Learning Objectives

A student will be able to:

- Find intervals where a function is increasing and decreasing.
- Apply the First Derivative Test to find extrema and sketch graphs.

Introduction

In this lesson we will discuss increasing and decreasing properties of functions, and introduce a method with which to study these phenomena, the First Derivative Test. This method will enable us to identify precisely the intervals where a function is either increasing or decreasing, and also help us to sketch the graph. Note on notation: The symbol ϵ and ∈ are equivalent and denote that a particular element is contained within a particular set.

Definition

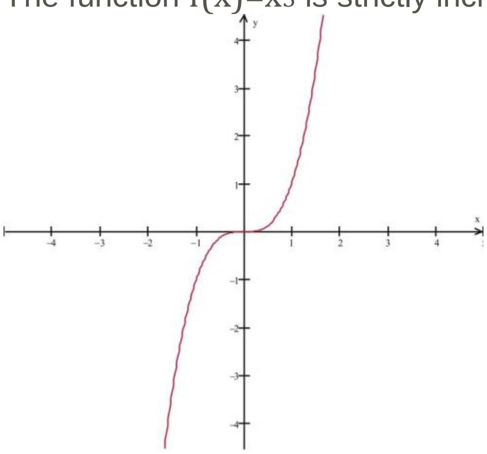
A function f is said to be ***increasing*** on $[a,b]$ contained in the domain of f if $f(x_1)\le f(x_2)$ whenever $x_1\le x_2$ for all $x_1,x_2\in[a,b]$. A function f is said to be ***decreasing*** on $[a,b]$ contained in the domain of f if $f(x_1)\ge f(x_2)$ whenever $x_1\ge x_2$ for all $x_1,x_2\in[a,b]$.

If $f(x_1)<f(x_2)$ whenever $x_1<x_2$ for all $x_1,x_2\in[a,b]$, then we say that f is ***strictly increasing*** on $[a,b]$. If $f(x_1)>f(x_2)$ whenever $x_1>x_2$ for all $x_1,x_2\in[a,b]$, then we say that f is ***strictly decreasing***on $[a,b]$.

We saw several examples in the Lesson on Extreme and the Mean Value Theorem of functions that had these properties.

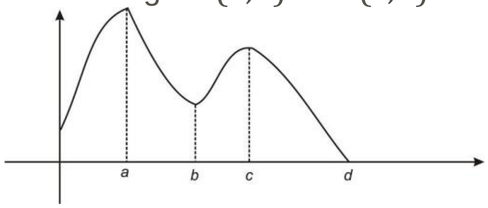
Example 1:

The function $f(x)=x^3$ is strictly increasing on $(-\infty,+\infty)$:



Example 2:

The function indicated here is strictly increasing on $(0,a)$ and (b,c) and strictly decreasing on (a,b) and (c,d).



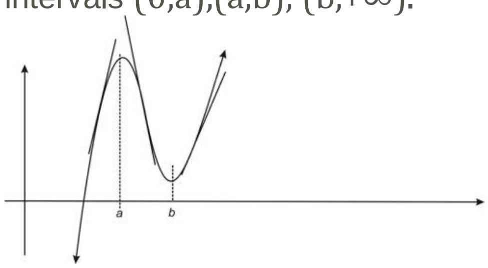
We can now state the theorems that relate derivatives of functions to the increasing/decreasing properties of functions.

Theorem: If f is continuous on interval $[a,b]$, then:

1. If $f'(x)>0$ for every $x\in[a,b]$, then f is strictly increasing in $[a,b]$.
2. If $f'(x)<0$ for every $x\in[a,b]$, then f is strictly decreasing in $[a,b]$.

Proof: We will prove the first statement. A similar method can be used to prove the second statement and is left as an exercise to the student.

Consider $x_1,x_2\in[a,b]$ with $x_1<x_2$. By the Mean Value Theorem, there exists $c\in(x_1,x_2)$ such that
$f(x_2)-f(x_1)=(x_2-x_1)f'(c)$.
By assumption, $f'(x)>0$ for every $x\in[a,b]$; hence $f'(c)>0$. Also, note that $x_2-x_1>0$.
Hence $f(x_2)-f(x_1)>0$ and $f(x_2)>f(x_1)$.
We can observe the consequences of this theorem by observing the tangent lines of the following graph. Note the tangent lines to the graph, one in each of the intervals $(0,a)$,(a,b), $(b,+\infty)$.



Note first that we have a relative maximum at $x=a$ and a relative minimum at $x=b$. The slopes of the tangent lines change from positive for $x\in(0,a)$ to negative for $x\in(a,b)$ and then back to positive for $x\in(b,+\infty)$. From this we example infer the following theorem:
First Derivative Test

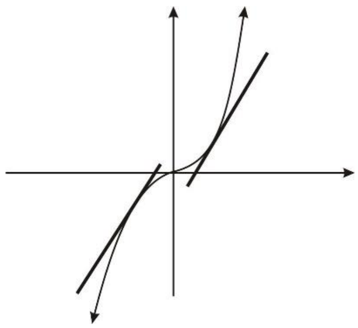
Suppose that f is a continuous function and that $x=c$ is a critical value of f. Then:

1. If f' changes from positive to negative at $x=c$, then f has a local maximum at $x=c$.
2. If f' changes from negative to positive at $x=c$, then f has a local minimum at $x=c$.
3. If f' does not change sign at $x=c$, then f has neither a local maximum nor minimum at $x=c$.

Proof of these three conclusions is left to the reader.

Example 3:

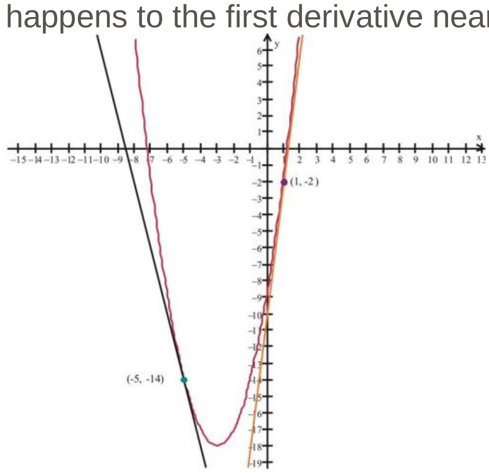
Our previous example showed a graph that had both a local maximum and minimum. Let's reconsider $f(x)=x^3$ and observe the graph around $x=0$. What happens to the first derivative near this value?



We observe that the tangent lines to the graph are positive on both sides of $x=0$. The first derivative test ($f'(x)=3x^2$) verifies this fact, and that the slopes of the tangent line are positive for all nonzero x. Although $f'(0)=0$, and so f has a critical value at $x=0$, the third part of the First Derivative Test tells us that the failure of f' to change sign at $x=0$means that f has neither a local minimum nor a local maximum at $x=0$.

Example 4:

Let's consider the function $f(x)=x^2+6x-9$ and observe the graph around $x=-3$. What happens to the first derivative near this value?



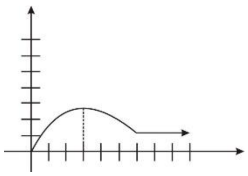
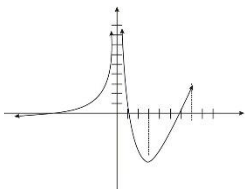
We observe that the slopes of the tangent lines to the graph change from negative to positive at $x=-3$. The first derivative test verifies this fact. Note that the slopes of the tangent lines to the graph are negative for $x\in(-\infty,-3)$ and positive for $x\in(-3,-\infty)$.

Lesson Summary

1. We found intervals where a function is increasing and decreasing.
2. We applied the First Derivative Test to find extrema and sketch graphs.

Review Questions

In problems #1–2, identify the intervals where the function is increasing, decreasing, or is constant. (Units on the axes indicate single units).

1. 
2. 
3. Give the sign of the following quantities for the graph in #2.
   a. $f'(-3)$
   b. $f'(1)$
   c. $f'(3)$
   d. $f'(4)$

For problems #4–6, determine the intervals in which the function is increasing and those in which it is decreasing. Sketch the graph.

4. $f(x)=x^2-1x$
5. $f(x)=(x^2-1)^5$
6. $f(x)=(x^2-1)^4$

For problems #7–10:

a. Use the First Derivative Test to find the intervals where the function increases and/or decreases

b. Identify all max, mins, or relative max and mins

c. Sketch the graph

7. $f(x)=-x^2-4x-1$
8. $f(x)=x^3+3x^2-9x+1$
9. $f(x)=x^{23}(x-5)$
10. $f(x)=2xx^2+1-----\sqrt{}$