5.4 Area of a Surface of a Revolution

1.

$$\begin{aligned} S &= \int_0^1 2\pi(3x)\sqrt{1+3^2}dx \\ &= 6\pi\int_0^1 x\sqrt{10}dx \\ &= 6\pi\left(\frac{\sqrt{10}x}{2}\right)\Bigg|_0^1 \\ &= 3\pi\sqrt{10} \end{aligned}$$

2.

$$\begin{aligned} S &= \int_1^9 2\pi\left(\sqrt{x}\right)\sqrt{1+\left(\frac{1}{2}x^{-\frac{1}{2}}\right)^2}dx \\ &= \int_1^9 2\pi\sqrt{x+\frac{x}{4x}}dx \\ &= 2\pi\int_1^9 \sqrt{x+\frac{1}{4}}dx \\ &= 2\pi\frac{(x+\frac{1}{4})^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_1^9 \\ &= \frac{4}{3}\pi\left[\left(9+\frac{1}{4}\right)^{\frac{3}{2}}-\left(1+\frac{1}{4}\right)^{\frac{3}{2}}\right] \\ &= \frac{4}{3}\pi\left[\left(\frac{37}{4}\right)^{\frac{3}{2}}-\left(\frac{5}{4}\right)^{\frac{3}{2}}\right] \\ &\approx 112 \end{aligned}$$

3.

$$\frac{d}{dx}[(4-x^2)^{\frac{1}{2}}] = \frac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x) = -\frac{x}{\sqrt{4-x^2}}$$

$$\begin{aligned}
S &= \int_{-1}^{1} 2\pi\left(\sqrt{4-x^2}\right)\sqrt{1+\left(-\frac{x}{\sqrt{4-x^2}}\right)^2}dx \\
&= \int_{-1}^{1} 2\pi\left(\sqrt{4-x^2}\right)\sqrt{1+\frac{x^2}{4-x^2}dx} \\
&= \int_{-1}^{1} 2\pi\left(\sqrt{4-x^2}\right)\sqrt{\frac{4-x^2}{4-x^2}+\frac{x^2}{4-x^2}dx} \\
&= \int_{-1}^{1} 2\pi\left(\sqrt{4-x^2}\right)\sqrt{\frac{4}{4-x^2}dx} \\
&= \int_{-1}^{1} 2\pi(2)dx \\
&= 4\pi x|_{-1}^{1} \\
&= 4\pi+4\pi \\
&= 8\pi
\end{aligned}$$

4.

$$\begin{aligned}
S &= \int_{0}^{3} 2\pi(7y+2)\sqrt{1+7^2}dy \\
&= 2\pi\sqrt{50}\int_{0}^{3}(7y+2)dy \\
&= 2\pi\sqrt{50}\left[\frac{7y^2}{2}+2y\right]_0^3 \\
&= 2\pi\sqrt{50}\left(\frac{63}{2}+6\right) \\
&= 75\pi\sqrt{50}
\end{aligned}$$

5.

$$\begin{aligned}
S &= \int_{0}^{8} 2\pi(y^3)\sqrt{1+(3y^2)^2}dy \\
&= 2\pi\int_{0}^{8}(y^3)\sqrt{1+9y^4}dy
\end{aligned}$$

Let

$$
\begin{aligned}
u &= 1+9y^4 \\
\frac{du}{dy} &= 36y^3 \\
\frac{1}{36}du &= y^3\ dy \\
x &= 0, u = 1 \\
x &= 8, u = 36,865
\end{aligned}
$$

Then

$$
\begin{aligned}
S &= \frac{2\pi}{36}\int_1^{36.825} u^{\frac{1}{2}} du \\
&= \frac{2\pi}{36}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_1^{36.825} \\
&= \frac{4\pi}{108}[(36.825)^{\frac{3}{2}} - 1] \\
&\approx 82,358.05
\end{aligned}
$$

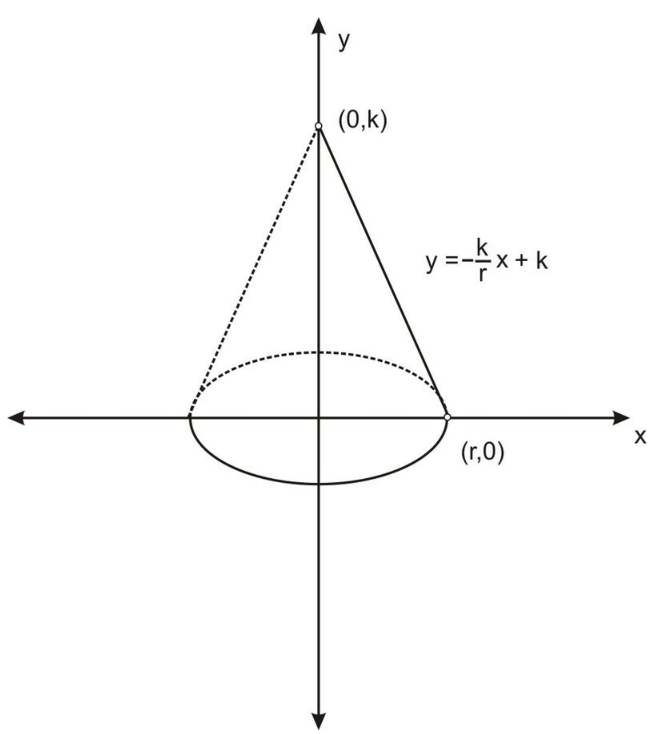
6.

$$
\begin{aligned}
\frac{d}{dy}[(9-y^2)^{\frac{1}{2}}] &= \frac{1}{2}(9-y^2)^{-\frac{1}{2}}(-2y) = -\frac{y}{\sqrt{9-y^2}} \\
S &= \int_{-2}^{2} 2\pi\left(\sqrt{9-y^2}\right)\sqrt{1+\left(-\frac{y}{\sqrt{9-y^2}}\right)^2}dy \\
&= \int_{-2}^{2} 2\pi\left(\sqrt{9-y^2}\right)\sqrt{1+\frac{y^2}{9-y^2}dy} \\
&= 2\pi\int_{-2}^{2} 2\pi\left(\sqrt{9-y^2}\right)\sqrt{1+\frac{y^2}{9-y^2}dy} \\
&= 2\pi\int_{-2}^{2} 2\pi\left(\sqrt{9-y^2}\right)\sqrt{\frac{9-y^2}{9-y^2}+\frac{y^2}{9-y^2}dy} \\
&= 2\pi\int_{-2}^{2} 2\pi\left(\sqrt{9-y^2}\right)\sqrt{\frac{9}{9-y^2}dy} \\
&= 2\pi\int_{-2}^{2} \sqrt{9}dy \\
&= 6\pi(y)|_{-2}^{2} \\
&= 12\pi + 12\pi = 24\pi
\end{aligned}
$$

7. Assume that the half circle is revolved around the $y-$axis. The proof that the surface area is $4\pi r^2$ for a half circle revolved around the $x-$axis is similar.

$$\begin{aligned}
S &= \int_{-r}^{r} 2\pi\left(\sqrt{r^2-y^2}\right)\sqrt{1+\left(-\frac{y}{\sqrt{r^2-y^2}}\right)^2}\,dy \\
&= \int_{-r}^{r} 2\pi\left(\sqrt{r^2-y^2}\right)\sqrt{1+\frac{y^2}{r^2-y^2}}\,dy \\
&= 2\pi\int_{-r}^{r} \left(\sqrt{r^2-y^2}\right)\sqrt{1+\frac{y^2}{r^2-y^2}}\,dy \\
&= 2\pi\int_{-r}^{r} \left(\sqrt{r^2-y^2}\right)\sqrt{\frac{r^2-y^2}{r^2-y^2}+\frac{y^2}{r^2-y^2}}\,dy \\
&= 2\pi\int_{-r}^{r} \left(\sqrt{r^2-y^2}\right)\sqrt{\frac{r^2}{r^2-y^2}}\,dy \\
&= 2\pi\int_{-r}^{r} \sqrt{r^2}\,dy \\
&= 2\pi r(y)|_{-r}^{r} \\
&= 2\pi r^2+2\pi r^2 = 4\pi r^2
\end{aligned}$$

8.



$$\begin{aligned}f(x) &= y = -\frac{k}{r}x + k \\ f'(x) &= -\frac{k}{r}\end{aligned}$$

$$\begin{aligned}S &= 2\pi \int_0^r x\sqrt{1 + |f'(x)|^2}dx \\ &= 2\pi \int_0^r x\sqrt{1 + \left(-\frac{k}{r}\right)^2}dx \\ &= 2\pi \int_0^r x\sqrt{1 + \frac{k^2}{r^2}}dx \\ &= 2\pi\sqrt{1 + \frac{k^2}{r^2}}\int_0^r x\,dx \\ &= 2\pi\sqrt{1 + \frac{k^2}{r^2}}\left(\frac{x^2}{2}\right)\Bigg|_0^r \\ &= 2\pi\sqrt{1 + \frac{k^2}{r^2}}\left(\frac{r^2}{2}\right) \\ &= \pi r^2\sqrt{\frac{r^2 + k^2}{r^2}} \\ &= \pi r\sqrt{r^2 + k^2}\end{aligned}$$