Calculus/Differentiation/Applications of Derivatives/Solutions

Relative Extrema

Find the relative maximum(s) and minimum(s), if any, of the following functions.

1. $f(x) = \frac{x}{x+1}$

$f'(x) = \frac{1}{(x+1)^2}$

There are no roots of the derivative. The derivative fails to exist when x=-1 , but the function also fails to exists at that point, so it is not an extremum. Thus, **the function has no relative extrema.**

2. $f(x) = (x-1)^{2/3}$

$f'(x) = \frac{2}{3}(x-1)^{-1/3} = \frac{2}{3\sqrt[3]{x-1}}$

There are no roots of the derivative. The derivative fails to exist at $x=1$. $f(1)=0$. **The point** $(1,0)$ **is a minimum** since $f(x)$ is nonnegative because of the even numerator in the exponent. **The function has no relative maximum.**

3. $f(x) = x^2 + \frac{2}{x}$

$f'(x) = 2x - \frac{2}{x^2}$

$2x - \frac{2}{x^2} = 0 \implies 2x^3 - 2 = 0 \implies x = 1$

$f''(x) = 2 + \frac{4}{x^3}$

$f''(1) = 6$

Since the second derivative is positive, $x=1$ **corresponds to a relative minimum.**

The derivative fails to exist when $x=0$, but so does the function. **There is no relative maximum.**

4. $f(s) = \frac{s}{1+s^2}$

$f'(x) = \frac{(1+s^2)-2s^2}{(1+s^2)^2} = \frac{1-s^2}{(1+s^2)^2}$

$\frac{1-s^2}{(1+s^2)^2} = 0 \implies s = \pm 1$

$f''(x) = \frac{-2s(1+s^2)^2 - 2(1+s^2)(2s)(1-s^2)}{(1+s^2)^4}$

$f''(-1) = \frac{-2(-1)(1+(-1)^2)^2 - 2(1+(-1)^2)(2(-1))(1-(-1)^2)}{(1+(-1)^2)^4} = \frac{8}{16} = \frac{1}{2}$

Since the second derivative of $f(x)$ at $x=-1$ is positive, $x=-1$ **corresponds to a relative mimimum.**

$f''(1) = \frac{-2(1+1^2)^2 - 2(1+1^2)(2)(1-1^2)}{(1+1^2)^4} = \frac{-8}{16} = -\frac{1}{2}$

Since the second derivative of $f(x)$ at $x=1$ is negative, $x=1$ **corresponds to a relative maximum.**

5. $f(x) = x^2 - 4x + 9$

$f'(x) = 2x - 4$

$2x - 4 = 0 \implies x = 2$

$f''(x) = 2$

Since the second derivative is positive, $x=2$ **corresponds to a relative minimum. There is no relative maximum.**

6. $f(x) = \frac{x^2+x+1}{x^2-x+1}$

$$f'(x) = \frac{(x^2-x+1)(2x+1)-(2x-1)(x^2+x+1)}{(x^2-x+1)^2}$$
$$= \frac{2x^3-2x^2+2x+x^2-x+1-2x^3-2x^2-2x+x^2+x+1}{(x^2-x+1)^2}$$
$$= \frac{-2x^2+2}{(x^2-x+1)^2}$$
$$\frac{-2x^2+2}{(x^2-x+1)^2} = 0 \implies x = \pm 1$$
$$f''(x) = \frac{(-4x)(x^2-x+1)^2-(2x-1)(-2x^2+2)}{(x^2-x+1)^4}$$
$$f''(-1) = \frac{(-4(-1))((-1)^2-(-1)+1)^2-(2(-1)-1)(-2(-1)^2+2)}{((-1)^2-(-1)+1)^4} = \frac{36}{81} = \frac{4}{9}$$
Since $f''(-1)$ is positive, $x=-1$ **corresponds to a relative minimum.**
$$f''(1) = \frac{-4}{1} = -4$$
Since $f''(1)$ is negative, $x=1$ **corresponds to a relative maximum.**

Range of Function

7. Show that the expression $x+1/x$ cannot take on any value strictly between 2 and -2.

$f(x) = x + \frac{1}{x}$
$f'(x) = 1 - \frac{1}{x^2}$
$1 - \frac{1}{x^2} = 0 \implies x = \pm 1$
$f''(x) = \frac{2}{x^3}$
$f''(-1) = -2$
Since $f''(-1)$ is negative, $x=-1$ corresponds to a relative maximum.
$f(-1) = -2$
$\lim_{x \to -\infty} f(x) = -\infty$
For $x < -1$, $f'(x)$ is positive, which means that the function is increasing. Coming from very negative x-values, f increases from a very negative value to reach a relative maximum of -2 at $x=-1$.
For $-1 < x < 1$, $f'(x)$ is negative, which means that the function is decreasing.
$\lim_{x \to 0^-} f(x) = -\infty$
$\lim_{x \to 0^+} f(x) = +\infty$
$f''(1) = 2$
Since $f''(1)$ is positive, $x=1$ corresponds to a relative minimum.
$f(1) = 2$
Between $[-1, 0)$ the function decreases from -2 to $-\infty$, then jumps to $+\infty$ and decreases until it reaches a relative minimum of 2 at $x=1$.
For $x > 1$, $f'(x)$ is positive, so the function increases from a minimum of 2.
The above analysis shows that there is a gap in the function's range between -2 and 2.

Absolute Extrema

Determine the absolute maximum and minimum of the following functions on the given domain

8. $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 1$ on $[0, 3]$

f is differentiable on $[0, 3]$, so the extreme value theorem guarantees the existence of an absolute maximum and minimum on $[0, 3]$. Find and check the critical points:
$f'(x) = x^2 - x$
$x^2 - x = 0 \implies x = 0, 1$
$f(0) = 1$
$f(1) = \frac{5}{6}$
Check the endpoint:
$f(3) = \frac{11}{2}$

Maximum at $(3, \frac{11}{2})$**; minimum at** $(1, \frac{5}{6})$

9. $f(x) = (\frac{4}{3}x^2 - 1)x$ on $[-\frac{1}{2}, 2]$

$f'(x) = (\frac{4}{3}x^2 - 1) + x(\frac{8}{3}x) = 4x^2 - 1$
$4x^2 - 1 = 0 \implies x = \pm\frac{1}{2}$
$f(-\frac{1}{2}) = (\frac{1}{3} - 1)(-\frac{1}{2}) = \frac{1}{3}$
$f(\frac{1}{2}) = (\frac{1}{3} - 1)(\frac{1}{2}) = -\frac{1}{3}$
$f(2) = (\frac{16}{3} - 1)(2) = \frac{26}{3}$
Maximum at $(2, \frac{26}{3})$**; minimum at** $(\frac{1}{2}, -\frac{1}{3})$

Determine Intervals of Change

Find the intervals where the following functions are increasing or decreasing

10. $f(x) = 10 - 6x - 2x^2$

$f'(x) = -6 - 4x$
$-6 - 4x = 0 \implies x = -\frac{3}{2}$

$f'(x)$ is the equation of a line with negative slope, so $f'(x)$ is positive for $x < -\frac{3}{2}$ and negative for $x > \frac{3}{2}$.

This means that the function is **increasing on** $(-\infty, -\frac{3}{2})$ **and decreasing on** $(\frac{3}{2}, +\infty)$.

11. $f(x) = 2x^3 - 12x^2 + 18x + 15$

$f'(x) = 6x^2 - 24x + 18$

$$\begin{aligned} 6x^2 - 24x + 18 = 0 &\implies x - 4x + 3 = 0 \\ &\implies (x-1)(x-3) = 0 \\ &\implies x = 1, 3 \end{aligned}$$

$f'(x)$ is the equation of a bowl-shaped parabola that crosses the x-axis at 1 and 3, so $f'(x)$ is negative for $1 < x < 3$ and positive elsewhere.

This means that the function is **decreasing on** $(1, 3)$ **and increasing elsewhere.**

12. $f(x) = 5 + 36x + 3x^2 - 2x^3$

$f'(x) = 36 + 6x - 6x^2$

$$\begin{aligned} 36 + 6x - 6x^2 = 0 &\implies 6 + x - x^2 = 0 \\ &\implies (x+2)(-x+3) = 0 \\ &\implies x = -2, 3 \end{aligned}$$

$f'(x)$ is the equation of a hill-shaped parabola that crosses the x-axis at -2 and 3, so $f'(x)$ is positive for $-2 < x < 3$ and negative elsewhere.

This means that the function is **increasing on** $(-2, 3)$ **and decreasing elsewhere.**

13. $f(x) = 8 + 36x + 3x^2 - 2x^3$

If you did the previous exercise then no calculation is required since this function has the same derivative as that function and thus is increasing and decreasing on the same intervals; i.e., the function is **increasing on** $(-2, 3)$ **and decreasing elsewhere**.

14. $f(x) = 5x^3 - 15x^2 - 120x + 3$

$f'(x) = 15x^2 - 30x - 120$

$$\begin{aligned} 15x^2 - 30x - 120 = 0 &\implies x^2 - 2x - 8 = 0 \\ &\implies (x+2)(x-4) = 0 \\ &\implies x = -2, 4 \end{aligned}$$

f' is negative on $(-2, 4)$ and positive elsewhere.
So f is **decreasing on** $(-2, 4)$ **and increasing elsewhere.**

15. $f(x) = x^3 - 6x^2 - 36x + 2$

$f'(x) = 3x^2 - 12x - 36$
$3x^2 - 12x - 36 = 0 \implies x^2 - 4x - 12 = 0$
$\implies (x+2)(x-6) = 0$
$\implies x = -2, 6$
f is **decreasing on $(-2, 6)$ and increasing elsewhere.**

Determine Intervals of Concavity

Find the intervals where the following functions are concave up or concave down

16. $f(x) = 10 - 6x - 2x^2$

$f'(x) = -6 - 4x$
$f''(x) = -4$
The function is **concave down everywhere.**

17. $f(x) = 2x^3 - 12x^2 + 18x + 15$

$f'(x) = 6x^2 - 24x + 18$
$f''(x) = 12x - 24$
$12x - 24 = 0 \implies x = 2$
When $x < 2$, $f''(x)$ is negative, and when $x > 2$, $f''(x)$ is positive.
This means that the function is **concave down on $(-\infty, 2)$ and concave up on $(2, +\infty)$.**

18. $f(x) = 5 + 36x + 3x^2 - 2x^3$

$f'(x) = 36 + 6x - 6x^2$
$f''(x) = 6 - 12x$
$6 - 12x = 0 \implies x = \frac{1}{2}$
$f''(x)$ is positive when $x < \frac{1}{2}$ and negative when $x > \frac{1}{2}$.
This means that the function is **concave up on $(-\infty, \frac{1}{2})$ and concave down on $(\frac{1}{2}, +\infty)$.**

19. $f(x) = 8 + 36x + 3x^2 - 2x^3$

If you did the previous exercise then no calculation is required since this function has the same second derivative as that function and thus is concave up and concave down on the same intervals; i.e., the function is **concave up on $(-\infty, \frac{1}{2})$ and concave down on $(\frac{1}{2}, +\infty)$.**

20. $f(x) = 5x^3 - 15x^2 - 120x + 3$

$f'(x) = 15x^2 - 30x - 120$
$f''(x) = 30x - 30$
$30x - 30 = 0 \implies x = 1$
$f''(x)$ is positive when $x > 1$ and negative when $x < 1$.
This means that the function is **concave down on $(-\infty, 1)$ and concave up on $(1, +\infty)$.**

21. $f(x) = x^3 - 6x^2 - 36x + 2$

$f'(x) = 3x^2 - 12x - 36$
$f''(x) = 6x - 12$
$6x - 12 = 0 \implies x = 2$
$f''(x)$ is positive when $x > 2$ and negative when $x < 2$.
This means that the function is **concave down on $(-\infty, 2)$ and concave up on $(2, +\infty)$.**

Word Problems

22. You peer around a corner. A velociraptor 64 meters away spots you. You run away at a speed of 6 meters per second. The raptor chases, running towards the corner you just left at a speed of $4t$ meters per second (time t measured in seconds after spotting). After you have run 4 seconds the raptor is 32 meters from the corner. At this time, how fast is death approaching your soon to be mangled flesh? That is, what is the rate of change in the

distance between you and the raptor?

Velocity is the rate in change of position with respect to time. The raptor's velocity relative to you is given by

$$\frac{dx}{dt} = v(t) = 4t - 6$$

After 4 seconds, the rate of change in position with respect to time is

$$v(4) = 16 - 6 = \mathbf{10\frac{m}{s}}$$

23. Two bicycles leave an intersection at the same time. One heads north going 12 mph and the other heads east going 5 mph. How fast are the bikes getting away from each other after one hour?

Set up a coordinate system with the origin at the intersection and the $+y$-axis pointing north. We assume that the position of the bike heading north is a function of the position of the bike heading east.

$$y = \frac{12}{5}x$$

The distance between the bikes is given by

$$s = \sqrt{x^2 + y^2} = \sqrt{x^2 + (\frac{12}{5}x)^2} = \sqrt{\frac{144+25}{25}x^2} = \sqrt{\frac{169}{25}x^2} = \frac{13}{5}x$$

Let t represent the elapsed time in hours. We want $\frac{ds}{dt}$ when $t = 1$. Apply the chain rule to s:

$$\frac{ds}{dt} = \frac{ds}{dx}\frac{dx}{dt} = \frac{13}{5} \cdot 5 = 13$$

Thus, the bikes are moving away from one another at **13 mph**.

24. You're making a can of volume 200 m^3 with a gold side and silver top/bottom. Say gold costs 10 dollars per m^2 and silver costs 1 dollar per m^2. What's the minimum cost of such a can?

The volume of the can as a function of the radius, r, and the height, h, is

$$V = \pi r^2 h$$

We are constricted to have a can with a volume of $200m^3$, so we use this fact to relate the radius and the height:

$$\pi r^2 h = 200 \implies h = \frac{200}{\pi r^2}$$

The surface area of the side is

$$A_{side} = 2\pi r h = 2\pi r \frac{200}{\pi r^2} = \frac{400}{r}$$

and the cost of the side is

$$C_{side} = \frac{4000}{r}$$

The surface area of the top and bottom (which is also the cost) is

$$A_{tb} = C_{tb} = 2\pi r^2$$

The total cost is given by

$$C = \frac{4000}{r} + 2\pi r^2$$

We want to minimize C, so take the derivative:

$$C' = -\frac{4000}{r^2} + 4\pi r$$

Find the critical points:

$$-\frac{4000}{r^2} + 4\pi r = 0 \implies 4\pi r = \frac{4000}{r^2} \implies 4\pi r^3 = 4000 \implies r = \sqrt[3]{\frac{1000}{\pi}} = \frac{10}{\sqrt[3]{\pi}}$$

Check the second derivative to see if this point corresponds to a maximum or minimum:

$$C'' = \frac{8000}{r^3} + 4\pi$$

$$C''(\frac{10}{\sqrt[3]{\pi}}) = \frac{8000}{1000/\pi} + 4\pi = 8\pi + 4\pi = 12\pi$$

Since the second derivative is positive, the critical point corresponds to a minimum. Thus, the minimum cost is

$$C(\frac{10}{\sqrt[3]{\pi}}) = \frac{4000}{10/\sqrt[3]{\pi}} + 2\pi\frac{10}{\sqrt[3]{\pi}} \approx \mathbf{\$878.76}$$