## Learning Objectives

A student will be able to:

- Use various techniques of differentiations to find the derivatives of various functions.
- Compute derivatives of higher orders.

Up to now, we have been calculating derivatives by using the definition. In this section, we will develop formulas and theorems that will calculate derivatives in more efficient and quick ways. It is highly recommended that you become very familiar with all of these techniques.

## The Derivative of a Constant

If $f(x)=c$ where $c$ is a constant, then $f'(x)=0$.
In other words, the derivative or slope of any constant function is zero.

*Proof:*

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{c-c}{h}=0$$

**Example 1:**

If $f(x)=16$ for all $x$, then $f'(x)=0$ for all $x$. We can also write $d/dx(16)=0$.

**The Power Rule**

If $n$ is a positive integer, then for all real values of $x$

$$\frac{d}{dx}[x^n]=nx^{n-1}$$

The proof of the power rule is omitted in this text, but it is available at http://en.wikipedia.org/wiki/Calculus_with_polynomials and also in video form at Khan Academy Proof of the Power Rule.
Note that this proof depends on using the binomial theorem from precalculus.

**Example 2:**

If $f(x)=x^3$, then

$$f'(x)=3x^{3-1}=3x^2$$

and

$$\frac{d}{dx}[x]=1\cdot x^{1-1}=x^0=1,$$
$$\frac{d}{dx}[\sqrt{x}]=\frac{d}{dx}[x^{1/2}]=\frac{1}{2}x^{1/2-1}=\frac{1}{2}x^{-1/2}=\frac{1}{2x^{1/2}}=\frac{1}{2\sqrt{x}},$$
$$\frac{d}{dx}\left[\frac{1}{x^3}\right]=\frac{d}{dx}[x^{-3}]=-3x^{-3-1}=-3x^{-4}=-\frac{3}{x^4},$$

**The Power Rule and a Constant**

If $c$ is a constant and $f$ is differentiable at all $x$, then

$$\frac{d}{dx}[cf(x)]=c\frac{d}{dx}[f(x)].$$

In simpler notation,

$(cf)'=c(f)'\cdot cf'$.

In other words, the derivative of a constant times a function is equal to the constant times the derivative of the function.

**Example 3:**

$ddx[4x^3]=4ddx[x^3]=4[3x^2]=12x^2$.

**Example 4:**

$ddx[-2x^4]=ddx[-2x^{-4}]=-2ddx[x^{-4}]=-2[-4x^{-4-1}]=-2[-4x^{-5}]=8x^{-5}=8x^5$.

**Derivatives of Sums and Differences**

If $f$ and $g$ are two differentiable functions at $x$, then

$ddx[f(x)+g(x)]=ddx[f(x)]+ddx[g(x)]$

and

$ddx[f(x)-g(x)]=ddx[f(x)]-ddx[g(x)]$.

In simpler notation,

$(f\pm g)'=f'\pm g'$.

**Example 5:**

$ddx[3x^2+2x]=ddx[3x^2]+ddx[2x]=3ddx[x^2]+2ddx[x]=3[2x]+2[1]=6x+2$.

**Example 6:**

$ddx[x^3-5x^2]=ddx[x^3]-5ddx[x^2]=3x^2-5[2x]=3x^2-10x$.

**The Product Rule**

If $f$ and $g$ are differentiable at $x$, then

$ddx[f(x)\cdot g(x)]=f(x)ddxg(x)+g(x)ddxf(x)$.

In a simpler notation,

$(f\cdot g)'=f\cdot g'+g\cdot f'$.

The derivative of the product of two functions is equal to the first times the derivative of the second plus the second times the derivative of the first.

Keep in mind that

$(f\cdot g)'\neq f'\cdot g'$.

**Example 7:**

Find $dydx$ for $y=(3x^4+2)(7x^3-1)$.

***Solution:***

There are two methods to solve this problem. One is to multiply the product and then use the derivative of the sum rule. The second is to directly use the product rule. Either rule will produce the same answer. We begin with the sum rule.

$$y=(3x^4+2)(7x^3-1)=21x^7-3x^4+14x^3-2.$$

Taking the derivative of the sum yields

$$\frac{dy}{dx}=147x^6-12x^3+42x^2+0=147x^6-12x^3+42x^2.$$

Now we use the product rule,

$$\frac{dy}{dx}=(3x^4+2)\cdot(7x^3-1)'+(3x^4+2)'\cdot(7x^3-1)=(3x^4+2)(21x^2)+(12x^3)(7x^3-1)=(63x^6+42x^2)+(84x^6-12x^3)=147x^6-12x^3+42x^2,$$

which is the same answer.

**The Quotient Rule**

If $f$ and $g$ are differentiable functions at $x$ and $g(x)\neq 0$, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}[f(x)]-f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

In simpler notation,

$$\left(\frac{f}{g}\right)'=\frac{g\cdot f'-f\cdot g'}{g^2}.$$

The derivative of a quotient of two functions is the bottom times the derivative of the top minus the top times the derivative of the bottom all over the bottom squared.

Keep in mind that the order of operations is important (because of the minus sign in the numerator) and

$$\left(\frac{f}{g}\right)'\neq\frac{f'}{g'}.$$

**Example 8:**

Find $dy/dx$ for

$$y=\frac{x^2-5}{x^3+2}$$

***Solution:***

$$\frac{dy}{dx}=\frac{d}{dx}\left[\frac{x^2-5}{x^3+2}\right]=\frac{(x^3+2)\cdot(x^2-5)'-(x^2-5)\cdot(x^3+2)'}{(x^3+2)^2}=\frac{(x^3+2)(2x)-(x^2-5)(3x^2)}{(x^3+2)^2}=\frac{2x^4+4x-3x^4+15x^2}{(x^3+2)^2}=\frac{-x^4+15x^2+4x}{(x^3+2)^2}=\frac{x(-x^3+15x+4)}{(x^3+2)^2}.$$

**Example 9:**

At which point(s) does the graph of $y=\frac{x}{x^2+9}$ have a horizontal tangent line?

***Solution:***

Since the slope of a horizontal line is zero, and since the derivative of a function signifies the slope of the tangent line, then taking the derivative and equating it to zero will enable us to find the points at which the slope of the tangent line equals to zero, i.e., the locations of the horizontal tangents.

$yy'=xx^2+9,=(x^2+9)(1)-x(2x)(x^2+9)^2=0.$
Multiplying by the denominator and solving for $x$,
$x^2+9-2x^2x^2x=0=9=\pm 3.$
Therefore the tangent line is horizontal at $x=-3,+3.$
**Higher Derivatives**

If the derivative $f'$ ***of the function*** $f$ ***is differentiable, then the derivative of*** $f'$, denoted by $f''$, is called the **second derivative** of $f$. We can continue the process of differentiating derivatives and obtain the third, fourth, fifth, and even higher derivatives of $f$. They are denoted by $f'''$, $f^{(4)}$, $f^{(5)}$, etc.
The second derivative, $f''$, can also be written as $d^2ydx^2$, and $f'''$ can be written as $d^3ydx^3$. For still higher derivatives, $f^{(n)}=d^nxdx^n$.
**Example 10:**

Find the fifth derivative of $f(x)=2x^4-3x^3+5x^2-x-1$.
***Solution:***

$f'(x)f''(x)f'''(x)f^{(4)}(x)f^{(5)}(x)=8x^3-9x^2+10x-1=24x^2-18x+10=48x-18=48=0$
**Example 11:**

Show that $y=x^3+3x+2$ satisfies the differential equation $y'''+xy''-2y'=0.$
***Solution:***

We need to obtain the first, second, and third derivatives and substitute them into the differential equation.

$yy'y''y'''=x^3+3x+2=3x^2+3=6x=6.$
Substituting,

$y'''+xy''-2y'=6+x(6x)-2(3x^2+3)=6+6x^2-6x^2-6=0$
which satisfies the equation.

## Review Questions

Use the results of this section to find the derivatives $dy/dx$.

1. $y=5x^7$
2. $y=12(x^3-2x^2+1)$
3. $y=2-\sqrt{}x^3-12-\sqrt{}x^2+2x+2-\sqrt{}$
4. $y=a^2-b^2+x^2-a-b+x$ (where $a$, $b$ are constants)
5. $y=x^{-3}+1x^7$

6. $y=(x^3-3x^2+x)(2x^3+7x^4)$
7. $y=(1x+1x^2)(3x^4-7)$
8. $y=x--\surd+1x--\surd$
9. $y=3x--\surd+3$
10. $y=4x+1x^2-9$
11. Newton’s Law of Universal Gravitation states that the gravitational force between two masses (say, the earth and the moon), $m$ and $M$, is equal to their product divided by the square of the distance $r$ between them. Mathematically, $F=GmMr^2$,where $G$ is the Universal Gravitational Constant $(1.602\times10^{-11}\ Nm^2kg^2)$. If the distance $r$ between the two masses is changing, find a formula for the instantaneous rate of change of $F$with respect to the separation distance $r$.
12. Find $dd\psi[\psi\psi_0+\psi^3 3-\psi_0]$where $\psi_0$ is a constant.
13. Find $d^3ydx^3||_{x=1}$ , where $y=2x^3$ .