Learning Objectives

A student will be able to:

- Know the chain rule and its proof.
- Apply the chain rule to the calculation of the derivative of a variety of composite functions.

We want to derive a rule for the derivative of a composite function of the form $f\circ g$ in terms of the derivatives of f and g. This rule allows us to differentiate complicated functions in terms of known derivatives of simpler functions.

The Chain Rule

If g is a differentiable function at x and f is differentiable at $g(x)$, then the composition function $f\circ g=f(g(x))$ is differentiable at x. The derivative of the composite function is:

$(f\circ g)'(x)=f'(g(x))g'(x)$.

Another way of expressing, if $u=u(x)$ and $f=f(u)$, then

$\frac{d}{dx}[f(u)]=f'(u)\frac{du}{dx}$.

And a final way of expressing the chain rule is the easiest form to remember: If y is a function of u and u is a function of x, then

$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$.

Example 1:

Differentiate $f(x)=(2x^3-4x^2+5)^2$.

Solution:

Using the chain rule, let $u=2x^3-4x^2+5$. Then

$\frac{d}{dx}[(2x^2-4x^2+5)^2]=\frac{d}{dx}[u^2]=2u\frac{du}{dx}=2(2x^3-4x^2+5)(6x^2-8x)$.

The example above is one of the most common types of composite functions. It is a power function of the type

$y=[u(x)]^n$.

The rule for differentiating such functions is called the **General Power Rule.** It is a special case of the Chain Rule.

The General Power Rule

if

$y=[u(x)]^n$

then

$\frac{dy}{dx}=n[u(x)]^{n-1}u'(x)$.

In simpler form, if

$y=u^n$
then

$y'=nu^{n-1}\cdot u'$.
Example 2:

What is the slope of the tangent line to the function $y=\sqrt{x^2-3x+2}$ that passes through point $x=3$?
Solution:

We can write $y=(x^2-3x+2)^{1/2}$. This example illustrates the point that n can be any real number including fractions. Using the General Power Rule,
$$\frac{dy}{dx}=\frac{1}{2}(x^2-3x+2)^{\frac{1}{2}-1}(2x-3)=\frac{1}{2}(x^2-3x+2)^{-1/2}(2x-3)=\frac{(2x-3)}{2\sqrt{x^2-3x+2}}$$
To find the slope of the tangent line, we simply substitute $x=3$ into the derivative:
$$\left.\frac{dy}{dx}\right|_{x=3}=\frac{2(3)-3}{2\sqrt{3^2-3(3)+2}}=\frac{3}{2\sqrt{2}}=\frac{3\sqrt{2}}{4}.$$
Example 3:

Find dy/dx for $y=\sin^3 x$.
Solution:

The function can be written as $y=[\sin x]^3$. Thus
$$\frac{dy}{dx}=3[\sin x]^2[\cos x]=3\sin^2 x\cos x$$
Example 4:

Find dy/dx for $y=5\cos(3x^2-1)$.
Solution:

Let $u=3x^2-1$. By the chain rule,
$$\frac{d}{dx}[f(u)]=f'(u)\frac{du}{dx}$$
where $f(u)=5\cos u$. Thus
$$\frac{dy}{dx}=5(-\sin u)(6x)=-5(6x)\sin u=-30x\sin(3x^2-1)$$
Example 5:

Find dy/dx for $y=[\cos(\pi x^2)]^3$.
Solution:

This example applies the chain rule twice because there are several functions embedded within each other.

Let u be the inner function and w be the innermost function.
$$y=(u(w))^3 \quad u(x)=\cos x \quad w(x)=\pi x^2.$$
Using the chain rule,

$$\frac{d}{dx}[f(u)] = \frac{d}{dx}[u^3] = f'(u)\frac{du}{dx} = \frac{d}{dx}[\cos^3(\pi x^2)] = \frac{d}{dx}[\cos(\pi x^2)]^3 = 3[\cos(\pi x^2)]^2[-\sin(\pi x^2)](2\pi x) = -6\pi x[\cos(\pi x^2)]^2\sin(\pi x^2).$$

Notice that we used the General Power Rule and, in the last step, we took the derivative of the argument.

Review Questions

Find $f'(x)$.

1. $f(x)=(2x^2-3x)^{39}$
2. $f(x)=(x^3-5x^2)^{-3}$
3. $f(x)=\sqrt{13x^2-6x+2}$
4. $f(x)=\sin^3 x$
5. $f(x)=\sin x^3$
6. $f(x)=\sin^3 x^3$
7. $f(x)=\tan(4x^5)$
8. $f(x)=\sqrt{4x-\sin^2 2x}$
9. $f(x)=\sin x\cos(3x-2)$
10. $f(x)=(5x+8)^3(x^3+7x)^{13}$
11. $f(x)=\left(\frac{x-3}{2x-5}\right)^3$