7.5 Trigonometric Substitutions

1. Let $x = 2\sin\theta$. Then $dx = 2\cos\theta\, d\theta$.

$$\begin{aligned}
\int \sqrt{4-x^2}dx &= \int \sqrt{4-4\ \sin^2\theta}2\cos\theta\, d\theta \\
&= \int \sqrt{4\cos^2\theta}2\cos\theta\, d\theta \\
&= \int 4\cos^2\theta d\theta \\
&= \int 4\left(\frac{1}{2}(1+\cos(2\theta))\right) d\theta \\
&= 2\int d\theta + 2\int \cos(2\theta)d\theta \\
&= 2\theta + 2\frac{\sin(2\theta)}{2} + C \\
&= 2\theta + \sin(2\theta) + C \\
&= 2\sin^{-1}\left(\frac{x}{2}\right) + 2\sin\theta\cos\theta + C \\
&= 2\sin^{-1}\left(\frac{x}{2}\right) + x\frac{\sqrt{4-x^2}}{2} + C
\end{aligned}$$

2. Let $x = 3\tan\theta$. Then $dx = 3\sec^2\theta\, d\theta$.

$$\begin{aligned}
\int \frac{1}{\sqrt{9+x^2}}dx &= \int \frac{3\sec^2\theta}{\sqrt{9+9\tan^2\theta}}d\theta \\
&= \int \frac{3(\sec^2\theta)}{\sqrt{9\sec^2\theta}}d\theta \\
&= \int \frac{3(\sec^2\theta)}{3\sec\theta}d\theta \\
&= \int \sec\theta d\theta \\
&= \ln|\sec\theta + \tan\theta| + C \\
&= \ln\left|\frac{1}{3}\sqrt{9+x^2} + x\right| + C
\end{aligned}$$

3. Let $x = \sin\theta$. Then $dx = \cos\theta\, d\theta$.

$$\begin{aligned}
\int \frac{x^3}{\sqrt{1-x^2}}dx &= \int \frac{\sin^3\theta\cos\theta}{\sqrt{1-\sin^2\theta}}d\theta \\
&= \int \frac{\sin^3\theta\cos\theta}{\cos\theta}d\theta \\
&= \int \sin^3\theta \, d\theta \\
&= \int \sin\theta(\sin^2\theta)d\theta \\
&= \int \sin(1-\cos^2\theta)d\theta \\
&= \int \sin \, d\theta - \int \sin\theta\cos^2\theta \, d\theta \\
&= -\cos\theta + \frac{\cos^3\theta}{3} + C \\
&= -\sqrt{1-x^2} + \frac{\left(\sqrt{1-x^2}\right)^3}{3} + C \\
&= -\sqrt{1-x^2} + \frac{(1-x^2)\sqrt{1-x^2}}{3} + C \\
&= -\sqrt{1-x^2}\frac{\sqrt{1-x^2}}{3} - \frac{x^2\sqrt{1-x^2}}{3} + C \\
&= -\frac{2\sqrt{1-x^2}}{3} - \frac{x^2\sqrt{1-x^2}}{3} + C
\end{aligned}$$

4. $\sqrt{1-9x^2} = 3\sqrt{\frac{1}{9}-x^2}$

Then $x = \frac{1}{3}\sin\theta$ and $dx = \frac{1}{3}\cos\theta \, d\theta$.

$$\begin{aligned}
\int \frac{1}{\sqrt{1-9x^2}}dx &= \int \frac{1}{3\sqrt{\frac{1}{9}-x^2}}dx \\
&= \frac{1}{3}\int \frac{\frac{1}{3}\cos\theta}{\sqrt{\frac{1}{9}-\frac{1}{9}\sin^2\theta}}d\theta \\
&= \frac{1}{3}\int \frac{\frac{1}{3}\cos\theta}{\sqrt{\frac{1}{9}\cos^2\theta}}d\theta \\
&= \frac{1}{3}\int \frac{\frac{1}{3}\cos\theta}{\frac{1}{3}\cos\theta}d\theta \\
&= \frac{1}{3}\int d\theta \\
&= \frac{1}{3}\theta \\
&= \frac{1}{3}\sin^{-1}(3x) + C
\end{aligned}$$

5. Let $x = 2\sin\theta$. Then $dx = 2\cos\theta$.

$$\begin{aligned}
\int x^3\sqrt{4-x^2}dx &= \int 8\sin^3\theta\sqrt{4-4\sin^2\theta}2\cos\theta\, d\theta \\
&= \int 8\sin^3\theta\sqrt{4\cos^2\theta}2\cos\theta\, d\theta \\
&= \int 8\sin^3\theta 4\cos^2\theta\, d\theta \\
&= \int 4\left(\frac{1}{2}(1+\cos(2\theta))\right)d\theta \\
&= 2\int d\theta + 2\int \cos(2\theta)d\theta \\
&= 2\theta + 2\frac{\sin(2\theta)}{2} + C \\
&= 2\theta + \sin(2\theta) + C \\
&= 2\sin^{-1}\left(\frac{x}{2}\right) + 2\sin\theta\cos\theta + C \\
&= 2\sin^{-1}\left(\frac{x}{2}\right) + x\frac{\sqrt{4-x^2}}{2} + C
\end{aligned}$$

6. Let $x = 6\sec\theta$. Then $dx = 6\sec\theta\tan\theta\, d\theta$.

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2-36}}dx &= \int \frac{6\sec\theta\tan\theta}{36\sec^2\theta\sqrt{36\sec^2\theta-36}}d\theta \\
&= \int \frac{6\tan\theta}{36\sec\theta\times 6\tan\theta}d\theta \\
&= \frac{1}{36}\int \frac{1}{\sec\theta}d\theta \\
&= \frac{1}{36}\int \cos\theta\, d\theta \\
&= \frac{1}{36}\sin\theta + C
\end{aligned}$$

Now $\sec\theta = \frac{x}{6}$ and thus, $\cos\theta = \frac{6}{x}$.

Then

$$\begin{aligned}
\sin^2\theta &= 1-\cos^2\theta \\
\sin\theta &= \sqrt{1-\frac{36}{x^2}} \\
&= \sqrt{\frac{x^2-36}{x^2}} \\
&= \frac{\sqrt{x^2-36}}{x}
\end{aligned}$$

Thus, $\int \frac{1}{x^2\sqrt{x^2-36}}dx = \frac{1}{36}\frac{\sqrt{x^2-36}}{x} + C$.

7. Let $x = 5\tan\theta$. Then $dx = 5\sec^2\theta\, d\theta$.

$$\begin{aligned}
\int \frac{1}{(x^2+25)^2}dx &= \int \frac{1}{\left(\sqrt{x^2+25}\right)^4}dx \\
&= \int \frac{5\sec^2\theta}{(\sqrt{25+\tan^2\theta+25})^4}d\theta \\
&= \int \frac{5\sec^2\theta}{(5\sec\theta)^4}d\theta \\
&= \int \frac{d\theta}{625\sec^2\theta} \\
&= \frac{1}{125}\int \cos^2\theta \, d\theta \\
&= \frac{1}{125}\left(\frac{\theta}{2}+\frac{1}{4}\sin 2\theta\right)+C \\
&= \frac{1}{125}\left(\frac{\theta}{2}+\frac{1}{4}2\sin\theta\cos\theta\right)+C \\
&= \frac{1}{250}\tan^{-1}\left(\frac{x}{5}\right)+\frac{1}{125}\left(\frac{1}{2}\right)\left(\frac{x}{\sqrt{x^2+25}}\right)\left(\frac{5}{\sqrt{x^2+25}}\right) \\
&= \frac{1}{250}\tan^{-1}\left(\frac{x}{5}\right)+\frac{1}{50}\left(\frac{x}{x^2+25}\right)
\end{aligned}$$

8. Let $x = 4\sin\theta$. Then $dx = 4\cos\theta \, d\theta$. For $x = 0$,

$$\begin{aligned}
\int_0^4 x^3\sqrt{16-x^2}dx &= \int_0^{\frac{\pi}{2}} 64\sin^3\theta\sqrt{16-16\sin^2\theta}(4\cos\theta)d\theta \\
&= \int_0^{\frac{\pi}{2}} 64\sin^3\theta(16\cos^2\theta)d\theta \\
&= 1024\int_0^{\frac{\pi}{2}} \sin^3\theta(\cos^2\theta)d\theta \\
&= 1024\left(\frac{\cos^5\theta}{5}-\frac{\cos^3\theta}{3}\right)_0^{\frac{\pi}{2}} \\
&= 1024\left(-\frac{1}{5}+\frac{1}{3}\right) \\
&= \frac{2048}{15}
\end{aligned}$$

9. Let $u = e^x$. Then $du = e^x \, dx$.

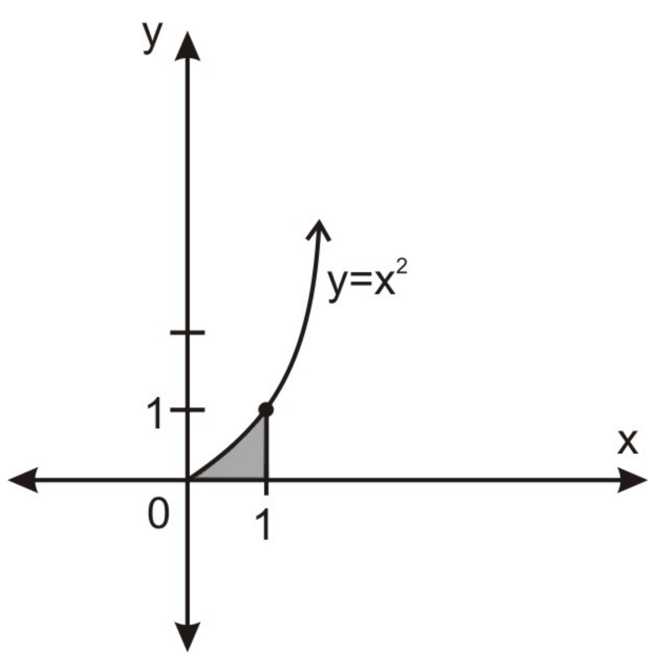
$$\int_{-\pi}^{0} e^x\sqrt{1-e^{2x}}dx = \int_{e^{-\pi}}^{1} \sqrt{1-u^2}du$$

Let $u = \sin\theta$. Then $du = \cos\theta \, d\theta$.

The integral becomes

$$\begin{aligned}\int_{\theta_1}^{\theta^2} \sqrt{1-\sin^2\theta}\cos\theta\, d\theta &= \int_{\theta_1}^{\theta^2} \cos\theta\cos\theta\, d\theta \\ &= \int_{\theta_1}^{\theta^2} \cos^2\theta\, d\theta \\ &= \frac{\theta}{2} + \frac{\sin(2\theta)}{4}\Big|_{\theta_1}^{\theta_1} \\ &= \frac{\theta}{2} + \frac{1}{2}\sin\theta\cos\theta\Big|_{\theta_1}^{\theta_1} \\ &= \frac{\sin^{-1}u}{2} + \frac{1}{2}u\sqrt{1-u^2}\Big|_{e^{-\pi}}^{1} \\ &= \frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{2}(1)(0) - \frac{\sin^{-1}e^{-\pi}}{2} - \frac{1}{2}e^{-\pi}\sqrt{1-e^{-2\pi}} \\ &= \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{\sin^{-1}e^{-\pi}}{2} - \frac{1}{2}e^{-\pi}\sqrt{1-e^{-2\pi}}\end{aligned}$$

10.



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\begin{aligned}
SA &= 2\pi \int_0^1 x^2(1+(2x)^2)dx \\
&= 2\pi \int_0^1 x^2(1+4x^2)dx \\
&= 2\pi \left[\frac{1}{16}x(1+4x^2)^{\frac{3}{2}} - \frac{1}{32}x\sqrt{1+4x^2} - \frac{1}{64}\ln(2x+\sqrt{1+4x^2})\right]_0^1 \\
&= 2\pi \left[\frac{1}{16}(1+4)^{\frac{3}{2}} - \frac{1}{32}\sqrt{1+4} - \frac{1}{64}\ln\left(2+\sqrt{1+4}\right)\right] \\
&= 2\pi \left[\frac{5}{16}\sqrt{6} - \frac{1}{32}\sqrt{5} - \frac{1}{64}\ln\left(2+\sqrt{1+4}\right)\right] \\
&= 2\pi \left[\frac{9}{32}\sqrt{5} - \frac{1}{64}\ln\left(2+\sqrt{1+4}\right)\right]
\end{aligned}
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