4.4 Definite Integrals

1. Each sub-interval of $[0,2]$ will have length of $\triangle x = \frac{2-0}{4} = \frac{1}{2}$. The midpoints are $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}$.

$$\begin{aligned} R_4 &= \left(2-\frac{1}{4}\right)\frac{1}{2} + \left(2-\frac{3}{4}\right)\frac{1}{2} + \left(2-\frac{5}{4}\right)\frac{1}{2} + \left(2-\frac{7}{4}\right)\frac{1}{2} \\ &= \left(\frac{7}{4}\right)\frac{1}{2} + \left(\frac{5}{4}\right)\frac{1}{2} + \left(\frac{3}{4}\right)\frac{1}{2} + \left(\frac{1}{4}\right)\frac{1}{2} \\ &= \frac{7+5+3+1}{8} \\ &= 2 \end{aligned}$$

2. From 0 to 2, the line forms a triangle with the $x-$axis from $(0,2)$ to $(2,0)$. The height of the triangle is 2. The width is $2-0=2$. The area of the triangle is $\frac{1}{2}(2)(2) = 2$.

3. Use the right endpoints to compute the integral. Divide $[0,2]$ into n subintervals of length $\triangle x = \frac{2-0}{n} = \frac{2}{n}$. Use the right endpoints: $\frac{2}{n}, \frac{4}{n}, K, \frac{2i}{n}, K, \frac{2n}{n}$.

$$\begin{aligned} \int_0^2 (2-x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) \\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(2-\frac{2i}{n}\right)\left(\frac{2}{n}\right) \\ &= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{4}{n} - \sum_{i=1}^{n}\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)\right] \\ &= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{4}{n} - \sum_{i=1}^{n}\left(\frac{4i}{n}\right)\right] \\ &= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{4}{n} - \frac{4}{n^2}\sum_{i=1}^{n} i\right] \\ &= \lim_{n\to\infty}\left[\frac{4n}{n} - \frac{4}{n^2}\frac{n^2+n}{2}\right] \\ &= \lim_{n\to\infty}\left[4-2-\frac{2}{n}\right] = \lim_{n\to\infty}\left[2-\frac{2}{n}\right] = 2 \end{aligned}$$

4. Each sub-interval of $[1,4]$ will have length of $\triangle x = \frac{4-1}{5} = \frac{3}{5}$. The left endpoints are $1, 1+\frac{3}{5} = \frac{8}{5}, 1+\frac{6}{5} = \frac{11}{5}, 1+\frac{9}{5} = \frac{14}{5}, 1+\frac{12}{5} = \frac{17}{5}$.

$$R_5 = \left(1^2-1\right)\frac{3}{5}+\left(\left(\frac{8}{5}\right)^2-\frac{8}{5}\right)\frac{3}{5}+\left(\left(\frac{11}{5}\right)^2-\frac{11}{5}\right)\frac{3}{5}+\left(\left(\frac{14}{5}\right)^2-\frac{14}{5}\right)\frac{3}{5}+\left(\left(\frac{17}{5}\right)^2-\frac{17}{5}\right)\frac{3}{5}$$
$$= 0+(2.56-1.6)\frac{3}{5}+(4.84-2.2)\frac{3}{5}+(7.84-2.8)\frac{3}{5}+(11.56-3.4)\frac{3}{5}$$
$$= 0.576+1.584+3.024+4.896$$
$$= 10.08$$

5. $\int\limits_1^4 \left(x^2-x\right)dx \lim_{n\to\infty}\sum_{i=1}^n f\left(1+\frac{3i}{n}\right)\left(\frac{3}{n}\right)$

$$= \lim_{n\to\infty}\sum_{i=1}^n\left[\left(1+\frac{3i}{n}\right)^2-\left(1+\frac{3i}{n}\right)\right]\left(\frac{3}{n}\right)$$
$$= \lim_{n\to\infty}\sum_{i=1}^n\left(1+\frac{6i}{n}+\frac{9i^2}{n^2}-1-\frac{3i}{n}\right)\left(\frac{3}{n}\right)$$
$$= \lim_{n\to\infty}\sum_{i=1}^n\left(\frac{3i}{n}+\frac{9i^2}{n^2}\right)\left(\frac{3}{n}\right)$$
$$= \lim_{n\to\infty}\left(\frac{3}{n}\right)\left[\sum_{i=1}^n\frac{3i}{n}+\sum_{i=1}^n i\frac{9i^2}{n^2}\right]$$
$$= \lim_{n\to\infty}\left(\frac{3}{n}\right)\left[\frac{3}{n}\sum_{i=1}^n+\frac{9}{n^2}\sum_{i=1}^n i^2\right]$$
$$= \lim_{n\to\infty}\left[\left(\frac{9}{n^2}\right)\left(\frac{n^2+n}{2}\right)+\left(\frac{27}{n^3}\right)\left(\frac{2n^3+3n^2+n}{6}\right)\right]$$
$$= \lim_{n\to\infty}\left[\frac{9n^2+9n}{2n^2}+\left(\frac{18n^3+27n^2+9n}{2n^3}\right)\right]$$
$$= \lim_{n\to\infty}\left[\frac{9}{2}+\frac{9}{2n}+9-\frac{27}{2n^3}+\frac{9}{2n^2}\right]$$
$$= 13.5$$

6. a. Each sub-interval of $[0,1]$ will have length of $\triangle x = \frac{1-0}{2} = \frac{1}{2}$. The right endpoints are $\frac{1}{2}, 1$.

$$R_2 = \left(3\left(\frac{1}{2}\right)^2\right)\frac{1}{2}+\left(3(1)^2\right)\frac{1}{2}$$
$$= \left(\frac{3}{4}\right)\frac{1}{2}+(3)\frac{1}{2}$$
$$= \frac{3}{8}+\frac{3}{2}$$
$$= 1.875$$

b. Each sub-interval of $[0,1]$ will have length of $\triangle x = \frac{1-0}{5} = \frac{1}{5}$. The right endpoints are $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$.

$$\begin{aligned} R_5 &= \left(3\left(\frac{1}{5}\right)^2\right)\frac{1}{5}+\left(3\left(\frac{2}{5}\right)^2\right)\frac{1}{5}+\left(3\left(\frac{3}{5}\right)^2\right)\frac{1}{5}+\left(3\left(\frac{4}{5}\right)^2\right)\frac{1}{5}+\left(3\,(1)^2\right)\frac{1}{5} \\ &= \frac{3}{5}\left(\frac{1+4+9+16}{25}+1\right) \\ &= \frac{3}{5}(2.2) \\ &= 1.32 \end{aligned}$$

c. Each sub-interval of $[0,1]$ will have length of $\triangle x = \frac{1-0}{10} = \frac{1}{10}$. The right endpoints are $\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}, \frac{9}{10}, 1$.

$$\begin{aligned} R_{10} &= \left(3\left(\frac{1}{10}\right)^2\right)\frac{1}{10}+\left(3\left(\frac{2}{10}\right)^2\right)\frac{1}{10}+\left(3\left(\frac{3}{10}\right)^2\right)\frac{1}{10}+\left(3\left(\frac{4}{10}\right)^2\right)\frac{1}{10}+\left(3\left(\frac{5}{10}\right)^2\right)\frac{1}{10} \\ &+\left(3\left(\frac{6}{10}\right)^2\right)\frac{1}{10}+\left(3\left(\frac{7}{10}\right)^2\right)\frac{1}{10}+\left(3\left(\frac{8}{10}\right)^2\right)\frac{1}{10}+\left(3\left(\frac{9}{10}\right)^2\right)\frac{1}{10}+\left(3\,(1)^2\right)\frac{1}{10} \\ &= \frac{3}{10}\left(\frac{1+4+9+16+25+36+49+64+81}{100}+1\right) \\ &= \frac{3}{10}\left(\frac{285}{100}+1\right) \\ &= 1.15 \end{aligned}$$

d. The area should be 1.

7. a. Each sub-interval of $[0,1]$ will have length of $\triangle x = \frac{1-0}{2} = \frac{1}{2}$. The right endpoints are $\frac{1}{2}, 1$.

$$\begin{aligned} R_2 &= \left(e^{0.5}\right)\frac{1}{2}+\left(e^{1}\right)\frac{1}{2} \\ &= (1.649)\frac{1}{2}+(2.718)\frac{1}{2} \\ &= 2.18 \end{aligned}$$

b. Each sub-interval of $[0,1]$ will have length of $\triangle x = \frac{1-0}{5} = \frac{1}{5}$. The right endpoints are $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$.

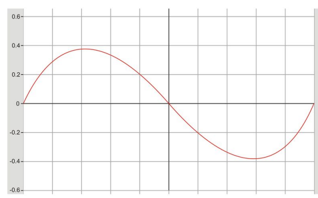
$$\begin{aligned} R_5 &= \left(e^{\frac{1}{5}}\right)\frac{1}{5}+\left(e^{\frac{2}{5}}\right)\frac{1}{5}+\left(e^{\frac{3}{5}}\right)\frac{1}{5}+\left(e^{\frac{4}{5}}\right)\frac{1}{5}+\left(e^{1}\right)\frac{1}{5} \\ &= \frac{1}{5}(1.221+1.492+1.822+2.226+2.718) \\ &= \frac{1}{5}(9.479) \\ &= 1.896 \end{aligned}$$

c. Each sub-interval of $[0,1]$ will have length of $\triangle x = \frac{1-0}{10} = \frac{1}{10}$. The right endpoints are $\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}, \frac{9}{10}, 1$
.

$$R_{10} = \left(e^{\frac{1}{10}}\right)\frac{1}{10} + \left(e^{\frac{2}{10}}\right)\frac{1}{10} + \left(e^{\frac{3}{10}}\right)\frac{1}{10} + \left(e^{\frac{4}{10}}\right)\frac{1}{10} + \left(e^{\frac{5}{10}}\right)\frac{1}{10}$$
$$+ \left(e^{\frac{6}{10}}\right)\frac{1}{10} + \left(e^{\frac{7}{10}}\right)\frac{1}{10} + \left(e^{\frac{8}{10}}\right)\frac{1}{10} + \left(e^{\frac{9}{10}}\right)\frac{1}{10} + \left(e^{1}\right)\frac{1}{10}$$
$$= \frac{1}{10}(1.105 + 1.221 + 1.35 + 1.492 + 1.649 + 1.822 + 2.014 + 2.226 + 2.46 + 2.718)$$
$$= \frac{1}{10}(18.06)$$
$$= 1.81$$

d. The area should be $e - 1 \approx 1.7$.

8.



The graph is symmetric around the origin. The area is 0.

9.

$$2\int_{-1}^{0}\left(x^3 - x\right)dx = 2\left[\int_{-1}^{0} x^3\,dx - \int_{-1}^{0} x\,dx\right]$$
$$= 2\left[\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{(-i)^3}{n^3}\right)\frac{1}{n} - \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{-i}{n}\right)\frac{1}{n}\right]$$
$$= 2\left[\lim_{n\to\infty}\left(-\frac{1}{n^4}\sum_{i=1}^{n} i^3 + \frac{1}{n^2}\sum_{i=1}^{n} i\right)\right]$$
$$= 2\left[\lim_{n\to\infty}\left(-\frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2 + \frac{1}{n^2}\left(\frac{n(n+1)}{2}\right)\right)\right]$$
$$= 2\left[\lim_{n\to\infty}\left(\left(-\frac{1}{n^2}\times\frac{n^2+2n+1}{4}\right) + \frac{n+1}{2n}\right)\right]$$
$$= 2\left[\lim_{n\to\infty}\left(-\frac{1}{4} - \frac{1}{2n} - \frac{1}{4n^2} + \frac{1}{2} + \frac{1}{2n}\right)\right]$$
$$= 2\left(-\frac{1}{4} + \frac{1}{2}\right)$$
$$= 2\left(\frac{1}{4}\right) = \frac{1}{2}$$

10.

$$
\begin{aligned}
y &= \sqrt{9-x^2} \\
y^2 &= 9-x^2 \\
x^2+y^2 &= 9 \\
x^2+y^2 &= 3^2
\end{aligned}
$$

From $x=0$ to $x=3, y=\sqrt{9-x^2}$ is a quarter of the circle around the origin with radius 3. The area is $\frac{\pi r^2}{4}=\frac{9\pi}{4}$.