5.3 The Length of a Plane Curve

1.

$$
\begin{aligned}
f'(x) &= \frac{1}{3} \times \frac{3}{2}(x^2+2)^{\frac{1}{2}}(2x) = x\sqrt{x^2+2} \\
L &= \int_0^3 \sqrt{1+\left(x\sqrt{x^2+2}\right)^2}\,dx \\
&= \int_0^3 \sqrt{1+x^2(x^2+2)}\,dx \\
&= \int_0^3 \sqrt{1+2x^2+x^4}\,dx \\
&= \int_0^3 \sqrt{(1+x^2)^2}\,dx \\
&= \int_0^3 (1+x^2)\,dx \\
&= \left(x+\frac{x^3}{3}\right)\Big|_0^3 \\
&= 3+9 \\
&= 12
\end{aligned}
$$

2.

$$
\begin{aligned}
y(x) &= \frac{1}{6}y^3 + \frac{1}{2}y^{-1} \\
y'(x) &= \frac{1}{2}y^2 - \frac{1}{2}y^{-2} \\
L &= \int_1^2 \sqrt{1+\left(\frac{1}{2}y^2 - \frac{1}{2y^2}\right)^2} dy \\
&= \int_1^2 \sqrt{1+\frac{1}{4}y^4 - \frac{1}{2} + \frac{1}{2}y^{-4}} dy \\
&= \int_1^2 \sqrt{\frac{1}{2}+\frac{1}{4}y^4 + \frac{1}{4y^4}} dy \\
&= \int_1^2 \sqrt{\left(\frac{1}{2}y^2 + \frac{1}{2y^2}\right)^2} dy \\
&= \int_1^2 \left(\frac{1}{2}y^2 + \frac{1}{2y^2}\right) dy \\
&= \left(\frac{y^3}{6} - \frac{1}{2y}\right)\Bigg|_1^2 \\
&= \frac{8}{6} - \frac{1}{4} - \frac{1}{6} + \frac{1}{2} \\
&= \frac{17}{12}
\end{aligned}
$$

3. $x = \int_0^y \sqrt{\sec^4 t - 1} dt$

By the Fundamental Theorem of Calculus, $\frac{dx}{dy} = \sqrt{\sec^4 y - 1}$.

$$\begin{aligned}
L &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{1+\left(\sqrt{\sec^4 y-1}\right)^2}dy \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{1+\sec^4 y-1}dy \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\sec^4 y}dy \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 y\, dy \\
&= \tan y|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
&= \tan\left(\frac{\pi}{4}\right)-\tan\left(-\frac{\pi}{4}\right) \\
&= 1-(-1) \\
&= 2
\end{aligned}$$

4.

$$\begin{aligned}
x^{\frac{2}{3}}+y^{\frac{2}{3}} &= 1 \\
y &= \left(1-x^{\frac{2}{3}}\right)^{\frac{3}{2}} \\
\frac{dy}{dx} &= x^{-\frac{1}{3}}\left(1-x^{\frac{2}{3}}\right)^{\frac{1}{2}}
\end{aligned}$$

Using symmetry,

$$\begin{aligned}
L &= 4\int_0^1 \left[1+\left(x^{-\frac{1}{3}}\left(1-x^{\frac{2}{3}}\right)^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}} dx \\
&= 4\int_0^1 \left[1+x^{-\frac{2}{3}}\left(1-x^{\frac{2}{3}}\right)\right]^{\frac{1}{2}} dx \\
&= 4\int_0^1 \left(x^{-\frac{2}{3}}\right)^{\frac{1}{2}} dx \\
&= 4\int_0^1 \left(x^{-\frac{1}{3}}\right) dx \\
&= 4\times\frac{3}{2}x^{\frac{2}{3}}\Big|_0^1 \\
&= 6
\end{aligned}$$

5. $y = kx^2$

Let $x = S$.

Then

$y = h$ and

$$\begin{aligned} h &= kS^2 \\ \frac{h}{S^2} &= k \\ \frac{dy}{dx} &= 2kx \\ &= 2\left(\frac{h}{S^2}\right)x \end{aligned}$$

Then

$$\begin{aligned} L &= \int_{-S}^{S} \sqrt{1+(2kx)^2}dx \\ &= 2\int_{0}^{S} \sqrt{1+4\frac{h^2}{S^4}x^2dx} \end{aligned}$$