Learning Objectives

A student will be able to:

- Demonstrate an understanding of the derivative of a function as a slope of the tangent line.
- Demonstrate an understanding of the derivative as an instantaneous rate of change.
- Understand the relationship between continuity and differentiability.

The function $f'(x)$ that we defined in the previous section is so important that it has its own name.

The Derivative

The function f' is defined by the new function

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

where f ***is called the*** *derivative* ***of*** f ***with respect to*** x. ***The domain of*** f consists of all the values of x for which the limit exists.

Based on the discussion in previous section, the derivative f' represents the slope of the tangent line at point x. Another way of interpreting it is to say that the function $y=f(x)$ has a derivative f' whose value at x is the instantaneous rate of change of y with respect to point x.

Example 1:

Find the derivative of $f(x)=\frac{x}{x+1}$.

Solution:

We begin with the definition of the derivative,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{1}{h}[f(x+h)-f(x)],$$

where

$$f(x)=\frac{x}{x+1} \qquad f(x+h)=\frac{x+h}{x+h+1}$$

Substituting into the derivative formula,

$$f'(x)=\lim_{h\to 0}\frac{1}{h}\left[\frac{x+h}{x+h+1}-\frac{x}{x+1}\right]=\lim_{h\to 0}\frac{1}{h}\left[\frac{(x+h)(x+1)-x(x+h+1)}{(x+h+1)(x+1)}\right]=\lim_{h\to 0}\frac{1}{h}\left[\frac{x^2+x+hx+h-x^2-xh-x}{(x+h+1)(x+1)}\right]=\lim_{h\to 0}\frac{1}{h}\left[\frac{h}{(x+h+1)(x+1)}\right]=\lim_{h\to 0}\frac{1}{(x+h+1)(x+1)}=\frac{1}{(x+1)^2}.$$

Example 2:

Find the derivative of $f(x)=\sqrt{x}$ and the equation of the tangent line at $x_0=1$.

Solution:

Using the definition of the derivative,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}=\lim_{h\to 0}\frac{1}{h}\frac{x+h-x}{\sqrt{x+h}+\sqrt{x}}=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}.$$

Thus the slope of the tangent line at $x_0=1$ is

$$f'(1)=\frac{1}{2\sqrt{1}}=\frac{1}{2}.$$

For $x_0=1$, we can find y_0 by simply substituting into $f(x)$.

$$f(x_0) \equiv y_0 \quad f(1) \quad y_0=\sqrt{1}=1.$$

Thus the equation of the tangent line is

$$y-y_0=m(x-x_0) \quad y-1=\frac{1}{2}(x-1) \quad y=\frac{1}{2}x+\frac{1}{2}.$$

Notation

Calculus, just like all branches of mathematics, is rich with notation. There are many ways to denote the derivative of a function $y=f(x)$ in addition to the most popular one, $f'(x)$. They are:

$$f'(x) \quad \frac{dy}{dx} \quad y' \quad \frac{df}{dx} \quad \frac{df(x)}{dx}$$

In addition, when substituting the point x_0 into the derivative we denote the substitution by one of the following notations:

$$f'(x_0) \quad \left.\frac{dy}{dx}\right|_{x=x_0} \quad \left.\frac{df}{dx}\right|_{x=x_0} \quad \frac{df(x_0)}{dx}$$

Existence and Differentiability of a Function

If, at the point $(x_0,f(x_0))$, the limit of the slope of the secant line does not exist, then the derivative of the function $f(x)$ at this point does not exist either. That is, if

$$m_{sec}=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\text{Does not exist}$$

then the derivative $f'(x)$ also fails to exist as $x\to x_0$. The following examples show four cases where the derivative fails to exist.

1. At a *corner*. For example $f(x)=|x|$, where the derivative on both sides of $x=0$ differ (Figure 4).
2. At a *cusp*. For example $f(x)=x^{2/3}$, where the slopes of the secant lines approach $+\infty$ on the right and $-\infty$ on the left (Figure 5).
3. A *vertical tangent*. For example $f(x)=x^{1/3}$, where the slopes of the secant lines approach $+\infty$ on the right and $-\infty$ on the left (Figure 6).
4. A *jump discontinuity*. For example, the step function (Figure 7)

$$f(x)=\begin{cases}-2, & x<0\\ 2, & x\geq 0\end{cases}$$

where the limit from the left is -2 and the limit from the right is 2.

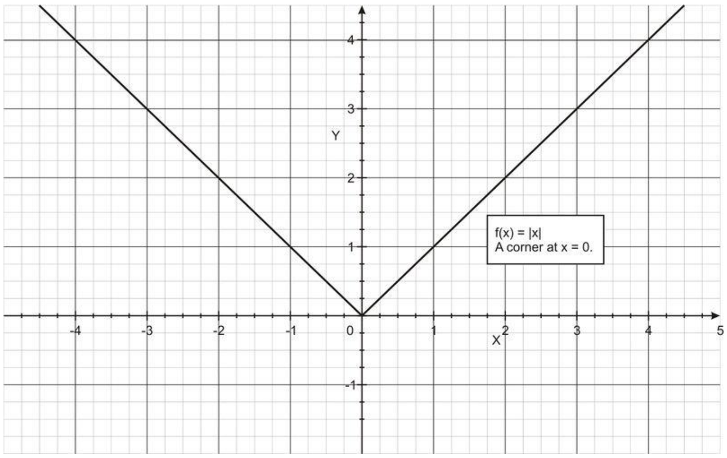

Figure 4

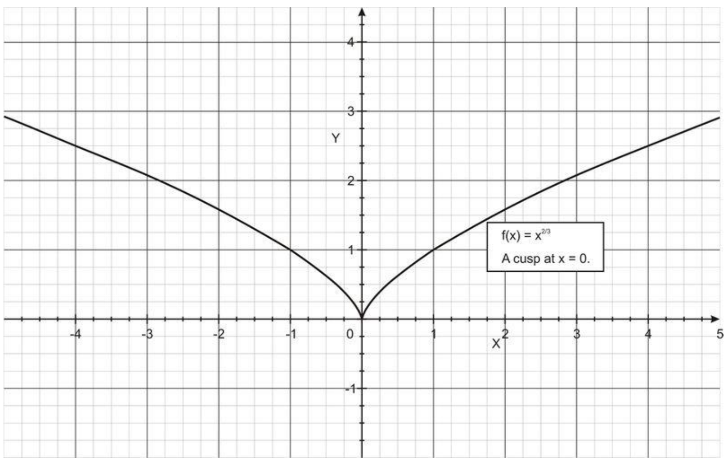

Figure 5

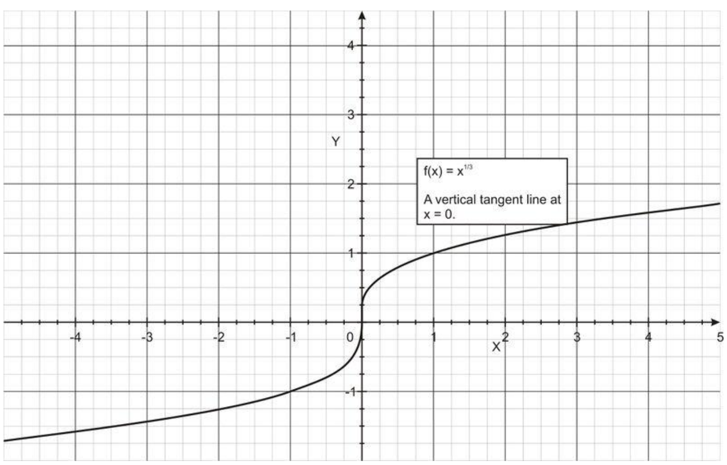

Figure 6

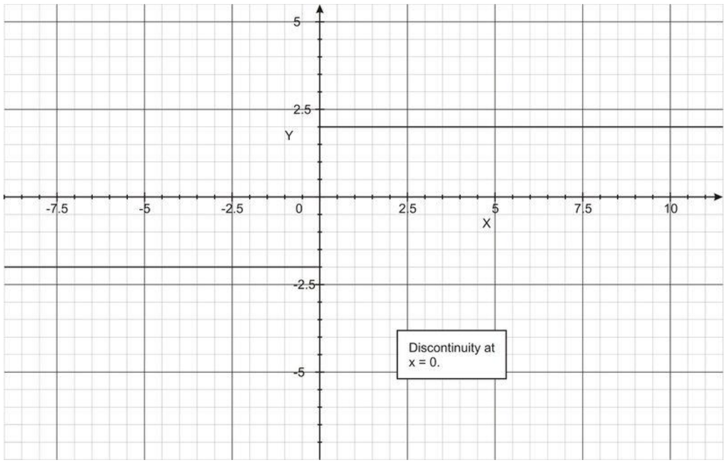

Figure 7

Many functions in mathematics do not have corners, cusps, vertical tangents, or jump discontinuities. We call them **differentiable functions**.

From what we have learned already about differentiability, it will not be difficult to show that continuity is an important condition for differentiability. The following theorem is one of the most important theorems in calculus:

Differentiability and Continuity

If f is differentiable at x_0, then f is also continuous at x_0.
The logically equivalent statement is quite useful: If f is *not* continuous at x_0, then f is not differentiable at x_0.
(The converse is not necessarily true.)

We have already seen that the converse is not true in some cases. The function can have a cusp, a corner, or a vertical tangent and still be continuous, but it is not differentiable.

Review Questions

In problems 1 through 6, use the definition of the derivative to find $f'(x)$ and then find the equation of the tangent line at $x=x_0$.

1. $f(x)=6x^2; x_0=3$
2. $f(x)=\sqrt{x+2}; x_0=8$
3. $f(x)=3x^3-2; x_0=-1$
4. $f(x)=\frac{1}{x+2}; x_0=-1$
5. $f(x)=ax^2-b$, (where a and b are constants); $x_0=b$

6. $f(x)=x^{1/3}; x_0=1$.
7. Find $dy/dx|_{x=1}$ given that $y=5x^2-2$.
8. Show that $f(x)=\sqrt[3]{x}$ is defined at $x=0$ but it is not differentiable at $x=0$. Sketch the graph.
9. Show that $f(x)=\begin{cases} x^2+1 & x\geq 1 \\ 2x & x>1 \end{cases}$ is continuous and differentiable at $x=1$. Hint: Take the limit from both sides. Sketch the graph of f.