Learning Objectives

A student will be able to:

- Find intervals where a function is concave upward or downward.
- Apply the Second Derivative Test to determine concavity and sketch graphs.

Introduction

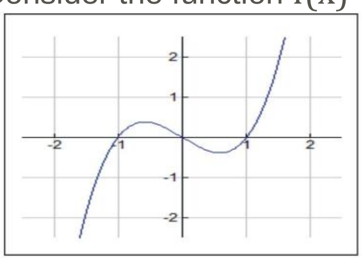
In this lesson we will discuss a property about the shapes of graphs called concavity, and introduce a method with which to study this phenomenon, the Second Derivative Test. This method will enable us to identify precisely the intervals where a function is either increasing or decreasing, and also help us to sketch the graph.

Definition

> A function f is said to be ***concave upward*** on $[a,b]$ contained in the domain of f if f' is an increasing function on $[a,b]$ and ***concave downward*** on $[a,b]$ if f' is a decreasing function on $[a,b]$.
> Here is an example that illustrates these properties.

Example 1:

Consider the function $f(x)=x^3-x$:



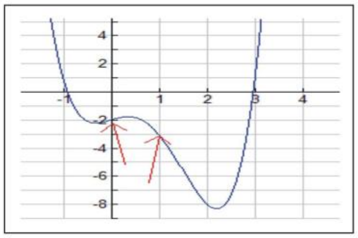
The function has zeros at $x=\pm 1,0$ and has a relative maximum at $x=-3-\sqrt{3}$ and a relative minimum at $x=3-\sqrt{3}$. Note that the graph appears to be concave down for all intervals in $(-\infty,0)$ and concave up for all intervals in $(0,+\infty)$. Where do you think the concavity of the graph changed from concave down to concave up? If you answered at $x=0$ you would be correct. In general, we wish to identify both the extrema of a function and also the points where the graph changes concavity. The following definition provides a formal characterization of such points.

Definition

> A point on a graph of a function f where the concavity changes is called an ***inflection point***.
> The example above had only one inflection point. But we can easily come up with examples of functions where there is more than one point of inflection.

Example 2:

Consider the function $f(x)=x^4-3x^3+x-2$.



We can see that the graph has two relative minimums, one relative maximum, and two inflection points (as indicated by arrows).

In general we can use the following two tests for concavity and determining where we have relative maximums, minimums, and inflection points.

Test for Concavity

Suppose that I is some interval $[a,b]$ in the domain of f and that f is continuous on I.

1. If $f''(x)>0$ for all $x\in I$, then the graph of f is concave upward on I.
2. If $f''(x)<0$ for all $x\in I$, then the graph of f is concave downward on I.

A consequence of this concavity test is the following test to identify extreme values of f.

Second Derivative Test for Extrema

Suppose that f is a continuous function near c and that c is a critical value of f. Then

1. If $f''(c)>0$, then f has a relative minimum at $x=c$.
2. If $f''(c)<0$, then f has a relative maximum at $x=c$.
3. If $f''(c)=0$, then the test is inconclusive and $x=c$ may be a point of inflection.

Recall the graph $f(x)=x^3$. We observed that $x=0$, and that there was neither a maximum nor minimum. The Second Derivative Test cautions us that this may be the case since at $f''(0)=0$ at $x=0$.

So now we wish to use all that we have learned from the First and Second Derivative Tests to sketch graphs of functions. The following table provides a summary of the tests and can be a useful guide in sketching graphs.

Signs of first and second derivatives	Information from applying First and Second Derivative Tests	Shape of the graphs
$f'(x)>0$ $f''(x)>0$	f is increasing f is concave upward	

Signs of first and second derivatives	**Information from applying First and Second Derivative Tests**	**Shape of the graphs**
$f'(x)>0$ $f''(x)<0$	f is increasing f is concave downward	
$f'(x)<0$ $f''(x)>0$	f is decreasing f is concave upward	
$f'(x)<0$ $f''(x)<0$	f is decreasing f is concave downward	

Lets' look at an example where we can use both the First and Second Derivative Tests to find out information that will enable us to sketch the graph.

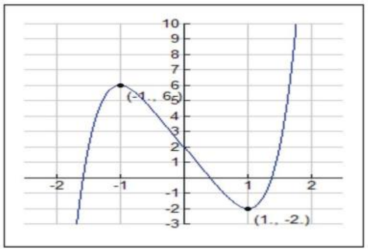
Example 3:

Let's examine the function $f(x)=x^5-5x+2$.
1. Find the critical values for which $f'(c)=0$.
$f'(x)=5x^4-5=0$, or
$x^4-1=0$ at $x=\pm 1$.
Note that $f''(x)=20x^3=0$ when $x=0$.
2. Apply the First and Second Derivative Tests to determine extrema and points of inflection.

We can note the signs of f' and f'' in the intervals partitioned by $x=\pm 1,0$.

Key intervals	$f'(x)$	$f''(x)$	***Shape of graph***
$x<-1$	+	−	Increasing, concave down
$-1<x<0$	−	−	Decreasing, concave down
$0<x<1$	−	+	Decreasing, concave up
$x>1$	+	+	Increasing, concave up

Also note that $f''(-1)=-20<0$. By the Second Derivative Test we have a relative maximum at $x=-1$, or the point $(-1,6)$.
In addition, $f''(1)=20>0$. By the Second Derivative Test we have a relative minimum at $x=1$, or the point $(1,-2)$. Now we can sketch the graph.



Lesson Summary

1. We learned to identify intervals where a function is concave upward or downward.
2. We applied the First and Second Derivative Tests to determine concavity and sketch graphs.

Multimedia Links

For a video presentation of the second derivative test to determine relative extrema **(9.0)**, see Math Video Tutorials by James Sousa, The Second Derivative Test (8:41).

Review Questions

1. Find all extrema using the Second Derivative Test. $f(x)=x^2 4+4x$
2. Consider $f(x)=x^2+ax+b$, with $f(1)=3$.
   a. Determine a and b so that $x=1$ is a critical value of the function f.
   b. Is the point $(1,3)$ a maximum, a minimum or neither?

In problems #3–6, find all extrema and inflection points. Sketch the graph.

3. $f(x)=x^3+x^2$
4. $f(x)=x^2+3x$
5. $f(x)=x^3-12x$
6. $f(x)=-14x^4+2x^2$
7. Use your graphing calculator to examine the graph of $f(x)=x(x-1)^3$ (Hint: you will need to change the $y$range in the viewing window)
   a. Discuss the concavity of the graph in the interval $(0,12)$.
   b. Use your calculator to find the minimum value of the function in the interval.
8. True or False: $f(x)=x^4+4x^3$ has a relative minimum at $x=-2$and a relative maximum at $x=0$?

9. If possible, provide an example of a non-polynomial function that has exactly one relative minimum.
10. If possible, provide an example of a non-polynomial function that is concave downward everywhere in its domain.