

7.3 Integration by Partial Fractions

1.

$$\frac{1}{x^2 - 1} = \frac{A}{x + 1} + \frac{B}{x - 1}$$

$$1 = A(x - 1) + B(x + 1)$$

Let $x = 1$.

$$1 = A(1 - 1) + B(x + 1)$$

$$1 = B(2)$$

$$\frac{1}{2} = B$$

Let $x = -1$.

$$1 = A(x - 1) + B(-1 + 1)$$

$$1 = A(-2)$$

$$-\frac{1}{2} = A$$

$$\int \frac{1}{x^2 - 1} dx = \int \left(\frac{A}{x + 1} + \frac{B}{x - 1} \right) dx$$

$$= \int \left(\frac{-\frac{1}{2}}{x + 1} + \frac{\frac{1}{2}}{x - 1} \right) dx$$

$$= -\frac{1}{2} \ln|x + 1| + \frac{1}{2} \ln|x - 1|$$

$$= \frac{1}{2} \left| \frac{x - 1}{x + 1} \right| + C$$

2.

$$\frac{x}{x^2 - 2x - 3} = \frac{A}{x - 3} + \frac{B}{x + 1}$$

$$x = A(x + 1) + B(x - 3)$$

Let $x = 3$.

$$3 = A(3 + 1) + B(3 - 3)$$

$$3 = A(4)$$

$$\frac{3}{4} = A$$

Let $x = -1$.

$$\begin{aligned}x &= A(x+1) + B(-1-3) \\-1 &= B(-4) \\ \frac{1}{4} &= B\end{aligned}$$

$$\begin{aligned}\int \frac{x}{x^2-2x-3} dx &= \int \frac{\frac{3}{4}}{x-3} dx + \int \frac{\frac{1}{4}}{x+1} dx \\ &= \frac{3}{4} \ln|x-3| + \frac{1}{4} \ln|x+1| + C\end{aligned}$$

3.

$$\begin{aligned}\frac{1}{x^3+x^2-2x} &= \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-1} \\ 1 &= A(x+2)(x-1) + Bx(x-1) + Cx(x+2)\end{aligned}$$

Let $x = 0$.

$$\begin{aligned}1 &= A(2)(-1) \\ -\frac{1}{2} &= A\end{aligned}$$

Let $x = -2$.

$$\begin{aligned}1 &= B(-2)(-3) \\ \frac{1}{6} &= B\end{aligned}$$

Let $x = 1$.

$$\begin{aligned}1 &= C(1)(3) \\ \frac{1}{3} &= C\end{aligned}$$

$$\begin{aligned}\int \frac{1}{x^3+x^2-2x} dx &= -\int \frac{\frac{1}{2}}{x} dx + \int \frac{\frac{1}{6}}{x+2} dx + \int \frac{\frac{1}{3}}{x-1} dx \\ &= -\frac{1}{2} \ln|x| + \frac{1}{6} \ln|x+2| + \frac{1}{3} \ln|x-1| + C\end{aligned}$$

4.

$$\begin{aligned}\int \frac{x^3}{x^2+4} dx &= \int x dx - \int \frac{4x}{x^2+4} dx \\ &= \frac{x^2}{2} - 2\ln|x^2+4| + C\end{aligned}$$

5.

$$\begin{aligned}\int_0^1 \frac{\phi}{1+\phi} d\phi &= \int_0^1 \left(1 - \frac{1}{1+\phi}\right) d\phi \\ &= [\phi - \ln|1+\phi|]_0^1 \\ &= 1 - \ln(2)\end{aligned}$$

6.

$$\begin{aligned}\frac{x-1}{x^2(x+1)} &= \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x+1} \\ x-1 &= A(x+1) + Bx(x+1) + Cx^2\end{aligned}$$

Let $x = -1$.

$$\begin{aligned}-2 &= C(1) \\ -2 &= C\end{aligned}$$

Let $x = 0$.

$$\begin{aligned}-1 &= A(1) \\ -1 &= A\end{aligned}$$

Pick any other value of x to find B .

$$\begin{aligned}x-1 &= A(x+1) + Bx(x+1) + Cx^2 \\ 2-1 &= -(2+1) + B(6) - (2)(4) \\ 1 &= -3 + 6B - 8 \\ 12 &= 6B \\ 2 &= B\end{aligned}$$

$$\begin{aligned}
 \int_1^5 \frac{x-1}{x^2(x+1)} dx &= -\int_1^5 \frac{1}{x^2} dx + \int_1^5 \frac{2}{x} dx - \int_1^5 \frac{2}{x+1} dx \\
 &= -(-x^{-1})\Big|_1^5 + 2\ln|x|\Big|_1^5 - 2\ln|x+1|\Big|_1^5 \\
 &= \frac{1}{5} - 1 + 2\ln 5 + 0 - 2(\ln 6 - \ln 2) \\
 &= -\frac{4}{5} + 2\ln 5 - 2\ln 3 \\
 &= -\frac{4}{5} + 2\ln \frac{5}{3}
 \end{aligned}$$

7. Let $u = \sin \theta$. Then $du = \cos \theta$.

$$\begin{aligned}
 \frac{1}{u^2 + 4u - 5} &= \frac{A}{u+5} + \frac{B}{u-1} \\
 1 &= A(u-1) + B(u+5)
 \end{aligned}$$

Let $u = 1$.

$$\begin{aligned}
 1 &= B(1+5) \\
 1 &= 6B \\
 \frac{1}{6} &= B
 \end{aligned}$$

Let $u = -5$.

$$\begin{aligned}
 1 &= A(-5-1) \\
 1 &= -6A \\
 -\frac{1}{6} &= A
 \end{aligned}$$

Then

$$\begin{aligned}
 \int \frac{1}{u^2 + 4u - 5} du &= -\int \frac{\frac{1}{6}}{u+5} du + \int \frac{\frac{1}{6}}{u-1} du \\
 &= -\frac{1}{6}(\ln|u+5| - \ln|u-1|) + C \\
 &= -\frac{1}{6}(\ln|\sin \theta + 5| - \ln|\sin \theta - 1|) + C \\
 &= \frac{1}{6} \ln \left| \frac{\sin \theta - 1}{\sin \theta + 5} \right| + C
 \end{aligned}$$

8. Let $u = e^\theta$ and $du = e^\theta d\theta$.

$$\int \frac{3e^\theta}{e^{2\theta} - 1} d\theta = \int \frac{3}{u^2 - 1} du$$

Then

$$\begin{aligned} \frac{3}{u^2 - 1} &= \frac{A}{u + 1} + \frac{B}{u - 1} \\ 3 &= A(u - 1) + B(u + 1) \end{aligned}$$

Let $u = 1$.

$$\begin{aligned} 3 &= B(1 + 1) \\ 3 &= 2B \\ \frac{3}{2} &= B \end{aligned}$$

Let $u = -1$.

$$\begin{aligned} 3 &= A(-2) \\ 3 &= -2A \\ -\frac{3}{2} &= A \end{aligned}$$

$$\begin{aligned} \int \frac{3}{u^2 - 1} du &= \int \frac{-\frac{3}{2}}{u + 1} du + \int \frac{\frac{3}{2}}{u - 1} du \\ &= -\frac{3}{2} \ln|u + 1| + \frac{3}{2} \ln|u - 1| + C \\ &= -\frac{3}{2} \ln|e^\theta + 1| + \frac{3}{2} \ln|e^\theta - 1| + C \end{aligned}$$

9.

$$\begin{aligned} \int_{-\ln 3}^{\ln 4} \frac{1}{2 + e^x} dx &= \int_{-\ln 3}^{\ln 4} \frac{1}{e^x(2e^{-x} + 1)} dx \\ &= \int_{-\ln 3}^{\ln 4} \frac{e^{-x}}{2e^{-x} + 1} dx \end{aligned}$$

Let $u = e^{-x}$. Then $du = -e^{-x} dx$.

$$\begin{aligned}
 \int_{-\ln 3}^{\ln 4} \frac{e^{-x}}{2e^{-x}+1} dx &= -\int_3^{\frac{1}{4}} \frac{du}{2u+1} \\
 &= -\frac{1}{2} \ln|2u+1| \Big|_3^{\frac{1}{4}} \\
 &= -\frac{1}{2} \left(\ln \frac{3}{2} - \ln 7 \right) \\
 &= \frac{1}{2} \left(\ln 7 - \ln \frac{3}{2} \right) \\
 &= \frac{1}{2} \ln \frac{14}{3}
 \end{aligned}$$

10.

$$\begin{aligned}
 \frac{1}{a^2-x^2} &= \frac{A}{a-x} + \frac{B}{a+x} \\
 1 &= A(a+x) + B(a-x)
 \end{aligned}$$

Let $x = b$.

$$\begin{aligned}
 1 &= A(a+a) \\
 \frac{1}{2a} &= A
 \end{aligned}$$

Let $x = -a$.

$$\begin{aligned}
 1 &= B(a - (-a)) \\
 1 &= 2aB \\
 \frac{1}{2a} &= B
 \end{aligned}$$

$$\begin{aligned}
 \int \frac{1}{a^2-x^2} dx &= \int \frac{\frac{1}{2a}}{a-x} dx + \int \frac{\frac{1}{2a}}{a+x} dx \\
 &= -\frac{1}{2a} \ln|a-x| + \frac{1}{2a} \ln|a+x| \\
 &= \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C
 \end{aligned}$$