Learning Objectives

A student will be able to:

- Solve problems that involve related rates.

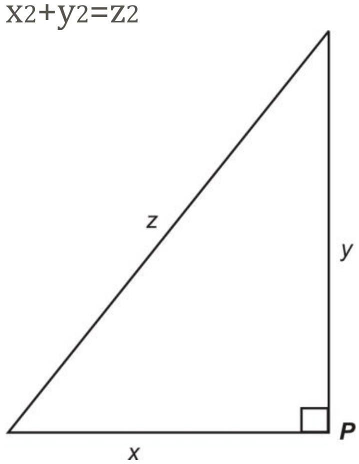
Introduction

In this lesson we will discuss how to solve problems that involve related rates. Related rate problems involve equations where there is some relationship between two or more derivatives. We solved examples of such equations when we studied implicit differentiation in Lesson 2.6. In this lesson we will discuss some real-life applications of these equations and illustrate the strategies one uses for solving such problems.

Let's start our discussion with some familiar geometric relationships.

Example 1: *Pythagorean Theorem*

$x^2+y^2=z^2$



We could easily attach some real-life situation to this geometric figure. Say for instance that x and y represent the paths of two people starting at point p and walking North and West, respectively, for two hours. The quantity z represents the distance between them at any time t. Let's now see some relationships between the various rates of change that we get by implicitly differentiating the original equation $x^2+y^2=z^2$ with respect to time t.

$x^2+y^2 \quad 2x\frac{dx}{dt}+2y\frac{dy}{dt}=z^2=2z\frac{dz}{dt}.$

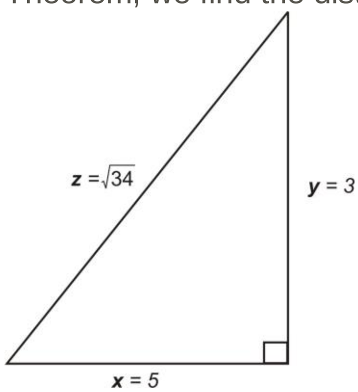
Simplifying, we have

Equation 1. $x\frac{dx}{dt}+y\frac{dy}{dt}=z\frac{dz}{dt}$

So we have relationships between the derivatives, and since the derivatives are rates, this is an example of ***related rates***. Let's say that person x is walking at 5mph and that person y is walking at 3mph. The rate at which the distance between the two walkers is

changing at any time is dependent on the rates at which the two people are walking. Can you think of any problems you could pose based on this information?
One problem that we could pose is at what rate is the distance between $x$and y increasing after one hour. That is, find dz/dt.
Solution:

Assume that they have walked for one hour. So $x=5mi$ and $y=3$.Using the Pythagorean Theorem, we find the distance between them after one hour is $z=34−−√=5.83miles$.



If we substitute these values into **Equation 1** along with the individual rates we get

$5(5)+3(3)343434−−√=34−−√dzdt=34−−√dzdt=dzdt$.
Hence after one hour the distance between the two people is increasing at a rate of $dzdt=3434−−√≈5.83mph$.
Our second example lists various formulas that are found in geometry.

As with the Pythagorean Theorem, we know of other formulas that relate various quantities associated with geometric shapes. These present opportunities to pose and solve some interesting problems

Example 2: ***Perimeter and Area of a Rectangle***

We are familiar with the formulas for Perimeter and Area:

$PA=2∗l+2∗w,=l∗w$.
Suppose we know that at an instant of time, the length is changing at the rate of $8ft/hour$ and the perimeter is changing at a rate of $24ft/hour$. At what rate is the width changing at that instant?
Solution:

If we differentiate the original equation, we have

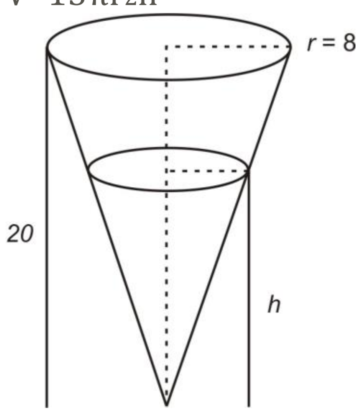
Equation 2: $dpdt=2∗dldt+2∗dwdt$.
Substituting our known information into Equation II, we have

$2484=(2*8)+2*dwdt=2*dwdt=dwdt$.
The width is changing at a rate of $4ft/hour$.
Okay, rather than providing a related rates problem involving the area of a rectangle, we will leave it to you to make up and solve such a problem as part of the homework (HW #1).

Let's look at one more geometric measurement formula.

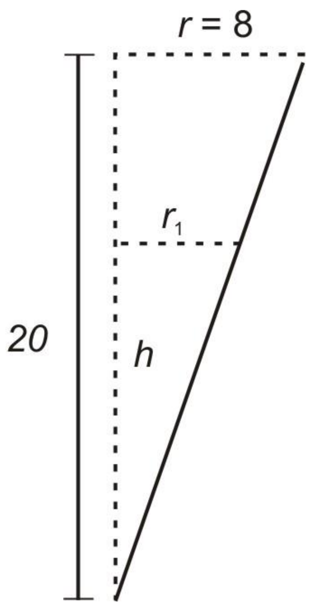
Example 3: ***Volume of a Right Circular Cone***

$V=13\pi r^2 h$



We have a water tank shaped as an inverted right circular cone. Suppose that water flows into the tank at the rate of $5ft^3/min$. At what rate is the water level rising when the height of the water in the tank is $6feet$?
Solution:

We first note that this problem presents some challenges that the other examples did not. Both r and h are functions of t, and so implicit differentiation of the $V(t)$ function is going to produce several variables in the form $r(t)$, $drdt$, $h(t)$, and $dhdt$, and that will give us too many variables to solve for $dhdt$. So we need to find a way to eliminate r and $drdt$.
When we differentiate the original equation, $V=(1/3)\pi r^2 h$, we get $dVdt=13\pi(h)(2r)drdt+13\pi r^2 dhdt$.
The difficulty here is that we have no information about the radius when the water level is at $6feet$. So we need to relate the radius a quantity that we do know something about. Starting with the original equation, let's find a relationship between h and r. Let r_1 be the radius of the surface of the water as it flows out of the tank.



Note that the two triangles are similar and thus corresponding parts are proportional. In particular,

$r_1 h r_1 = 820 = 8h20 = 2h5.$

Now we can solve the problem by substituting $r_1 = (2h/5)$ into the original equation:

$V = 13\pi(2h5)^2 h = 4\pi 75 h^3.$

Hence $dVdt = 12\pi 75 h^2 dhdt$, and by substitution,

$5dhdt = 12\pi 75(36)dhdt = 375432\pi \approx 0.28 ftmin.$

Lesson Summary

1. We learned to solve problems that involved related rates.

Review Questions

1.
   a. Make up a related rates problem about the area of a rectangle.
   b. Illustrate the solution to your problem.
2. Suppose that a particle is moving along the curve $4x^2 + 16y^2 = 32$. When it reaches the point $(2,1)$, the $x-$coordinate is increasing at a rate of $3 ft/sec$. At what rate is the $y-$coordinate changing at that instant?
3. A regulation softball diamond is a square with each side of length $60 ft$. Suppose a player is running from first base to second base at a speed of $18 ft/sec$. At what rate is the distance between the runner and home plate changing when the runner is $2/3$ of the way from first to second base?

4. At a recent Hot Air Balloon festival, a hot air balloon was released. Upon reaching a height of 300ft, it was rising at a rate of $20\text{ft}/sec$. Mr. Smith was 100ft away from the launch site watching the balloon. At what rate was the distance between Mr. Smith and the balloon changing at that instant?
5. Two trains left the St. Louis train station in the late morning. The first train was traveling East at a constant speed of 65mph. The second train traveled South at a constant speed of 75mph. At 3 PM, the first train had traveled a distance of 120miles while the second train had traveled a distance of 130miles. How fast was the distance between the two trains changing at that time?
6. Suppose that a 17ft ladder is sliding down a wall at a rate of $-6\text{ft}/sec$. At what rate is the distance between the bottom of the ladder and the wall increasing when the top is 8ft from the ground?
7. Suppose that the length of a rectangle is increasing at the rate of $6\text{ft}/\text{min}$ and the width is increasing at a rate of $2\text{ft}/\text{min}$. At what rate is the area of the rectangle changing when its length is 25ft and its width is 15ft?
8. Suppose that the quantity demand of new $40''$ plasma TVs is related to its unit price by the formula $\text{p}+\text{x}^2=1200$, where p is measured in dollars and x is measured in units of one thousand. How is the quantity demand changing when $\text{x}=20$, $\text{p}=1500$, and the price per TV is decreasing at a rate of $\$10$ per week?
9. The volume of a cube with side s is changing. At a certain instant, the sides of the cube are 6inches and increasing at the rate of $1/4\text{in}/\text{min}$. How fast is the volume of the cube increasing at that time?
10.
    a. Suppose that the area of a circle is increasing at a rate of $24\text{in}^2/\text{min}$. How fast is the radius increasing when the area is $36\pi\text{in}^2$?
    b. How fast is the circumference changing at that instant?