

4.7 Integration by Substitution

1. Let $u = \ln x$ and $dv = x$. Then $du = \frac{1}{x}dx$ and $v = \frac{x^2}{2}$.

$$\begin{aligned}\int x \ln x \, dx &= uv - \int v \, du \\ &= \frac{x^2 \ln x}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx \\ &= \frac{x^2 \ln x}{2} - \int \frac{x}{2} dx \\ &= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C\end{aligned}$$

2. Let $u = \ln x$ and $dv = \sqrt{x}$. Then $du = \frac{1}{x}dx$ and $v = \frac{2}{3}x^{\frac{3}{2}}$.

$$\begin{aligned}\int x \ln x \, dx &= uv - \int v \, du \\ \int_1^3 \sqrt{x} \ln x \, dx &= \frac{2x^{\frac{3}{2}} \ln x}{3} - \frac{2}{3} \int_1^3 x^{\frac{3}{2}} \cdot \frac{1}{x} dx \\ &= \frac{2x^{\frac{3}{2}} \ln x}{3} - \frac{2}{3} \int_1^3 x^{\frac{1}{2}} dx \\ &= \left(\frac{2x^{\frac{3}{2}} \ln x}{3} - \frac{4x^{\frac{3}{2}}}{9} \right) \Big|_1^3 \\ &= \frac{2 \times 3^{\frac{3}{2}} \ln 3}{3} - \frac{4 \times 3^{\frac{3}{2}}}{9} - \left(0 - \frac{4}{9} \right) \\ &= \frac{2 \times 3^{\frac{3}{2}} \ln 3}{3} - \frac{4 \times 3^{\frac{3}{2}}}{9} + \frac{4}{9}\end{aligned}$$

3. Let $u^2 = 2x + 1$. Then

$$\begin{aligned}u^2 &= 2x + 1 \\ u^2 - 1 &= 2x \\ \frac{u^2 - 1}{2} &= x \\ u \, du &= dx\end{aligned}$$

$$\begin{aligned}
\int \frac{x}{\sqrt{2x+1}} dx &= \int \frac{1}{u} \left(\frac{u^2-1}{2} \right) u du \\
&= \int \frac{u^2-1}{2} du \\
&= \frac{1}{2} \int (u^2-1) du \\
&= \frac{1}{2} \left[\frac{u^3}{3} - u \right] \\
&= \frac{1}{2} \left[\frac{(\sqrt{2x+1})^3}{3} - \sqrt{2x+1} \right] \\
&= \frac{1}{2} \sqrt{2x+1} \left(\frac{2x+1}{3} - 1 \right) + C \\
&= \frac{1}{2} \cdot \frac{\sqrt{2x+1}(2x-2)}{3} + C \\
&= \frac{\sqrt{2x+1}(x-1)}{3} + C
\end{aligned}$$

4. Let $u = 1 - x^2$. Then

$$\begin{aligned}
du &= -2x dx \\
-\frac{1}{2} du &= x dx \\
\text{and} \\
u - 1 &= -x^2 \\
-u + 1 &= x^2
\end{aligned}$$

When using u -substitution, just put limits as u_1 and u_2 as placeholders on the integral. After u is replaced by the function of x , put back the original limits of integration.

$$\begin{aligned}
\int_0^1 x^3 \sqrt{1-x^2} dx &= \int_{u_1}^{u_2} x^2 (x \sqrt{1-x^2}) dx \\
&= \frac{1}{2} \int_{u_1}^{u_2} -(-u+1) \sqrt{u} du \\
&= \frac{1}{2} \int_{u_1}^{u_2} (u^{\frac{3}{2}} - u^{\frac{1}{2}}) du \\
&= \frac{1}{2} \left(\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right) \\
&= \frac{1}{2} \left[\frac{2}{5} (1-x^2)^{\frac{5}{2}} - \frac{2}{3} (1-x^2)^{\frac{3}{2}} \right] \Big|_0^1 \\
&= 0 - \frac{1}{2} \left(\frac{2}{5} - \frac{2}{3} \right) \\
&= \frac{2}{15}
\end{aligned}$$

5. Let $u = x$ and $dv = \cos x dx$. Then $du = dx$ and $v = \sin x$.

$$\begin{aligned}
\int x \cos x dx &= x \sin x - \int \sin x dx \\
&= x \sin x - (-\cos x) + C \\
&= x \sin x + \cos x + C
\end{aligned}$$

6. Let $u = x^3 + 9$. Then

$$\begin{aligned}
du &= 3x^2 dx \\
\frac{1}{3} du &= x^2 dx
\end{aligned}$$

$$\begin{aligned}
\int_0^1 x^2 \sqrt{x^3+9} dx &= \int_{u_1}^{u_2} \frac{1}{3} \sqrt{u} du \\
&= \frac{1}{3} \int_{u_1}^{u_2} u^{\frac{1}{2}} du \\
&= \frac{1}{3} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \\
&= \frac{2}{9} \sqrt{(x^3+9)^3} \Big|_0^1 \\
&= \frac{2}{9} (\sqrt[3]{10^2} - 27)
\end{aligned}$$

7. Let

$$\begin{aligned}u &= \frac{1}{x} = x^{-1} \\du &= -x^{-2} dx \\-du &= \frac{1}{x^2} dx\end{aligned}$$

$$\begin{aligned}\int \left(\frac{1}{x^2} \times e^{\frac{1}{x}} \right) dx &= \int e^u du \\&= -e^u + C \\&= e^{\frac{1}{x}} + C\end{aligned}$$

8. Let $u = x^2$ and $dv = xe^{x^2} dx$. Then $du = 2x dx$ and $v = \frac{1}{2}e^{x^2}$.

$$\begin{aligned}\int x^3 e^{x^2} dx &= \frac{1}{2} x^2 e^{x^2} (2x) dx \\&= \frac{1}{2} x^2 e^{x^2} - \int e^{x^2} (x) dx \\&= \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C\end{aligned}$$

9. Let $u = \ln x$ and $dv = \frac{1}{x^{\frac{5}{2}}} dx = x^{-\frac{5}{2}} dx$. Then $du = \frac{1}{x}$ and $v = -\frac{2}{3}x^{-\frac{3}{2}}$.

$$\begin{aligned}\int \frac{\ln x}{x^{\frac{5}{2}}} dx &= \frac{-2}{3} \ln x \left(x^{-\frac{3}{2}} \right) - \int \left(\frac{-2}{3} \right) \frac{x^{-\frac{3}{2}}}{x} dx \\&= \frac{-2 \ln x}{3x^{\frac{3}{2}}} + \frac{2}{3} \int x^{-\frac{5}{2}} dx \\&= \frac{-2 \ln x}{3x^{\frac{3}{2}}} + \frac{2}{3} \frac{x^{-\frac{3}{2}}}{\left(-\frac{3}{2}\right)} + C \\&= \frac{-2 \ln x}{3x^{\frac{3}{2}}} - \frac{4}{9x^{\frac{3}{2}}} + C\end{aligned}$$

10. $\int_1^e \frac{1}{x} dx = \ln x \Big|_1^e = \ln e - \ln 1 = \ln e = 1$