Learning Objectives

A student will be able to:

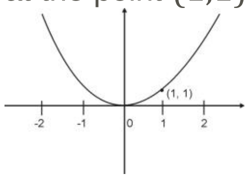
- Use linear approximations to study the limit process.
- Compute approximations for the slope of tangent lines to a graph.
- Introduce applications of differential calculus.

Introduction

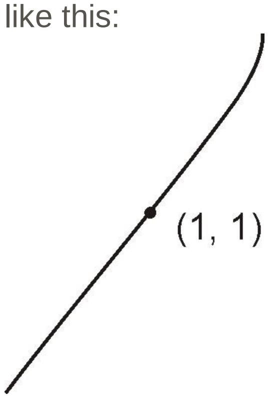
In this lesson we will begin our discussion of the key concepts of calculus. They involve a couple of basic situations that we will come back to time and again throughout the book. For each of these, we will make use of some basic ideas about how we can use straight lines to help approximate functions.

Let's start with an example of a simple function to illustrate each of the situations.

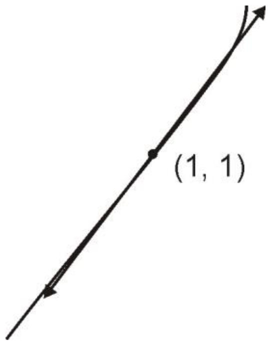
Consider the quadratic function $f(x)=x^2$. We recall that its graph is a parabola. Let's look at the point $(1,1)$ on the graph.



Suppose we magnify our picture and zoom in on the point $(1,1)$. The picture might look like this:



We note that the curve now looks very much like a straight line. If we were to overlay this view with a straight line that intersects the curve at $(1,1)$,our picture would look like this:



We can make the following observations. First, this line would appear to provide a good estimate of the value of $f(x)$ for $x-$values very close to $x=1$. Second, the approximations appear to be getting closer and closer to the actual value of the function as we take points on the line closer and closer to the point $(1,1)$. This line is called ***the tangent line to***$f(x)$ ***at*** $(1,1)$. This is one of the basic situations that we will explore in calculus.

Tangent Line to a Graph

Continuing our discussion of the tangent line to $f(x)$ at $(1,1)$, we next wish to find the equation of the tangent line. We know that it passes though $(1,1)$, but we do not yet have enough information to generate its equation. What other information do we need? ***(Answer: The slope of the line.)***

Yes, we need to find the slope of the line. We would be able to find the slope if we knew a second point on the line. So let's choose a point P on the line, very close to $(1,1)$. We can approximate the coordinates of Pusing the function $f(x)=x^2$; hence $P(x,x^2)$. Recall that for points very close to $(1,1)$, the points on the line are close approximate points of the function. Using this approximation, we can compute the slope of the tangent as follows:

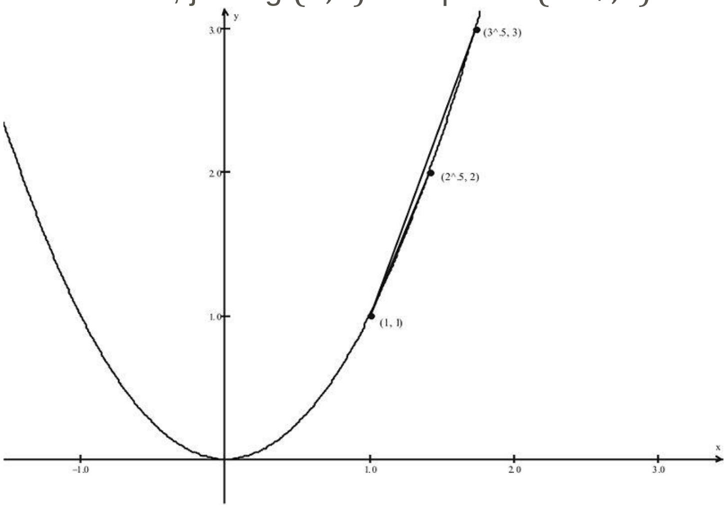
$m=(x^2-1)/(x-1)=x+1$ (Note: We choose points very close to $(1,1)$ but not the point itself, so $x\neq1$).

In particular, for $x=1.25$ we have $P(1.25,1.5625)$ and $m=x+1=2.25$. Hence the equation of the tangent line, in point slope form is $y-1=2.25(x-1)$. We can keep getting closer to the actual value of the slope by taking P closer to $(1,1)$, or x closer and closer to $x=1$, as in the following table:

$P(x,y)(1.2,1.44)(1.15,1.3225)(1.1,1.21)(1.05,1.1025)(1.005,1.010025)(1.0001,1.00020001)m2.22.152.12.052.0052.0001$

As we get closer to $(1,1)$, we get closer to the actual slope of the tangent line, the value 2. We call the slope of the tangent line at the point $(1,1)$ **the derivative of the function** $f(x)$ **at the point** $(1,1)$**.**

Let's make a couple of observations about this process. First, we can interpret the process graphically as finding secant lines from $(1,1)$ to other points on the graph. From the diagram we see a sequence of these secant lines and can observe how they begin to approximate the tangent line to the graph at $(1,1)$. The diagram shows a pair of secant lines, joining $(1,1)$ with points $(2-\sqrt{},2)$ and $(3-\sqrt{},3)$.

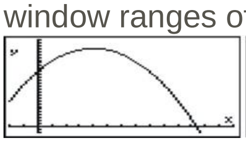
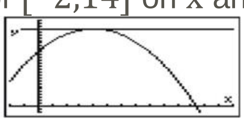


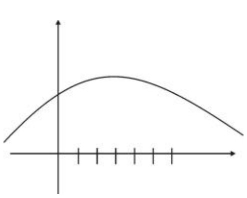
Second, in examining the sequence of slopes of these secants, we are systematically observing ***approximate slopes of the function*** as point Pgets closer to $(1,1)$. Finally, producing the table of slope values above was an inductive process in which we generated some data and then looked to deduce from our data the value to which the generated results tended. In this example, the slope values appear to approach the value 2.This process of finding how function values behave as we systematically get closer and closer to particular $x-$values is the process of finding ***limits***. In the next lesson we will formally define this process and develop some efficient ways for computing limits of functions.

Applications of Differential Calculus

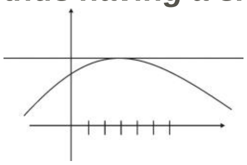
Maximizing and Minimizing Functions

Recall from Lesson 1.3 our example of modeling the number of Food Stamp recipients. The model was found to be $y=-0.5x^2+4x+19$with graph as follows: (Use viewing window ranges of $[-2,14]$ on x and $[-2,30]$ on y)



We note that the function appears to attain a maximum value about an $x-$value somewhere around $x=4$. Using the process from the previous example, what can we say about the tangent line to the graph for that $x$value that yields the maximum y value (the point at the top of the parabola)? ***(Answer: the tangent line will be horizontal, thus having a slope of*** 0***.)***



Hence we can use calculus to model situations where we wish to maximize or minimize a particular function. This process will be particularly important for looking at situations from business and industry where polynomial functions provide accurate models.

Velocity of a Falling Object

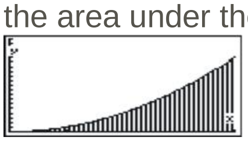
We can use differential calculus to investigate the velocity of a falling object. Galileo found that the distance traveled by a falling object was proportional to the square of the time it has been falling:

$s(t)=4.9t^2$.

The average velocity of a falling object from $t=a$ to $t=b$ is given by $(s(b)-s(a))/(b-a)$. HW Problem #10 will give you an opportunity to explore this relationship. In our discussion, we saw how the study of tangent lines to functions yields rich information about functions. We now consider the second situation that arises in Calculus, the central problem of ***finding the area under the curve of a function*** $f(x)$.

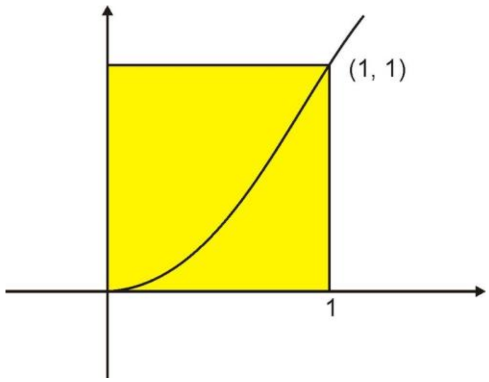
Area Under a Curve

First let's describe what we mean when we refer to the area under a curve. Let's reconsider our basic quadratic function $f(x)=x^2$. Suppose we are interested in finding the area under the curve from $x=0$ to $x=1$.

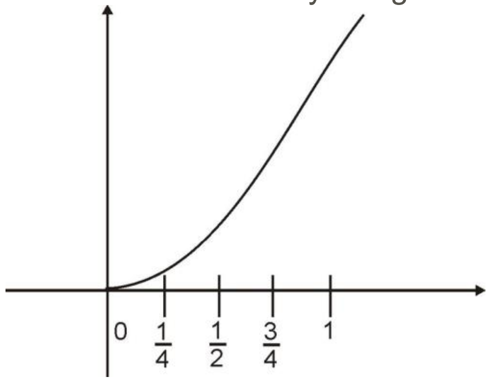


We see the cross-hatched region that lies between the graph and the $x-$axis. That is the area we wish to compute. As with approximating the slope of the tangent line to a function, we will use familiar linear methods to approximate the area. Then we will repeat the iterative process of finding better and better approximations.

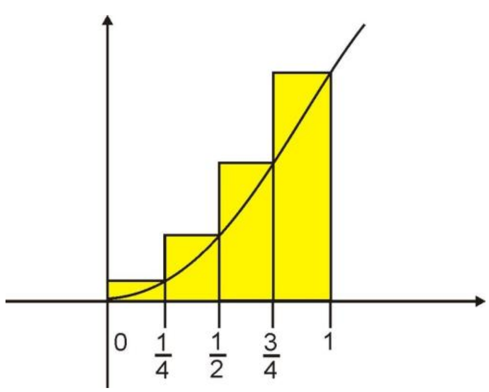
Can you think of any ways that you would be able to approximate the area? (Answer: One ideas is that we could compute the area of the square that has a corner at $(1,1)$ to be $A=1$ and then take half to find an area $A=1/2$. This is one estimate of the area and it is actually a pretty good first approximation.)



We will use a variation of this covering of the region with quadrilaterals to get better approximations. We will do so by dividing the $x-$interval from $x=0$ to $x=1$ into equal sub-intervals. Let's start by using four such subintervals as indicated:



We now will construct four rectangles that will serve as the basis for our approximation of the area. The subintervals will serve as the width of the rectangles. We will take the length of each rectangle to be the maximum value of the function in the subinterval. Hence we get the following figure:



If we call the rectangles R1–R4, from left to right, then we have the areas

$R_1R_2R_3R_4=14*f(14)=164,=14*f(12)=116,=14*f(34)=964,=14*f(1)=14,$
and $R_1+R_2+R_3+R_4=3064=1532.$

Note that this approximation is very close to our initial approximation of $1/2$. However, since we took the maximum value of the function for a side of each rectangle, this process tends to overestimate the true value. We could have used the minimum value of the function in each sub-interval. Or we could have used the value of the function at the midpoint of each sub-interval.

Can you see how we are going to improve our approximation using successive iterations like we did to approximate the slope of the tangent line? ***(Answer: we will sub-divide the interval from*** $x=0$ ***to*** $x=1$ ***into more and more sub-intervals, thus creating successively smaller and smaller rectangles to refine our estimates.)***

Example 1:

The following table shows the areas of the rectangles and their sum for rectangles having width $w=1/8$.

Rectangle R_i	**Area of** R_i
R_1	1512
R_2	4512
R_3	9512
R_4	16512
R_5	25512
R_6	36512
R_7	49512
R_8	64512

$A=\sum R_i=195512$. This value is approximately equal to $.3803$. Hence, the approximation is now quite a bit less than $.5$. For sixteen rectangles, the value is 14324096 which is approximately equal to $.34$. Can you guess what the true area will approach? ***(Answer: using our successive approximations, the area will approach the value*** $1/3$.***)***

We call this process of finding the area under a curve ***integration of*** $f(x)$ ***over the interval*** $[0,1]$.

Applications of Integral Calculus

We have not yet developed any computational machinery for computing ***derivatives*** and ***integrals*** so we will just state one popular application of integral calculus that relates the derivative and integrals of a function.

Example 2:

There are quite a few applications of calculus in business. One of these is the cost function $C(x)$ of producing x items of a product. It can be shown that the derivative of the cost function that gives the slope of the tangent line is another function that that gives the cost to produce an additional unit of the product. This is called the ***marginal cost*** and is a very important piece of information for management to have. Conversely, if one knows the marginal cost as a function of x, then finding the area under the curve of the function will give back the cost function $C(x)$.

Lesson Summary

1. We used linear approximations to study the limit process.
2. We computed approximations for the slope of tangent lines to a graph.
3. We analyzed applications of differential calculus.
4. We analyzed applications of integral calculus.

Review Questions

1. For the function $f(x)=x^2$ approximate the slope of the tangent line to the graph at the point $(3,9)$.
   a. Use the following set of $x-$values to generate the sequence of secant line slopes: $x=2.9, 2.95, 2.975, 2.995, 2.999$.
   b. What value does the sequence of slopes approach?
2. Consider the function $f(x)=x^2$.
   a. For what values of x would you expect the slope of the tangent line to be negative?
   b. For what value of x would you expect the tangent line to have slope $m=0$?
   c. Give an example of a function that has two different horizontal tangent lines?
3. Consider the function $p(x)=x^3-x$. Generate the graph of $p(x)$using your calculator.
   a. Approximate the slope of the tangent line to the graph at the point $(2,6)$. Use the following set of $x-$values to generate the sequence of secant line slopes. $x=2.1, 2.05, 2.005, 2.001, 2.0001$.
   b. For what values of x do the tangent lines appear to have slope of 0? (Hint: Use the calculate function in your calculator to approximate the $x-$values.)
   c. For what values of x do the tangent lines appear to have positive slope?
   d. For what values of x do the tangent lines appear to have negative slope?

4. The cost of producing x Hi−Fi stereo receivers by Yamaha each week is modeled by the following function: $C(x)=850+200x-.3x^2$.
   a. Generate the graph of $C(x)$ using your calculator. (Hint: Change your viewing window to reflect the high y values.)
   b. For what number of units will the function be maximized?
   c. Estimate the slope of the tangent line at $x=200,300,400$.
   d. Where is marginal cost positive?
5. Find the area under the curve of $f(x)=x^2$ from $x=1$ to $x=3$. Use a rectangle method that uses the minimum value of the function within sub-intervals. Produce the approximation for each case of the subinterval cases.
   a. four sub-intervals.
   b. eight sub-intervals.
   c. Repeat part a. using a Mid-Point Value of the function within each sub-interval.
   d. Which of the answers in a. - c. provide the best estimate of the actual area?
6. Consider the function $p(x)=-x^3+4x$.
   a. Find the area under the curve from $x=0$ to $x=1$.
   b. Can you find the area under the curve from $x=-1$ to $x=0$. Why or why not? What is problematic for this computation?
7. Find the area under the curve of $f(x)=x--\sqrt{}$ from $x=1$ to $x=4$. Use the Max Value rectangle method with six sub-intervals to compute the area.
8. The Eiffel Tower is $320 meters$ high. Suppose that you drop a ball off the top of the tower. The distance that it falls is a function of time and is given by $s(t)=4.9t^2$.

Find the velocity of the ball after $4 seconds$. (Hint: the average velocity for a time interval is **average velocity** = **change in distance/change in time**. Investigate the average velocity for t intervals close to $t=4$ such as $3.9\leq t\leq 4$ and closer and see if a pattern is evident.)