7.6 Improper Integrals

Review Questions

1. a. This is improper because there is an infinite discontinuity between the limits of integration at $x = 3$.

b. This is not improper.

c. This is improper because there is an infinite discontinuity at $x = 0$.

d. This is improper because the integral has an infinite interval of integration.

e. This is not improper.

2. $\int\limits_{l}^{\infty} \frac{1}{x^{2.001}} dx = \lim_{l \to \infty} \frac{-x^{1.001}}{1.001}\Big|_{1}^{l} = \frac{1}{1.001}$

3.

$$\begin{aligned}
\int\limits_{-\infty}^{-2} \left[\frac{1}{x-1} - \frac{1}{x+1}\right] dx &= \lim_{l \to -\infty} [\ln|x-1| - \ln|x+1|]_{l}^{-2} \\
&= \ln 3 - 0 - \lim_{l \to -\infty} [\ln|l-1| - \ln|l+1|] \\
&= \ln 3 - \ln\left|\frac{l-1}{l+1}\right| \\
&= \ln 3 - \ln\left|\frac{l\left(1-\frac{1}{l}\right)}{l\left(1+\frac{1}{l}\right)}\right| \\
&= \ln 3 - 0 \\
&= \ln 3
\end{aligned}$$

4.

$$\begin{aligned}
\int\limits_{-\infty}^{0} e^{5x} dx &= \lim_{l \to -\infty} \left(\frac{1}{5} e^{5x}\right)\Big|_{l}^{0} \\
&= \lim_{l \to -\infty} \left(\frac{1}{5} \times l - \frac{1}{5} e^{5l}\right) \\
&= \frac{1}{5} - 0 \\
&= \frac{1}{5}
\end{aligned}$$

5. The integral is divergent.

6. $\int\limits_{\frac{-\pi}{2}}^{\frac{-\pi}{2}} \tan x = 0$ (Look at the symmetry of the graph on the interval.)

7.

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x\Big|_0^1$$
$$= \sin^{-1}1 - \sin^{-1}0$$
$$= \frac{\pi}{2} - 0$$
$$= \frac{\pi}{2}$$

8. a.

$$V = \pi \int_0^\infty (e^{-x})^2 dx$$
$$= \pi \int_0^\infty e^{-2x} dx$$
$$= \pi \left(\frac{-e^{-2x}}{2}\right)_0^\infty$$
$$= \pi \left(0 + \frac{1}{2}\right)$$
$$= \frac{\pi}{2}$$

b.

$$A = 2\pi \int_0^\infty f(x)\sqrt{1+[f'(x)]^2}dx$$
$$= 2\pi \int_0^\infty e^{-x}\sqrt{1+[-e^{-x}]^2}dx$$
$$= 2\pi \int_0^\infty e^{-x}\sqrt{1+e^{-2x}}dx$$

Let $e^{-x} = \tan u$. Then $-e^{-x}dx = \sec^2 u\, du$.

$$
\begin{aligned}
A &= 2\pi \int_{0}^{\infty} e^{-x}\sqrt{1+e^{-2x}}dx \\
&= 2\pi \int_{\infty}^{0} \sec^2 u\sqrt{1+\tan^2 u}\, du \\
&= 2\pi \int_{\infty}^{0} \sec^3 u\, du \\
&= 2\pi\left(\frac{1}{2}\sec u\tan u+\frac{1}{2}\ln|\sec u+\tan u|\right) \\
&= 2\pi\left((1+e^{-2x})^{\frac{1}{2}}e^{-x}+\ln\left|(1+e^{-2x})^{\frac{1}{2}}+e^{-x}\right|_{\infty}^{0}\right) \\
&= \pi[2+\ln(\sqrt{2}+1)]
\end{aligned}
$$