Learning Objectives

A student will be able to:

- Demonstrate an understanding of the slope of the tangent line to the graph.
- Demonstrate an understanding of the instantaneous rate of change.

A car speeding down the street, the inflation of currency, the number of bacteria in a culture, and the AC voltage of an electric signal are all examples of quantities that change with time. In this section, we will study the rate of change of a quantity and how is it related to the tangent lines on a curve.

The Tangent Line

If two points $P(x_0,y_0)$ and $Q(x_1,y_1)$ are two different points of the curve $y=f(x)$, then the slope of the secant line connecting the two points is given by

$$m_{sec}=\frac{y_1-y_0}{x_1-x_0}=\frac{f(x_1)-f(x_0)}{x_1-x_0} \quad (1)$$

Now if we let the point x_1 approach x_0, Q will approach P along the graph f. Thus the slope of the secant line will gradually approach the slope of the tangent line as x_1 approaches x_0. Therefore (1) becomes

$$m_{tan}=\lim_{x_1\to x_0}\frac{f(x_1)-f(x_0)}{x_1-x_0}. \quad (2)$$

If we let $h=x_1-x_0$, then $x_1=x_0+h$ and $h\to 0$ becomes equivalent to $x_1\to x_0$, so (2) becomes

$$m_{tan}=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}.$$

If the point $P(x_0,y_0)$ is on the curve f, then the tangent line at P has a slope that is given by

$$m_{tan}=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

provided that the limit exists.

Recall from algebra that the *point-slope* form for the tangent line is given by

$$y-y_0=m_{tan}(x-x_0).$$

Example 1:

Find the slope of the tangent line to the curve $f(x)=x^3$ passing through point $P(2,8)$.

Solution:

Since $P(x_0,y_0)=(2,8)$, using the slope of the tangent equation,

$$m_{tan}=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

we get

$$m_{tan}=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{(h^3+6h^2+12h+8)-8}{h}=\lim_{h\to 0}\frac{h^3+6h^2+12h}{h}=\lim_{h\to 0}(h^2+6h+12)=12.$$

Thus the slope of the tangent line is 12. Using the point-slope formula above,
$y-8=12(x-2)$
or

$y=12x-16$
Next we are interested in finding a formula for the slope of the tangent line at *any point* on the curve f. Such a formula would be the same formula that we are using except we replace the constant x_0 by the variable x. This yields
$m_{tan}=\lim_{h\to 0} f(x+h)-f(x)h$.
We denote this formula by

$f'(x)=\lim_{h\to 0} f(x+h)-f(x)h,$
where $f(x)$ is read "f prime of x." The next example illustrate its usefulness.
Example 2:

If $f(x)=x^2-3$, find $f(x)$ and use the result to find the slope of the tangent line at $x=2$ and $x=-1$.
Solution:

Since

$f'(x)=\lim_{h\to 0} f(x+h)-f(x)h,$
then

$f'(x)=\lim_{h\to 0}[(x+h)^2-3]-[x^2-3]h=\lim_{h\to 0} x^2+2xh+h^2-3-x^2+3h=\lim_{h\to 0} 2xh+h^2h=\lim_{h\to 0}(2x+h)=2x$
To find the slope, we simply substitute $x=2$ into the result $f(x)$,
$f'(x)f'(2)=2x=2(2)=4$
and

$f'(x)f'(-1)=2x=2(-1)=-2$
Thus slopes of the tangent lines at $x=2$ and $x=-1$ are 4 and -2,respectively.
Example 3:

Find the slope of the tangent line to the curve $y=1x$ that passes through the point $(1,1)$.
Solution:

Using the slope of the tangent formula

$f'(x)=\lim_{h\to 0} f(x+h)-f(x)h$
and substituting $y=1x$,

$$y' = \lim_{h\to 0}\frac{\left(\frac{1}{x+h}\right)-\frac{1}{x}}{h} = \lim_{h\to 0}\frac{\frac{x-x-h}{x(x+h)}}{h} = \lim_{h\to 0}\frac{x-x-h}{hx(x+h)} = \lim_{h\to 0}\frac{-h}{hx(x+h)} = \lim_{h\to 0}\frac{-1}{x(x+h)} = \lim_{h\to 0}\frac{-1}{x(x+h)} = -\frac{1}{x^2}.$$

Substituting $x=1$,

$$y' = -\frac{1}{1} = -1.$$

Thus the slope of the tangent line at $x=1$ for the curve $y=\frac{1}{x}$ is $m=-1$. To find the equation of the tangent line, we simply use the point-slope formula,

$$y-y_0=m(x-x_0),$$

where $(x_0,y_0)=(1,1)$.

$$y-1=-1(x-1)=-x+1+1=-x+2,$$

which is the equation of the tangent line.

Average Rates of Change

The primary concept of calculus involves calculating the rate of change of a quantity with respect to another. For example, speed is defined as the rate of change of the distance travelled with respect to time. If a person travels 120miles in four hours, his speed is $120/4=30 \text{mi/hr}$. This speed is called the *average speed* or the *average rate of change* of distance with respect to time. Of course the person who travels 120milesat a rate of 30mi/hr for four hours does not do so continuously. He must have slowed down or sped up during the four-hour period. But it does suffice to say that he traveled for four hours at an average rate of 30miles per hour. However, if the driver strikes a tree, it would not be his average speed that determines his survival but his speed at the *instant of the collision*. Similarly, when a bullet strikes a target, it is not the average speed that is significant but its *instantaneous speed* at the moment it strikes. So here we have two distinct kinds of speeds, average speed and instantaneous speed.

The average speed of an object is defined as the object's displacement Δx divided by the time interval Δt during which the displacement occurs:

$$v=\frac{\Delta x}{\Delta t}=\frac{x_1-x_0}{t_1-t_0}.$$

Notice that the points (t_0,x_0) and (t_1,x_1) lie on the position-versus-time curve, as Figure 1 shows. This expression is also the expression for the slope of a secant line connecting the two points. Thus we conclude that the average velocity of an object between time t_0 and t_1 is represented geometrically by the slope of the secant line connecting the two points (t_0,x_0) and (t_1,x_1). If we choose t_1 close to t_0, then the average velocity will closely approximate the instantaneous velocity at time t_0.

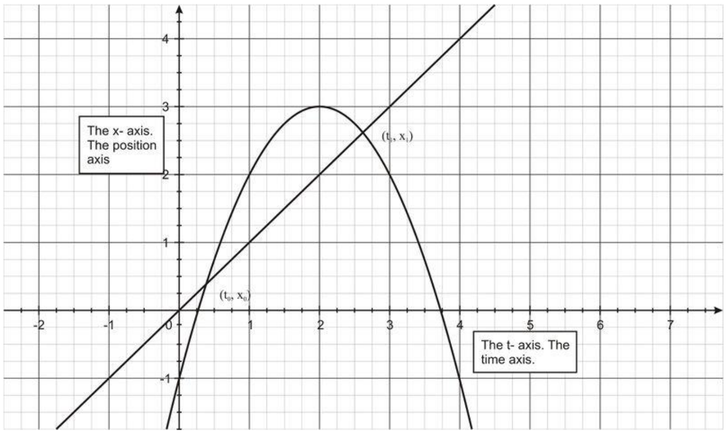

Figure 1

Geometrically, the average rate of change is represented by the slope of a secant line and the instantaneous rate of change is represented by the slope of the tangent line (Figures 2 and 3).

Average Rate of Change (such as the *average velocity*)

The average rate of change of $x=f(t)$ over the time interval $[t_0,t_1]$ is the slope m_{sec} of the secant line to the points $(t_0,f(t_0))$ and $(t_1,f(t_1))$on the graph (Figure 2).

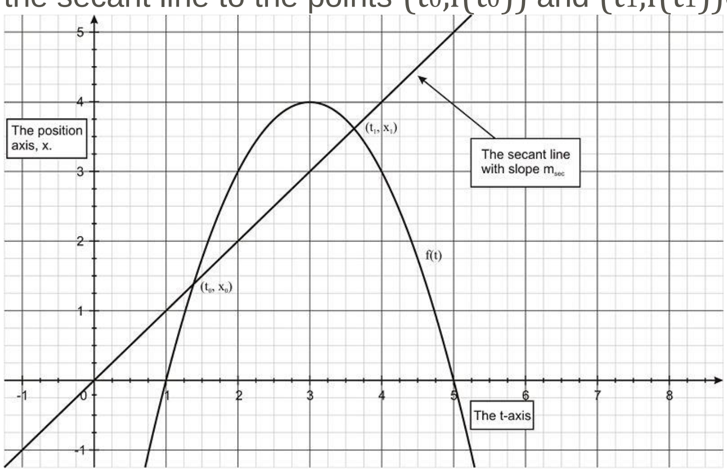

Figure 2

$$m_{sec}=\frac{f(t_1)-f(t_0)}{t_1-t_0}$$

Instantaneous Rate of Change

The instantaneous rate of change of $x=f(t)$ at the time t_0 is the slope m_{tan} of the tangent line at the time t_0 on the graph.

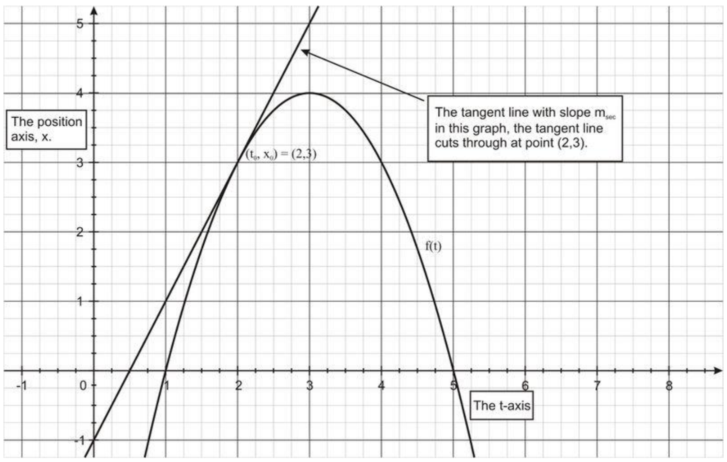

Figure 3

$$m_{tan}=f'(t_0)=\lim_{t_1\to t_0}\frac{f(t_1)-f(t_0)}{t_1-t_0}$$

Example 4:

Suppose that $y=x^2-3$.

1. Find the average rate of change of y with respect to x over the interval $[0,2]$.
2. Find the instantaneous rate of change of y with respect to x at the point $x=-1$.

Solution:

1. Applying the formula for Average Rate of Change with $f(x)=x^2-3$ and $x_0=0$ and $x_1=2$ yields

$$m_{sec}=\frac{f(x_1)-f(x_0)}{x_1-x_0}=\frac{f(2)-f(0)}{2-0}=\frac{1-(-3)}{2}=2$$

This means that the average rate of change of y is 2 units per unit increase in x over the interval $[0,2]$.

2. From the example above, we found that $f'(x)=2x$, so

$$m_{tan}=f'(x_0)=f'(-1)=2(-1)=-2$$

This means that the instantaneous rate of change is negative. That is, y is decreasing at $x=-1$. It is decreasing at a rate of 2 units per unit increase in x.

Review Questions

1. Given the function $y=1/2\ x^2$ and the values of $x_0=3$ and $x_1=4$, find
   a. The average rate of change of y with respect to x over the interval $[x_0,x_1]$.
   b. The instantaneous rate of change of y with respect to x at x_0.
   c. The slope of the tangent line at x_1.
   d. The slope of the secant line between points x_0 and x_1.
   e. Make a sketch of $y=1/2\ x^2$ and show the secant and tangent lines at their respective points.

2. Repeat problem #1 for $f(x)=1/x$ and the values $x_0=2$ and $x_1=3$.
3. Find the slope of the graph $f(x)=x^2+1$ at a general point x. What is the slope of the tangent line at $x_0=6$?
4. Suppose that $y=1/x--\sqrt{}$.
    a. Find the average rate of change of y with respect to x over the interval $[1,3]$.
    b. Find the instantaneous rate of change of y with respect to x at point $x=1$.
5. A rocket is propelled upward and reaches a height (in meters) of $h(t)=4.9t^2$ in $t \mathrm{seconds}$.
    a. How high does it reach in $35\mathrm{seconds}$?
    b. What is the average velocity of the rocket during the first $35\mathrm{seconds}$?
    c. What is the average velocity of the rocket during the first $200\mathrm{meters}$?
    d. What is the instantaneous velocity of the rocket at the end of the $35\mathrm{seconds}$?
6. A particle moves in the positive direction along a straight line so that after $t\mathrm{nanoseconds}$, its traversed distance is given by $\chi(t)=9.9t^3\mathrm{nanometers}$.
    a. What is the average velocity of the particle during the first $2\mathrm{nanoseconds}$?
    b. What is the instantaneous velocity of the particle at $t=2\mathrm{nanoseconds}$?