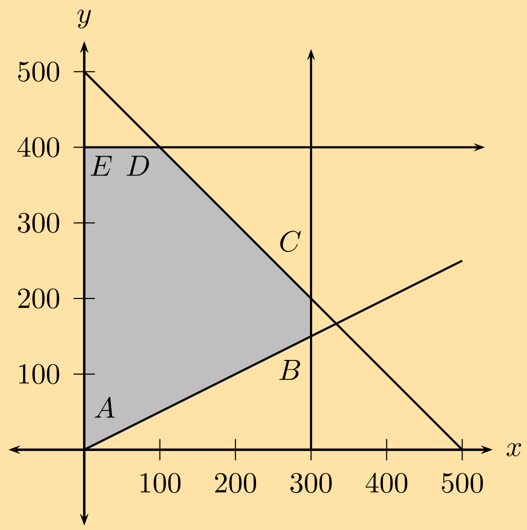
$$y \leq 400$$

The demand is likely to be such that the number of chicken burgers sold is at least half the number of hamburgers sold:

$$y \geq (0.5)x$$

Each burger of both types is sold in a packet. There are 500 packets available:

$$x + y \leq 500$$



b)

$$\begin{aligned} P =& (\text{profit per hamburger}) \times (\text{number of hamburgers sold}) \\ &+ (\text{profit per chicken burger}) \times (\text{number of chicken burgers sold}) \\ P =& (3)x + (2)y \end{aligned}$$

c) The vertices of the feasible region are as follows:

$$\text{Point A: } (0;0)$$
$$\text{Point B: } (300;150)$$
$$\text{Point C: } (300;200)$$
$$\text{Point D: } (100;400)$$
$$\text{Point E: } (0;400)$$

The cost at each vertex is as follows:

$$\text{Point A: } P = 3(0) + 2(0) = 0$$
$$\text{Point B: } P = 3(300) + 2(150) = 1200$$
$$\text{Point C: } P = 3(300) + 2(200) = 1300$$
$$\text{Point D: } P = 3(100) + 2(400) = 1100$$
$$\text{Point E: } P = 3(0) + 2(400) = 800$$

To maximize the objective function select **Point C**.

The maximum possible profit of **R 1300** can be made if 300 hamburgers and 200 chicken burgers are sold.

5. Fashion-Cards is a small company that makes two types of cards, type X and type Y. With the available labour and material, the company can make at most 150 cards of type X and at most 120 cards of type Y per week. Altogether they cannot make more than 200 cards per week.

   There is an order for at least 40 type X cards and 10 type Y cards per week. Fashion-Cards makes a profit of R 5 for each type X card sold and R 10 for each type Y card.

   Let the number of type X cards manufactured per week be x and the number of type Y cards manufactured per week be y.

   a) One of the constraint inequalities which represents the restrictions above is $0 \leq x \leq 150$. Write the other constraint inequalities.
   b) Represent the constraints graphically and shade the feasible region.
   c) Write the equation that represents the profit P (the objective function), in terms of x and y.
   d) Calculate the maximum weekly profit.

   Solution:

   a) The company can not make more than 150 cards of type X per week:

   $$x \leq 150$$

   The company can make not more than 120 cards of type Y per week:

   $$y \leq 120$$

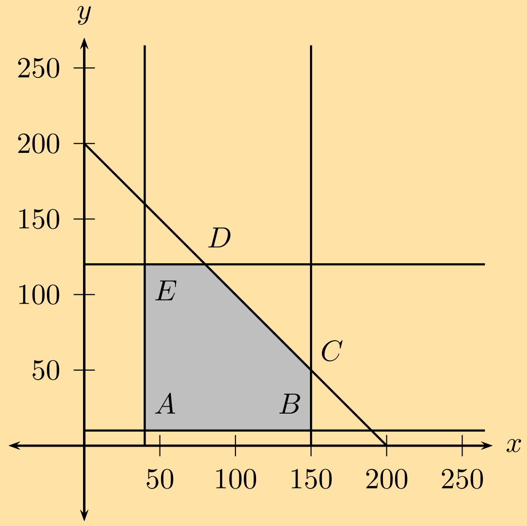
   Altogether they cannot make more than 200 cards per week:

   $$x + y \leq 200$$

   There is an order for at least 40 type X cards and 10 type Y cards per week:

   $$x \geq 40$$
   $$y \geq 10$$

b)



c)

$$\begin{aligned} P =& (\text{profit per card X}) \times (\text{number of card X sold}) \\ & + (\text{profit per card Y}) \times (\text{number of card Y sold}) \\ P =& (5)x + (10)y \end{aligned}$$

d) The vertices of the feasible region are as follows:

Point A: $(40; 10)$
Point B: $(150; 10)$
Point C: $(150; 50)$
Point D: $(80; 120)$
Point E: $(40; 120)$

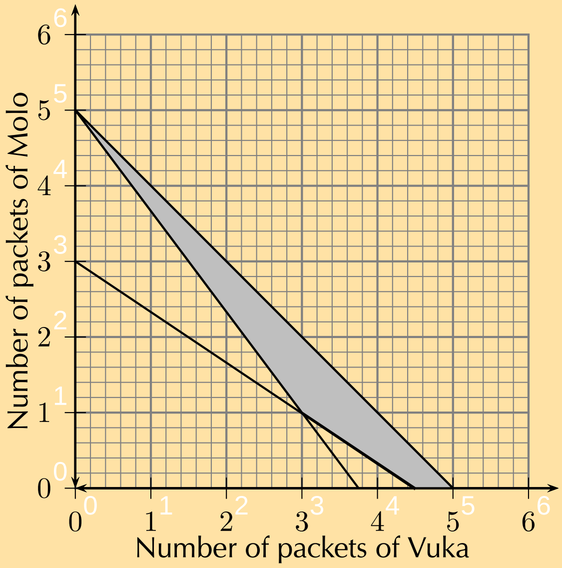
The cost at each vertex is as follows:

Point A: $P = 5(40) + 10(10) = 300$
Point B: $P = 5(150) + 10(10) = 850$
Point C: $P = 5(150) + 10(50) = 1250$
Point D: $P = 5(80) + 10(120) = 1600$
Point E: $P = 5(40) + 10(120) = 1400$

To maximize the objective function select **Point D**.

The maximum possible weekly profit of **R 1600** can be made if 80 of card X and 120 of card Y are sold.

6. To meet the requirements of a specialised diet a meal is prepared by mixing two types of cereal, Vuka and Molo. The mixture must contain x packets of Vuka cereal and y packets of Molo cereal. The meal requires at least 15 g of protein and at least 72 g of carbohydrates. Each packet of Vuka cereal contains 4 g of protein and 16 g of carbohydrates. Each packet of Molo cereal contains 3 g of protein and 24 g of carbohydrates. There are at most 5 packets of cereal available. The feasible region is shaded on the attached graph paper.



a) Write down the constraint inequalities.

b) If Vuka cereal costs R 6 per packet and Molo cereal also costs R 6 per packet, use the graph to determine how many packets of each cereal must be used so that the total cost for the mixture is a minimum.

c) Use the graph to determine how many packets of each cereal must be used so that the total cost for the mixture is a maximum (give all possibilities).

Solution:

a) Adding the protein:

$$(4)x + (3)y \geq 15$$

Adding the carbohydrates:

$$(16)x + (24)y \geq 72$$

There are at most 5 packets of cereal available:

$$x + y \leq 5$$

b) The objective function can be written as follows:

$$P = \text{(profit per packet of Vuka)} \times \text{(number of packets of Vuka)}$$
$$+ \text{(profit per packet of Molo)} \times \text{(number of packets of Molo)}$$
$$P = (6)x + (6)y$$

The vertices of the feasible region are as follows:

Point A: $(3; 1)$

Point B: $(4{,}5; 0)$

Point C: $(5; 0)$

Point D: $(0; 5)$

The cost at each vertex is as follows:

$$\text{Point A: } P = 6(3) + 6(1) = 24$$
$$\text{Point B: } P = 6(4{,}5) + 6(0) = 27$$
$$\text{Point C: } P = 6(5) + 6(0) = 30$$
$$\text{Point D: } P = 6(0) + 6(5) = 30$$

To minimize the objective function select Point A. The minimum possible cost of R 24 can be made if 3 packets of Vuka and 1 packets of Molo are sold.

c) To maximize the objective function select Point C, or Point D. The maximum possible cost of R 30 can be made if either 5 packets of Vuka and 0 packets of Molo are sold, or 0 packets of Vuka and 5 packets of Molo are sold.

VERSION 1 CAPS

GRADE 11

MATHEMATICS

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