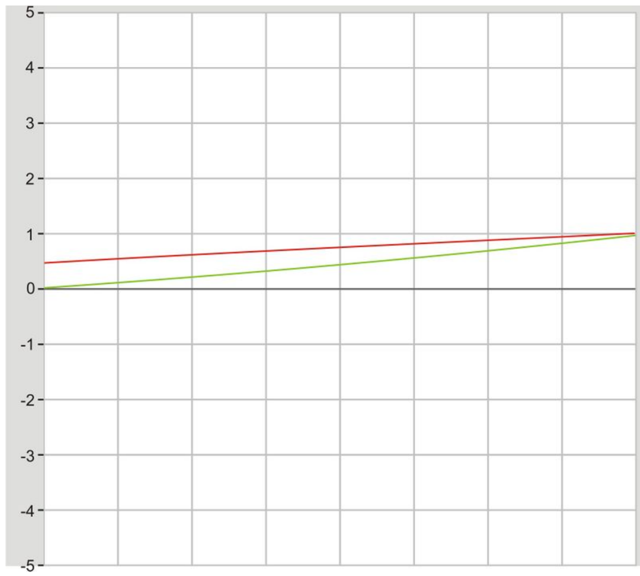
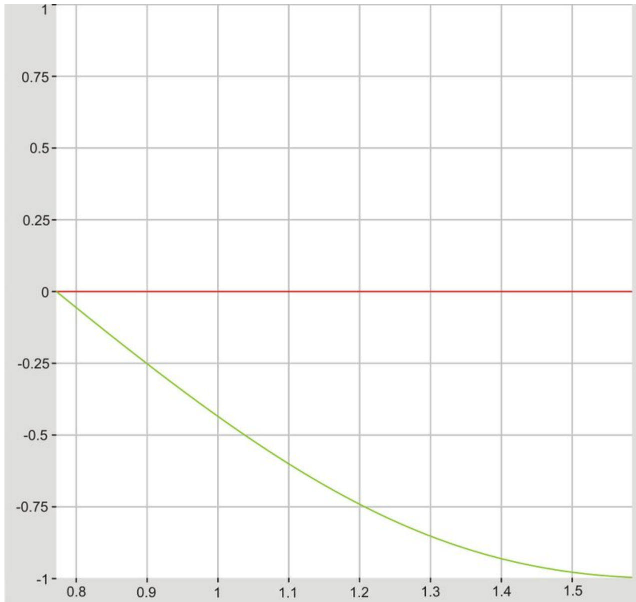
5.1 Area Between Two Curves

1.



$$
\begin{aligned}
\text{Area} &= \int_{0.25}^{1} \left(\sqrt{x} - x^2\right) dx \\
&= \int_{0.25}^{1} x^{\frac{1}{2}} dx - \int_{0.25}^{1} x^2 dx \\
&= \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3}\right)\Bigg|_{0.25}^{1} \\
&= \left(\frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3}\right)\Bigg|_{0.25}^{1} \\
&= \left(\frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3}\right)\Bigg|_{\frac{1}{4}}^{1} \\
&= \frac{2}{3} - \frac{1}{3} - \left(\frac{2\left(\sqrt{\frac{1}{4}}\right)^3}{3} - \frac{\left(\frac{1}{4}\right)^3}{3}\right) \\
&= \frac{1}{3} - \frac{2}{24} + \frac{1}{192} \\
&= \frac{64 - 16 + 1}{192} \\
&= \frac{49}{192}
\end{aligned}
$$

2.



$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (0 - \cos\ 2x)dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos\ 2x\ dx$$

Let $u = 2x$.

Then $du = 2dx$

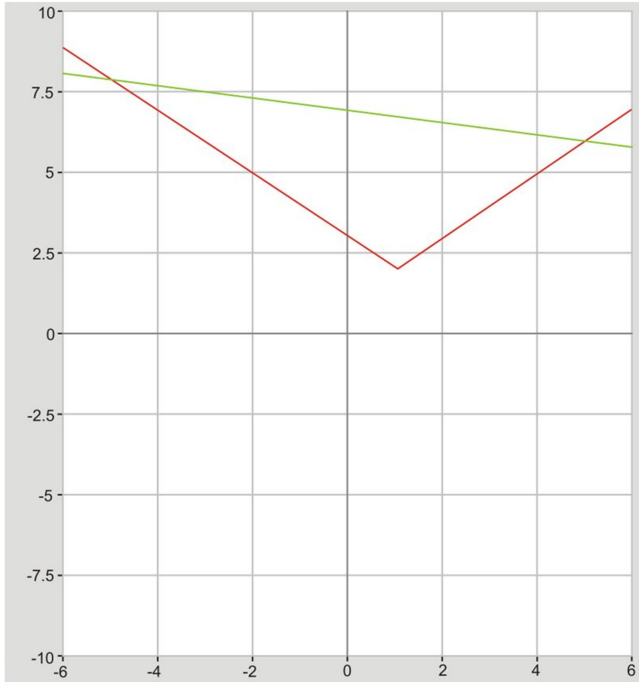
$$\frac{1}{2}du = dx$$

Integrating, $\frac{1}{2}\int \cos u\ du = \frac{1}{2}\sin u$.

Then

$$\begin{aligned}
\text{Area} &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos\ 2x\ dx \\
&= \frac{1}{2}\sin 2x|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= \frac{1}{2}\left(\sin\pi - \sin\frac{\pi}{2}\right) \\
&= \frac{1}{2}(0-(1)) \\
&= -\frac{1}{2}
\end{aligned}$$

3.



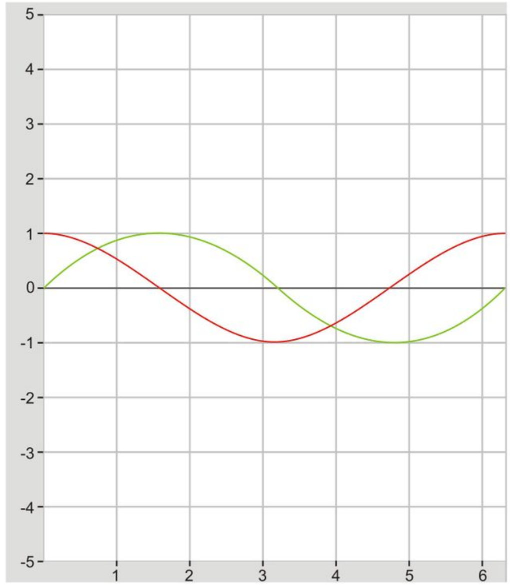
When $x < 1$, then $|-1+x|+2 = 1-x+2 = -x+3$.

When $x > 1$, then $|-1+x|+2 = -1+x+2 = x+1$.

Then

$$\begin{aligned}
\text{Area} &= \int_{-5}^{1} \left[-\frac{1}{5}x+7-(-x+3)\right] dx + \int_{1}^{5} \left[-\frac{1}{5}x+7-(x+1)\right] dx \\
&= \int_{-5}^{1} \left(\frac{4}{5}x+4\right) dx + \int_{1}^{5} \left(-\frac{6}{5}x+6\right) dx \\
&= \left(\frac{4}{10}x^2+4x\right)\Bigg|_{-5}^{1} + \left(-\frac{6}{10}x^2+6x\right)\Bigg|_{1}^{5} \\
&= \frac{4}{10}+4-10+20-15+30+\frac{6}{10}-6 \\
&= 24
\end{aligned}$$

4.



$$\text{Area} = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)dx + \int_{\frac{5\pi}{4}}^{2\pi} (\cos x - \sin x)dx$$

$$= [\sin x - (-cosc)]|_0^{\frac{\pi}{4}} + [-\cos x - \sin x]|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} + [\sin x - (-\cos x)]|_{\frac{5\pi}{4}}^{2\pi}$$

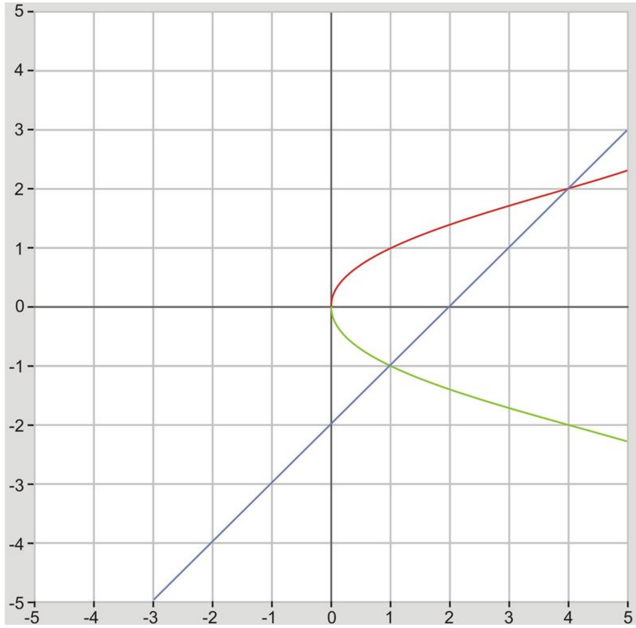
$$-\sin\frac{\pi}{4} + \cos\frac{\pi}{4} - \sin 0 - \cos 0 - \cos\frac{5\pi}{4} - \sin\frac{5\pi}{4} + \cos\frac{\pi}{4} + \sin\frac{\pi}{4} + \sin 2\pi + \cos 2\pi - \sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1 - \left(-\frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 0 + 1 - \left(-\frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right)$$

$$= \frac{8\sqrt{2}}{2} = 4\sqrt{2}$$

5.

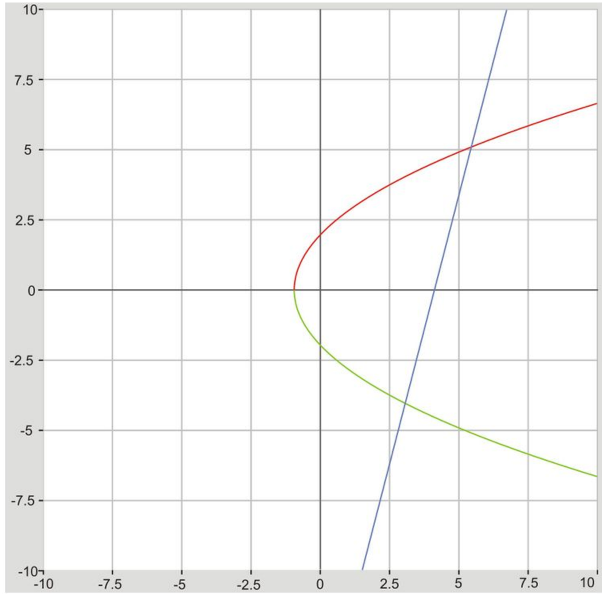
$$x = y^2$$
$$y = x - 2$$
$$y + 2 = x$$



$$\begin{aligned}
\text{Area} &= \int_{-1}^{2} y+2-(y^2)dy \\
&= \left(\frac{y^2}{2}+2y-\frac{y^3}{2}\right)\Big|_{-1}^{2} \\
&= \frac{4}{2}+4-\frac{8}{3}-\left(\frac{1}{2}-2+\frac{1}{3}\right) \\
&= 2+4-\frac{8}{3}-\frac{1}{2}+2-\frac{1}{3} \\
&= \frac{9}{2}
\end{aligned}$$

6.

$$\begin{aligned}
y^2 &= 4x+4 \\
y^2-4 &= 4x \\
\frac{1}{4}y^2-1 &= x \\
4x &= 16+y \\
x &= 4+\frac{1}{4}y
\end{aligned}$$

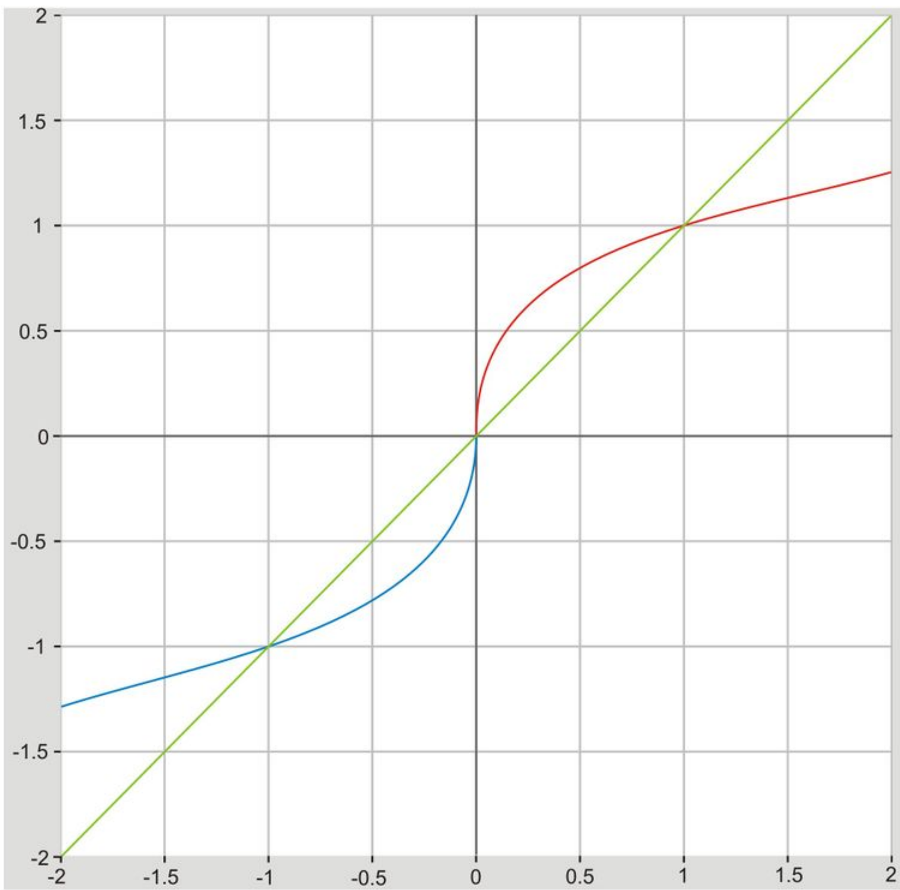


$$
\begin{aligned}
\text{Area} &= \int_{-4}^{5} \left[4 + \frac{1}{4}y - \left(\frac{1}{4}y^2 - 1\right)\right] dy \\
&= \int_{-4}^{5} \left(5 + \frac{1}{4}y - \frac{1}{4}y^2\right) dy \\
&= \left[5y + \frac{1}{8}y - \frac{1}{12}y^3\right]\Bigg|_{-4}^{5} \\
&= 25 + \frac{5}{8} - \frac{125}{12} - \left(-20 - \frac{4}{8} + \frac{64}{12}\right) \\
&= \frac{243}{8} = 30\frac{3}{8}
\end{aligned}
$$

7.

$$\begin{aligned}\text{Area} &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (8\cos x - \sec^2 x)dx + \\ &= [8\sin x - \tan x]|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \\ &= 8\sin\frac{\pi}{3} - \tan\frac{\pi}{3} - 8\sin\left(-\frac{\pi}{3}\right) + \tan\left(-\frac{\pi}{3}\right) \\ &= 8\left(\frac{\sqrt{3}}{2}\right) - \sqrt{3} - 8\left(-\frac{\sqrt{3}}{2}\right) - \sqrt{3} \\ &= \frac{12\sqrt{3}}{2} = 6\sqrt{3}\end{aligned}$$

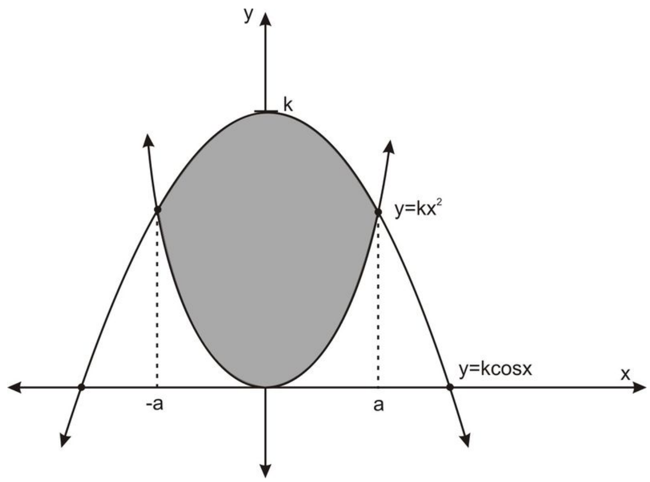
8.



$$\begin{aligned}\text{Area} &= \int_{0}^{1} (y - y^3)dy \\ &= \left[\frac{y^2}{2} - \frac{y^4}{4}\right]\Big|_0^1 \\ &= \frac{1}{4}\end{aligned}$$

The area from $y = -1$ to $y = 1$ is $2\left(\frac{1}{4}\right) = \frac{1}{2}$.

9.



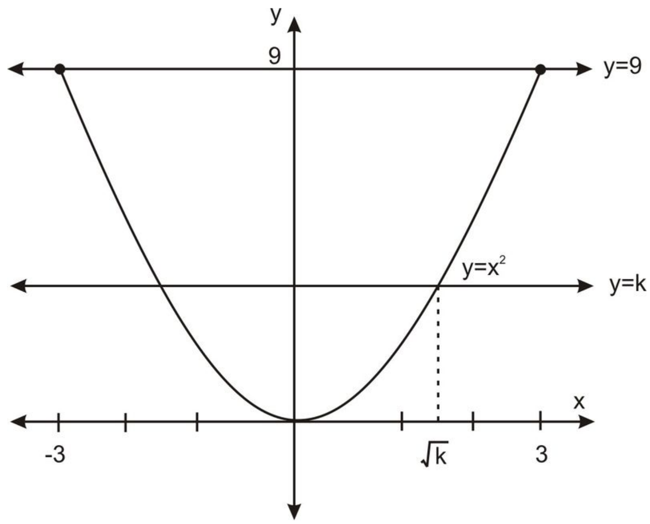
By symmetry, the two functions intersect at $x = -a$ and $x = a$.

$$\begin{aligned} k\ \cos x &= kx^2 \\ \cos x &= x^2 \\ \cos x - x^2 &= 0 \end{aligned}$$

Using a graphing calculator to find the zero, we find $a \approx 0.824$ radians.

$$\begin{aligned} &2 = 2\int_0^a (k\ \cos x - kx^2)dx \\ &2 = 2\left[k\ sinx - \frac{kx^3}{3}\right]\Bigg|_0^a \\ &1 = k\ sina - \frac{ka^3}{3} - 0 - 0 \\ &1 = k\left(\sin\ a - \frac{a^3}{3}\right) \\ &\frac{1}{k} = \sin(0.824) - \frac{0.842^3}{3} \\ &\frac{1}{k} \approx 0.547 \\ &k \approx 1.83 \end{aligned}$$

10.



First, find the area between $y = x^2$ and $y = 9$.

The two functions intersect at $x = -3$ and $x = 3$.

$$\int_{-3}^{3} (9 - x^2)dx = \left(9x - \frac{x^3}{3}\right)\Big|_{-3}^{3}$$
$$= 27 - 9 + 27 - 9$$
$$= 36$$

Then one-half of the area is 18.

Find k such that the area between $y = k$ and $y = x^2$ is 18. Note that the line $y = k$ intersects with $y = x^2$ at $y = -\sqrt{k}$ and $y = \sqrt{k}$.

$$
\begin{aligned}
18 &= \int_{-\sqrt{k}}^{\sqrt{k}} (k - x^2)dx \\
18 &= \left(kx - \frac{x^3}{3}\right)\Bigg|_{-\sqrt{k}}^{\sqrt{k}} \\
18 &= k\sqrt{k} - \frac{\left(\sqrt{k}\right)^3}{3} - \left(-k\sqrt{k} - \frac{\left(\sqrt{k}\right)^3}{3}\right) \\
18 &= 2k\sqrt{k} - 2\frac{(\sqrt{k})^3}{3} \\
9 &= \left(\sqrt{k}\right)^3 - \frac{\left(\sqrt{k}\right)^3}{3} \\
9 &= \frac{2}{3}(\sqrt{k})^3 \\
\frac{27}{2} &= \left(\sqrt{k}\right)^3 \\
\frac{3}{\sqrt[3]{2}} &= \sqrt{k} \\
\frac{9}{\sqrt[3]{4}} &= k
\end{aligned}
$$