

4.8 Numerical Integration

$$1. \Delta x = \frac{1-0}{8} = \frac{1}{8}$$

$$\begin{aligned} & \int_0^1 x^2 e^{-x} dx \\ &= \frac{1}{16} \left[f(0) + 2f\left(\frac{1}{8}\right) + 2f\left(\frac{2}{8}\right) + 2f\left(\frac{3}{8}\right) + 2f\left(\frac{4}{8}\right) + 2f\left(\frac{5}{8}\right) + 2f\left(\frac{6}{8}\right) + 2f\left(\frac{7}{8}\right) + f(1) \right] \\ &= \frac{1}{16} \left[0 + 2\left(\frac{1}{8}\right)^2 e^{-\frac{1}{8}} + 2\left(\frac{2}{8}\right)^2 e^{-\frac{2}{8}} + 2\left(\frac{3}{8}\right)^2 e^{-\frac{3}{8}} + 2\left(\frac{4}{8}\right)^2 e^{-\frac{4}{8}} + 2\left(\frac{5}{8}\right)^2 e^{-\frac{5}{8}} + 2\left(\frac{6}{8}\right)^2 e^{-\frac{6}{8}} + 2\left(\frac{7}{8}\right)^2 e^{-\frac{7}{8}} + (1)^2 e^{-1} \right] \\ &= \frac{1}{16} [0.028 + 0.097 + 0.193 + 0.303 + 0.148 + 0.531 + 0.638 + 0.368] \\ &\approx 0.16 \end{aligned}$$

$$2. \Delta x = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2}$$

$$\begin{aligned} & \int_1^4 \ln \sqrt{x} dx = \frac{1}{4} \left[f(1) + 2f\left(\frac{3}{2}\right) + 2f(2) + 2f\left(\frac{5}{2}\right) + 2f(3) + 2f\left(\frac{7}{2}\right) + f(4) \right] \\ &= \frac{1}{4} \left[0 + 2\ln\left(\sqrt{\frac{3}{2}}\right) + 2\ln(\sqrt{2}) + 2\ln\left(\sqrt{\frac{5}{2}}\right) + 2\ln(\sqrt{3}) + 2\ln\left(\sqrt{\frac{7}{2}}\right) + \ln(\sqrt{4}) \right] \\ &= \frac{1}{4} [0.405 + 0.693 + 0.916 + 1.099 + 1.253 + 0.693] \\ &\approx 1.26 \end{aligned}$$

$$3. \Delta x = \frac{1-0}{4} = \frac{1}{4}$$

$$\begin{aligned} & \int_0^1 \sqrt{1+x^4} dx = \frac{1}{8} \left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) + f(1) \right] \\ &= \frac{1}{8} \left[1 + 2\sqrt{1+\left(\frac{1}{4}\right)^4} + 2\sqrt{1+\left(\frac{1}{2}\right)^4} + 2\sqrt{1+\left(\frac{3}{4}\right)^4} + \sqrt{1+(1)^4} \right] \\ &= \frac{1}{8} [1 + 2.004 + 2.061 + 2.295 + 1.414] \\ &\approx 1.10 \end{aligned}$$

$$4. \Delta x = \frac{3-1}{8} = \frac{2}{8} = \frac{1}{4}$$

$$\begin{aligned}
 \int_1^3 \frac{1}{x} dx &= \frac{1}{8} \left[f(1) + 2f\left(\frac{5}{4}\right) + 2f\left(\frac{6}{4}\right) + 2f\left(\frac{7}{4}\right) + 2f(2) + 2f\left(\frac{9}{4}\right) + 2f\left(\frac{10}{4}\right) + 2f\left(\frac{11}{4}\right) + f(3) \right] \\
 &= \frac{1}{8} \left[1 + 2\left(\frac{4}{5}\right) + 2\left(\frac{4}{6}\right) + 2\left(\frac{4}{7}\right) + 2\left(\frac{1}{2}\right) + 2\left(\frac{4}{9}\right) + 2\left(\frac{4}{10}\right) + 2\left(\frac{4}{11}\right) + \frac{1}{3} \right] \\
 &= \frac{1}{8} [1 + 1.6 + 1.333 + 1.143 + 1 + 0.889 + 0.8 + 0.727 + 0.667] \\
 &\approx 1.10
 \end{aligned}$$

5. Find n such that $|\text{Error}_{\text{Trapezoidal}}| \leq 0.001$. $|f''(x)| = \left|\frac{2}{x^3}\right| \leq 2$ because the maximum value of the second derivative is 2 on the interval $[1, 3]$.

$$\begin{aligned}
 \frac{2(3-1)^3}{12n^2} &\leq 0.001 \\
 \frac{16}{12n^2} &\leq 0.001 \\
 16 &\leq 0.012n^2 \\
 \frac{16}{0.012} &\leq n^2 \\
 \sqrt{\frac{16}{0.012}} &\leq n \\
 36.5 &\leq n
 \end{aligned}$$

Thus, you can choose $n = 37$ in order for the Trapezoidal Estimate to be within 0.001 of the actual integral.

$$6. \Delta x = \frac{1-0}{8} = \frac{1}{8}$$

$$\begin{aligned}
 &\int_0^1 x^2 e^{-x} dx \\
 &= \frac{1}{24} \left[f(0) + 4f\left(\frac{1}{8}\right) + 2f\left(\frac{2}{8}\right) + 4f\left(\frac{3}{8}\right) + 2f\left(\frac{4}{8}\right) + 4f\left(\frac{5}{8}\right) + 2f\left(\frac{6}{8}\right) + 4f\left(\frac{7}{8}\right) + f(1) \right] \\
 &= \frac{1}{24} \left[0 + 4\left(\frac{1}{8}\right)^2 e^{-\frac{1}{8}} + 2\left(\frac{2}{8}\right)^2 e^{-\frac{2}{8}} + 4\left(\frac{3}{8}\right)^2 e^{-\frac{3}{8}} + 2\left(\frac{4}{8}\right)^2 e^{-\frac{4}{8}} + 4\left(\frac{5}{8}\right)^2 e^{-\frac{5}{8}} + 2\left(\frac{6}{8}\right)^2 e^{-\frac{6}{8}} + 4\left(\frac{7}{8}\right)^2 e^{-\frac{7}{8}} + (1)^2 e^{-1} \right] \\
 &= \frac{1}{24} [0.055 + 0.097 + 0.387 + 0.303 + 0.836 + 0.531 + 1.276 + 0.368] \\
 &\approx 0.16
 \end{aligned}$$

$$7. \Delta x = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2}$$

$$\begin{aligned}
 \int_1^4 \sqrt{x} \ln x \, dx &= \frac{1}{6} \left[f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + 2f(4) \right] \\
 &= \frac{1}{6} \left[0 + 4\sqrt{\frac{3}{4}} \ln\left(\frac{3}{2}\right) + 2\sqrt{2} \ln(2) + 4\sqrt{\frac{5}{2}} \ln\left(\frac{5}{2}\right) + 2\sqrt{3} \ln(3) + 4\sqrt{\frac{7}{2}} \ln\left(\frac{7}{2}\right) + \sqrt{4} \ln(4) \right] \\
 &= \frac{1}{6} [0 + 1.986 + 1.961 + 5.795 + 3.805 + 9.374 + 2.773] \\
 &\approx 4.28
 \end{aligned}$$

$$8. \Delta x = \frac{2-0}{6} = \frac{2}{6} = \frac{1}{3}$$

$$\begin{aligned}
 \int_0^2 \sqrt{x^4+1} \, dx &= \frac{1}{9} \left[f(0) + 4f\left(\frac{1}{3}\right) + 2f\left(\frac{2}{3}\right) + 4f(1) + 2f\left(\frac{4}{3}\right) + 4f\left(\frac{5}{3}\right) + f(2) \right] \\
 &= \frac{1}{9} \left[1 + 4\sqrt{\left(\frac{1}{3}\right)^4+1} + 2\sqrt{\left(\frac{2}{3}\right)^4+1} + 4\sqrt{(1)^4+1} + 2\sqrt{\left(\frac{4}{3}\right)^4+1} + 4\sqrt{\left(\frac{5}{3}\right)^4+1} + 1 + \sqrt{(2)^4+1} \right] \\
 &= \frac{1}{9} [1 + 4.025 + 2.189 + 5.657 + 4.080 + 11.809 + 4.123] \\
 &\approx 3.65
 \end{aligned}$$

$$9. \Delta x = \frac{1-0}{4} = \frac{1}{4}$$

$$\begin{aligned}
 \int_0^1 \sqrt{1+x^4} \, dx &= \frac{1}{12} \left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{2}{4}\right) + 4f\left(\frac{3}{4}\right) + f(1) \right] \\
 &= \frac{1}{12} \left[1 + 4\sqrt{1+\left(\frac{1}{4}\right)^4} + 2\sqrt{1+\left(\frac{2}{4}\right)^4} + 4\sqrt{1+\left(\frac{3}{4}\right)^4} + \sqrt{1+(1)^4} \right] \\
 &= \frac{1}{12} [1 + 4.008 + 2.062 + 4.589 + 1.414] \\
 &\approx 1.09
 \end{aligned}$$

10. The problem needs to be fixed. The fourth derivative of e is 0 and thus this problem cannot be solved as it is written.