Learning Objectives

A student will be able to:

- Identify functions from various relationships.
- Review function notation.
- Determine domains and ranges of particular functions.
- Identify key properties of some basic functions.
- Sketch graphs of basic functions.
- Sketch variations of basic functions using transformations.
- Compose functions.

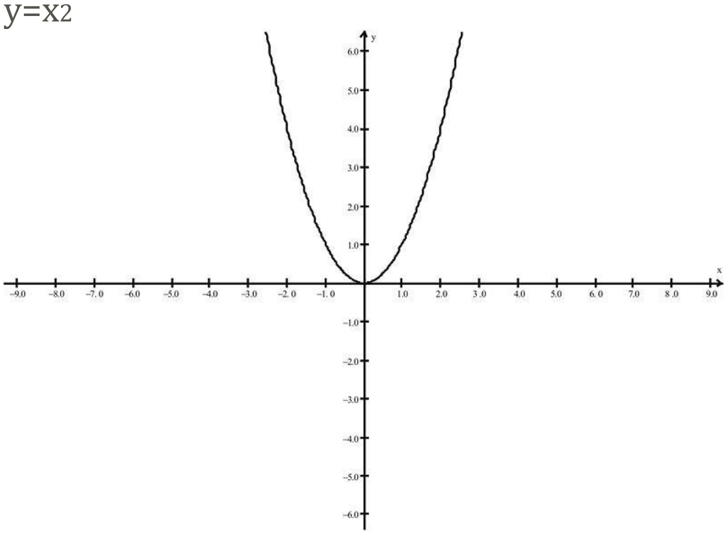
Introduction

In our last lesson we examined a variety of mathematical equations that expressed mathematical relationships. In this lesson we will focus on a particular class of relationships called functions, and examine their key properties. We will then review how to sketch graphs of some basic functions that we will revisit later in this class. Finally, we will examine a way to combine functions that will be important as we develop the key concepts of calculus.

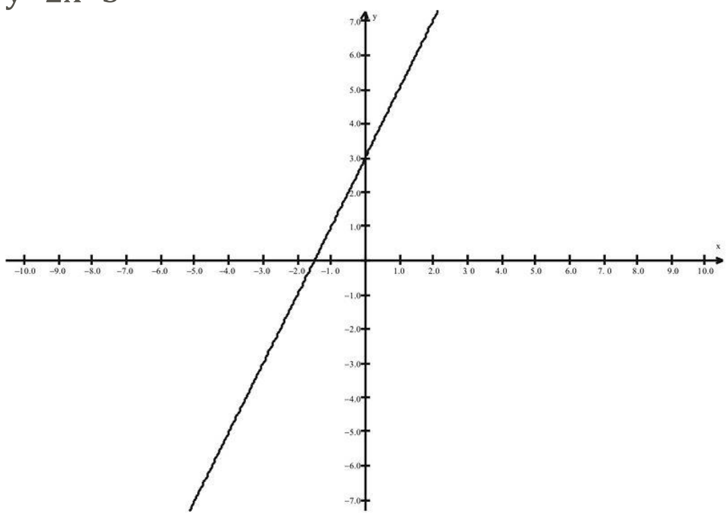
Let’s begin our discussion by reviewing four types of equations we examined in our last lesson.

Example 1:

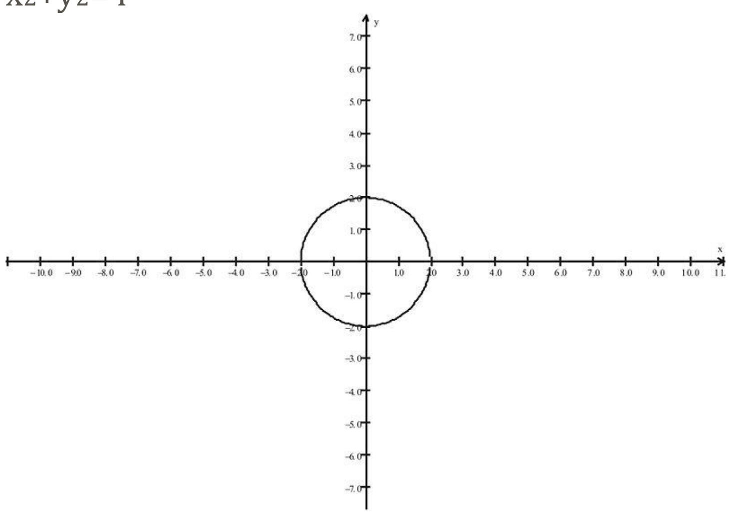
$y=x^2$



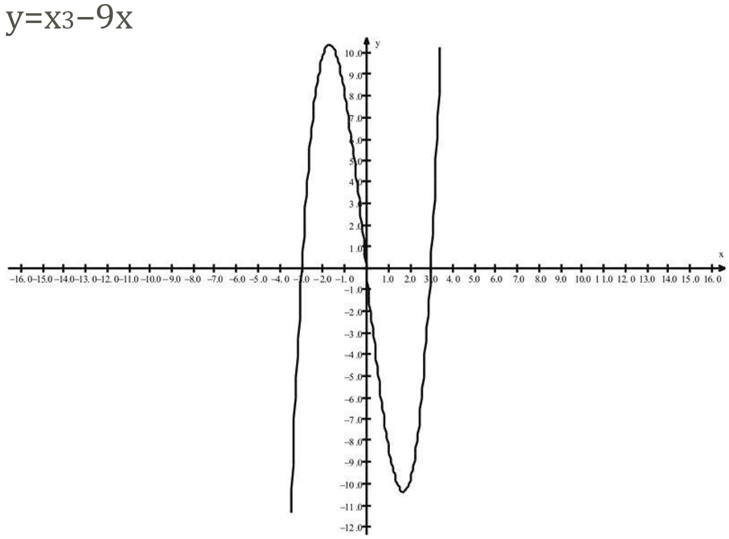
$y=2x+3$



$x^2+y^2=4$



$y=x^3-9x$

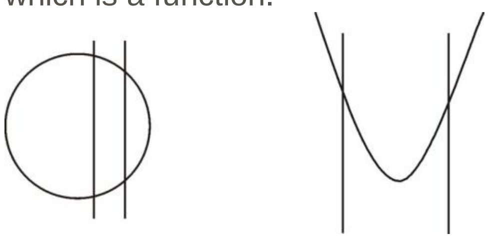


Of these, the circle has a quality that the other graphs do not share. Do you know what it is?

Solution:

The circle's graph includes points where a particular $x-$value has two points associated with it; for example, the points $(1,3-\surd)$ and $(1,-3-\surd)$are both solutions to the equation $x2+y2=4$. For each of the other relationships, a particular $x-$value has exactly one $y-$value associated with it.

The relationships that satisfy the condition that for each $x-$value there is a unique $y-$value are called **functions**. Note that we could have determined whether the relationship satisfied this condition by a graphical test, the vertical line test. Recall the relationships of the circle, which is not a function. Let's compare it with the parabola, which is a function.



If we draw vertical lines through the graphs as indicated, we see that the condition of a particular $x-$value having exactly one $y-$value associated with it is equivalent to having at most one point of intersection with any vertical line. The lines on the circle intersect the graph in more than one point, while the lines drawn on the parabola intersect the graph in exactly one point. So this vertical line test is a quick and easy way to check whether or not a graph describes a function.

We want to examine properties of functions such as function notation, their domain and range (the sets of x and y values that define the function), graph sketching techniques, how we can combine functions to get new functions, and also survey some of the basic functions that we will deal with throughout the rest of this book.

Let's start with the notation we use to describe functions. Consider the example of the linear function $y=2x+3$. We could also describe the function using the symbol $f(x)$ and read as "f *of* x" to indicate the $y-$value of the function for a particular $x-$value. In particular, for this function we would write $f(x)=2x+3$ and indicate the value of the function at a particular value, say $x=4$ as $f(4)$ and find its value as follows: $f(4)=2(4)+3=11$. This statement corresponds to the solution $(4,11)$ as a point on the graph of the function. It is read, "f *of* 4*is* 11."

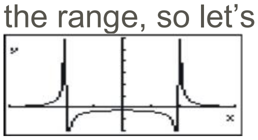
We can now begin to discuss the properties of functions, starting with the ***domain*** and the ***range*** of a function. The ***domain*** refers to the set of $x-$values that are inputs in the function, while the ***range*** refers to the set of $y-$values that the function takes on. Recall our examples of functions:

Linear Function $g(x)=2x+3$

Quadratic Function $f(x)=x^2$
Polynomial Function $p(x)=x^3-9x$
We first note that we could insert any real number for an $x-$value and a well-defined $y-$value would come out. Hence each function has the set of all real numbers as a domain and we indicate this in interval form as $D:(-\infty,\infty)$. Likewise we see that our graphs could extend up in a positive direction and down in a negative direction without end in either direction. Hence we see that the set of $y-$values, or the range, is the set of all real numbers $R:(-\infty,\infty)$.
Example 2:

Determine the domain and range of the function.

$f(x)=1/(x^2-4)$.
Solution:

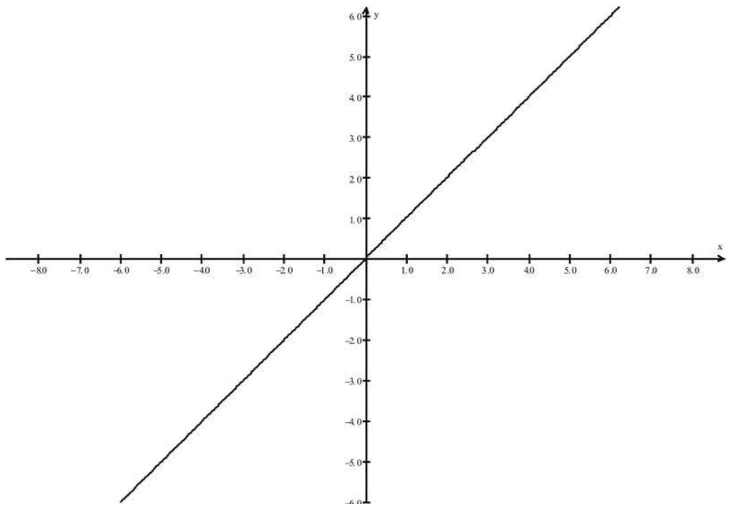
We note that the condition for each $y-$value is a fraction that includes an x term in the denominator. In deciding what set of $x-$values we can use, we need to exclude those values that make the denominator equal to 0.Why? **(Answer: division by 0 is not defined for real numbers.)** Hence the set of all permissible $x-$values, is all real numbers except for the numbers $(2,-2)$, which yield division by zero. So on our graph we will not see any points that correspond to these $x-$values. It is more difficult to find the range, so let's find it by using the graphing calculator to produce the graph.


From the graph, we see that every $y\neq 0$ value in $(-\infty,\infty)$ (or "All real numbers") is represented; hence the range of the function is $\{-\infty,0\}\cup\{0,\infty\}$. This is because a fraction with a non-zero numerator never equals zero.
Eight Basic Functions

We now present some basic functions that we will work with throughout the course. We will provide a list of eight basic functions with their graphs and domains and ranges. We will then show some techniques that you can use to graph variations of these functions.

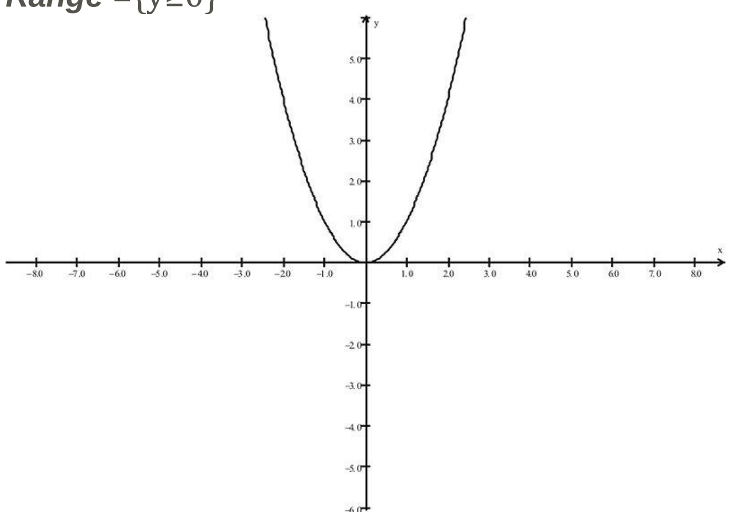
Linear

$f(x)=x$
Domain = ***All reals***
Range = ***All reals***



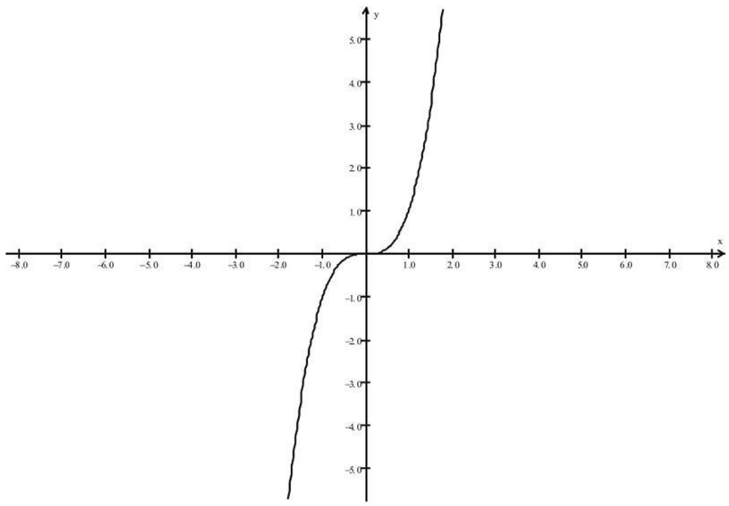
Square (Quadratic)

$f(x)=x^2$
***Domain** = All reals*
Range $=\{y\geq 0\}$



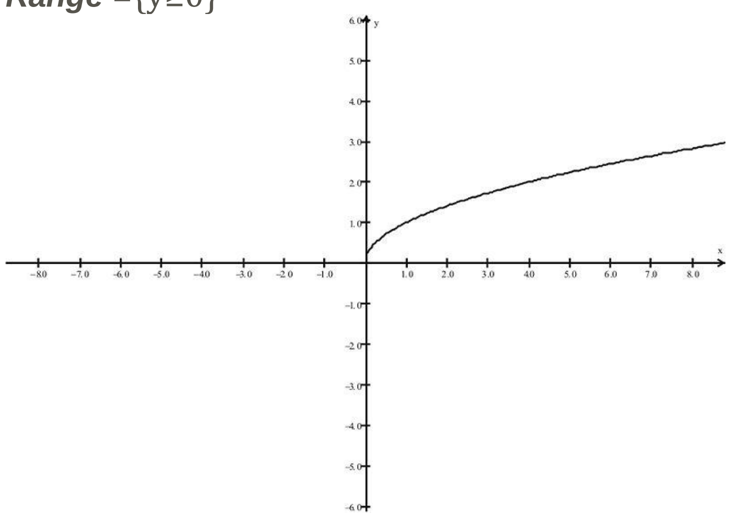
Cube (Polynomial)

$f(x)=x^3$
***Domain** = All reals*
***Range** = All reals*



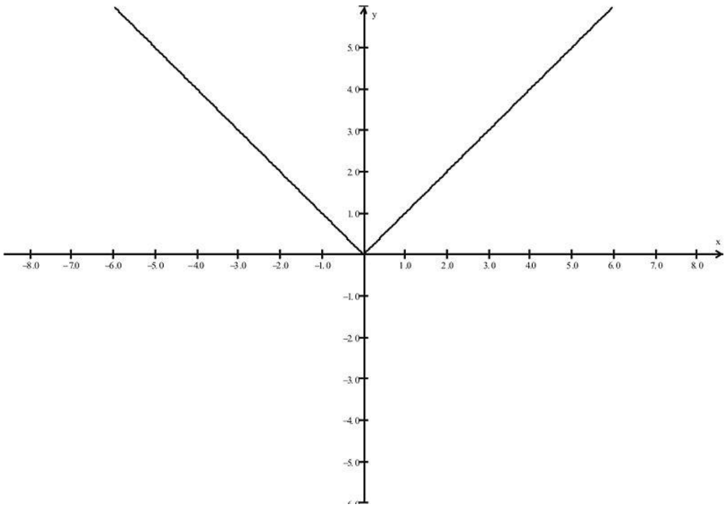
Square Root

$f(x)=\sqrt{x}$
Domain $=\{x\geq 0\}$
Range $=\{y\geq 0\}$



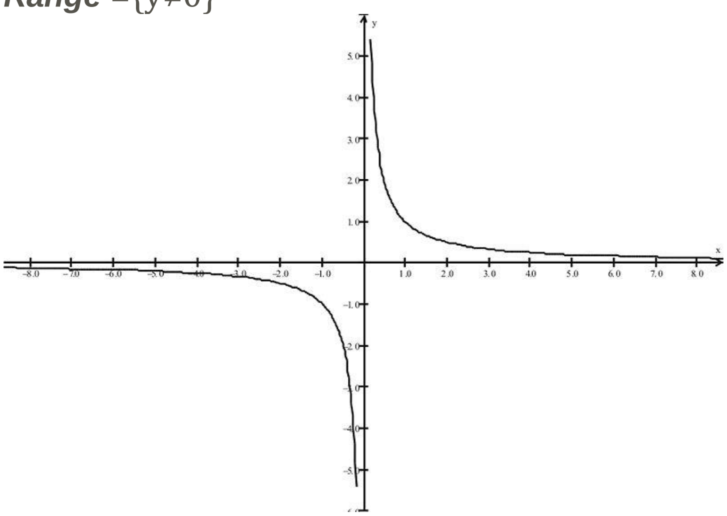
Absolute Value

$f(x)=|x|$
Domain = ***All reals***
Range $=\{y\geq 0\}$



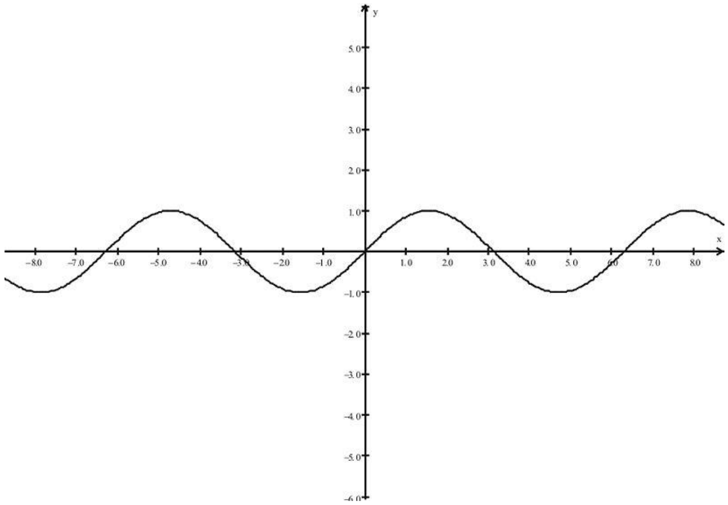
Reciprocal

$f(x)=1/x$
Domain $=\{x\neq 0\}$
Range $=\{y\neq 0\}$



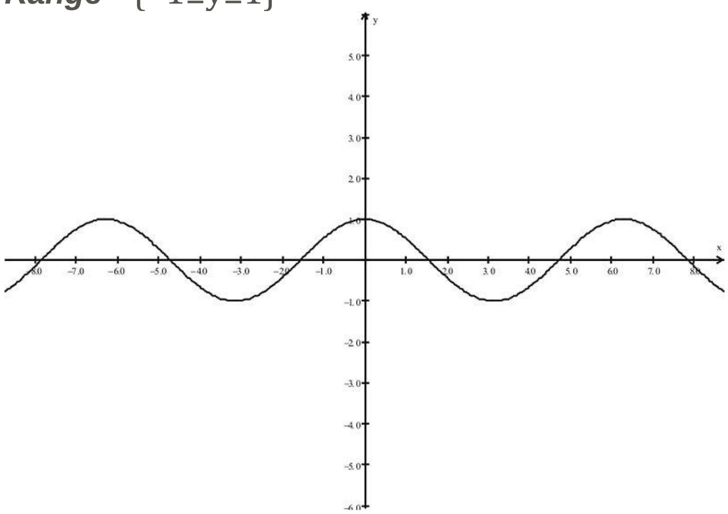
Sine

$f(x)=\sin x$
***Domain** = All reals*
Range $=\{-1\leq y\leq 1\}$



Cosine

$f(x)=\cos x$
Domain** = **All reals
Range $=\{-1\leq y\leq 1\}$

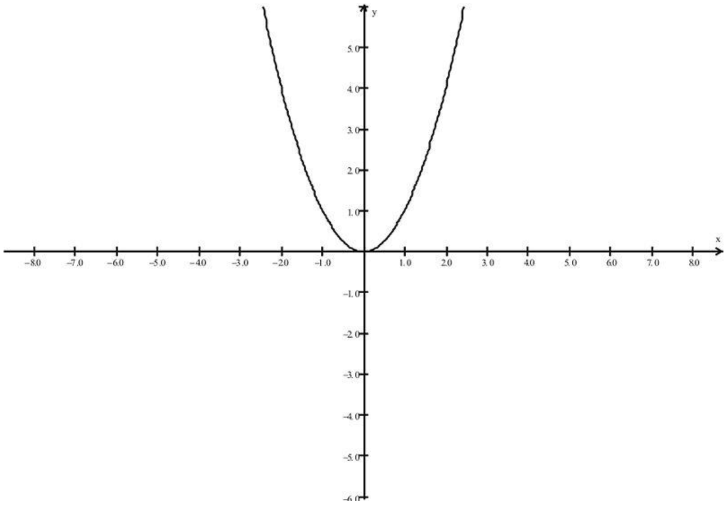


Graphing by Transformations

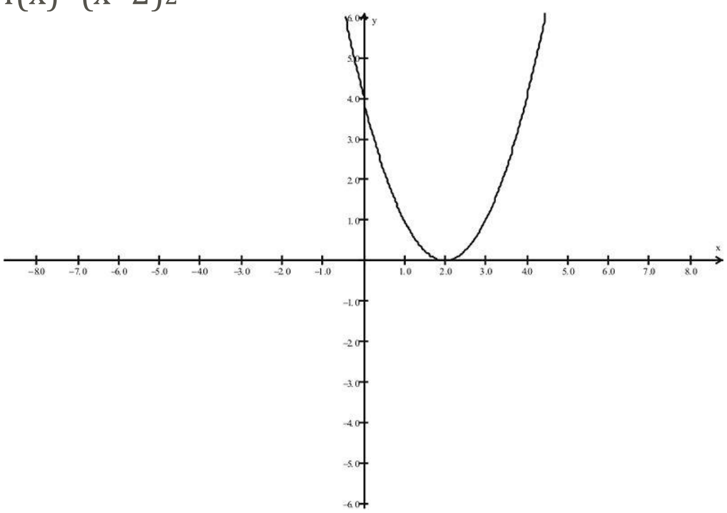
Once we have the basic functions and each graph in our memory, we can easily sketch variations of these. In general, if we have $f(x)$, and c is some constant value, then the graph of $f(x-c)$ is just the graph of $f(x)$shifted $c$units to the right. Similarly, the graph of $f(x+c)$ is just the graph of $f(x)$ shifted $c$units to the left.

Example 3:

$f(x)=x^2$



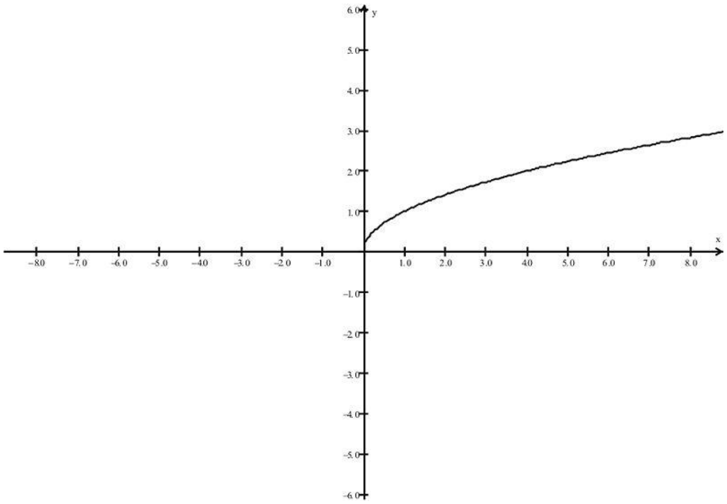
$f(x)=(x-2)^2$



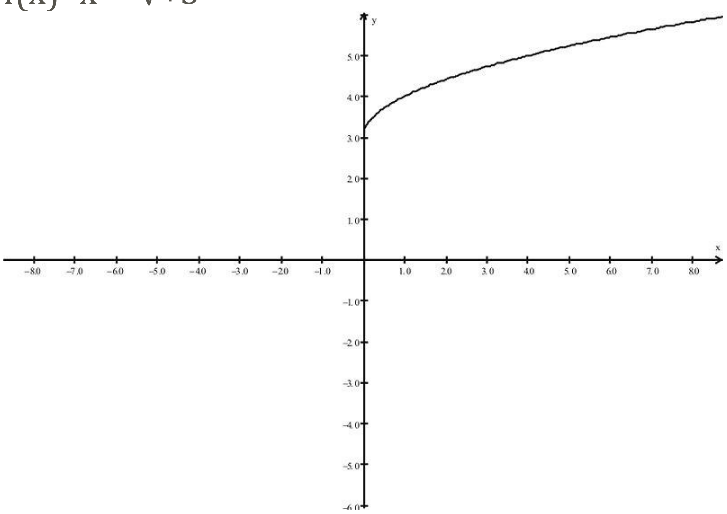
In addition, we can shift graphs up and down. In general, if we have $f(x)$,and c is some constant value, then the graph of $f(x)+c$ is just the graph of $f(x)$ shifted c units up on the y−axis. Similarly, the graph of $f(x)-c$ is just the graph of $f(x)$ shifted c units down on the y−axis.

Example 4:

$f(x)=\sqrt{x}$

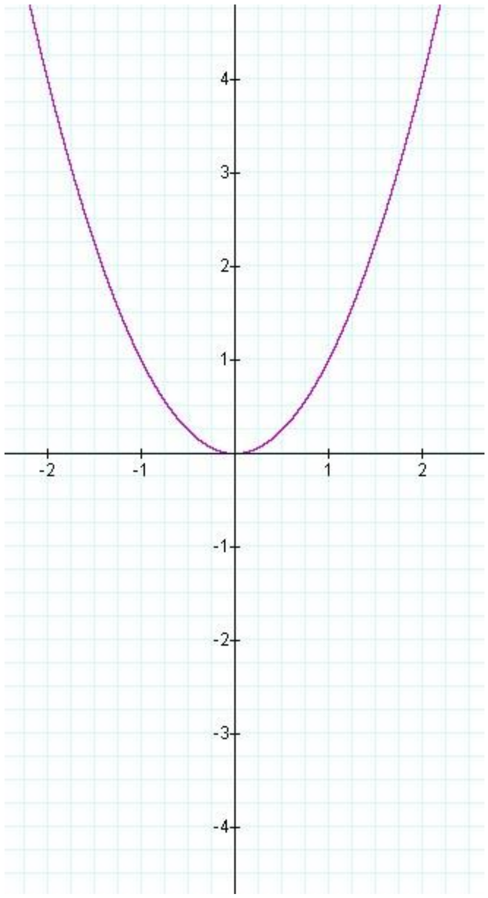


$f(x)=\sqrt{x}+3$

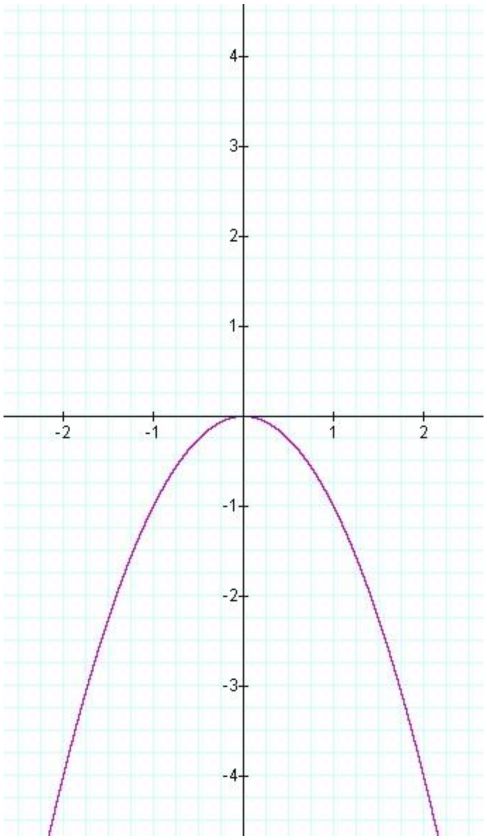


We can also flip graphs in the x−axis by multiplying by a negative coefficient.

$f(x)=x^2$



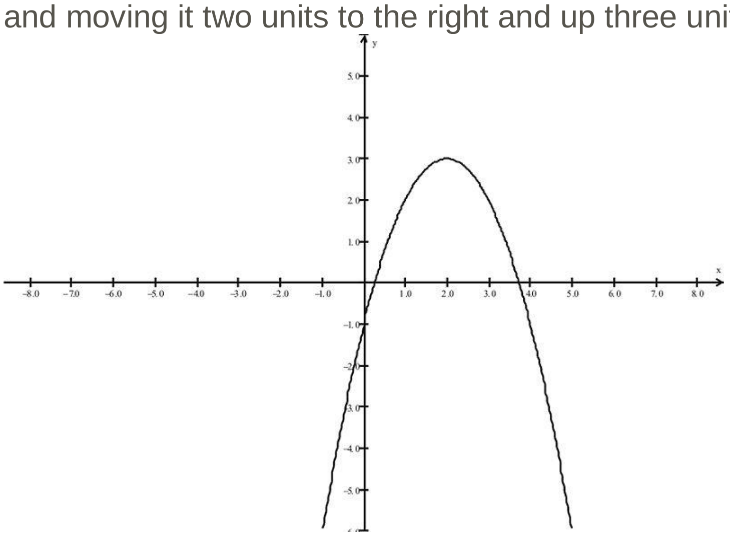
$f(x) = -x^2$



Finally, we can combine these transformations into a single example as follows.

Example 5:

$f(x)=-(x-2)^2+3$. The graph will be generated by taking $f(x)=x^2$, flipping in the $y-$axis, and moving it two units to the right and up three units.



Function Composition

The last topic for this lesson involves a way to combine functions called **function composition**. Composition of functions enables us to consider the effects of one function followed by another. Our last example of graphing by transformations provides a nice illustration. We can think of the final graph as the effect of taking the following steps:

$x \rightarrow -(x-2)^2 \rightarrow -(x-2)^2+3$

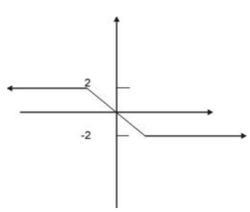
We can think of it as the application of two functions. First, $g(x)$ takes $x$to $-(x-2)^2$ and then we apply a second function, $f(x)$ to those $y-$values, with the second function adding $+3$ to each output. We would write the functions as $f(g(x))=-(x-2)^2+3$ where $g(x)=-(x-2)^2$ and $f(x)=x+3$. We call this operation the composing of f with g and use notation $f \circ g$. Note that in this example, $f \circ g \neq g \circ f$. Verify this fact by computing $g \circ f$ right now. (Note: this fact can be verified algebraically, by showing that the expressions $f \circ g$ and $g \circ f$ differ, or by showing that the different function decompositions are not equal for a specific value.)

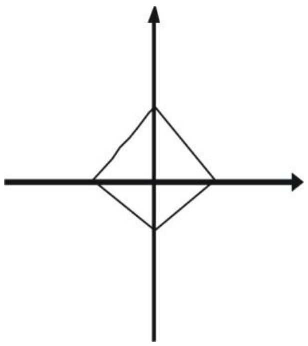
Lesson Summary

1. Learned to identify functions from various relationships.
2. Reviewed the use of function notation.
3. Determined domains and ranges of particular functions.
4. Identified key properties of basic functions.
5. Sketched graphs of basic functions.
6. Sketched variations of basic functions using transformations.
7. Learned to compose functions.

Review Questions

In problems 1 - 2, determine if the relationship is a function. If it is a function, give the domain and range of the function.

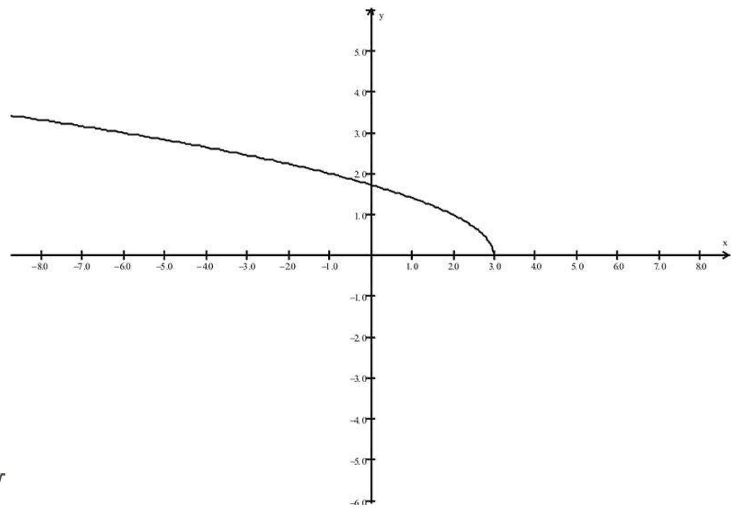
1. 



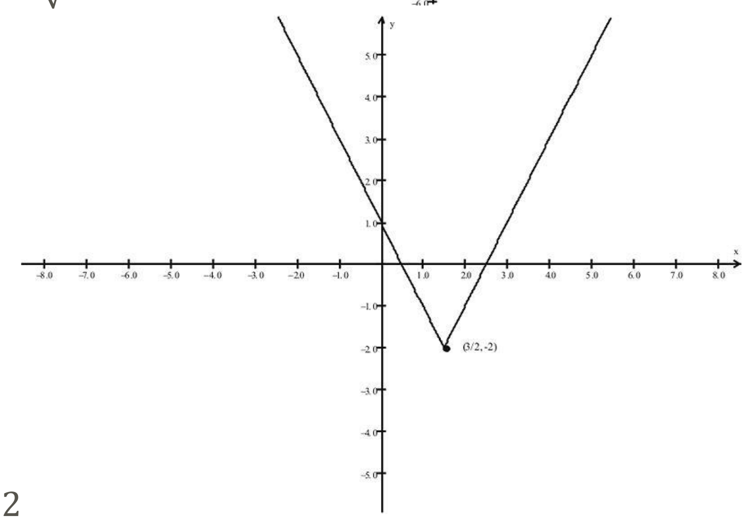
2.

In problems 3 - 5, determine the domain and range of the function and sketch the graph if no graph is provided.

3. f(x)=3x2x2−1



4. y=−x+3−−−−−−√



5. f(x)=|2x−3|−2

In problems 6 - 8, sketch the graph using transformations of the graphs of basic functions.

6. f(x)=−(x+2)2+5

7. $f(x)=-1x-2+3$
8. $y=--x-2------\surd+3$
9. Find the composites, $f \circ g$ and $g \circ f$ for the following functions. $f(x)=-3x+2, g(x)=x--\surd$
10. Find the composites, $f \circ g$ and $g \circ f$ for the following functions. $f(x)=x2, g(x)=x--\surd$