My monumental ignorance: proofs I wish I knew, and the challenge of negativity

Posted on March 14, 2012 by Jonas Reitz

As with most things, the farther you go in mathematics the sharper your sense of ignorance becomes. There is just too much math out there, and too little time to follow every thread that crosses your path. I'd like to share a few of the math facts that I've accepted for for much of my mathematical life, but which I (lamentably) have never seen proven. I choose these (out of the domain that our former Secretary of Defense Donald Rumsfeld would describe as my "known unknowns") because they are common facts, they are interesting and accessible even at the undergraduate level (or earlier), and they seem hard (in the sense that their proofs are not easily communicable in a few sentences, or twenty minutes at a chalkboard). Here goes:

1. That π is transcendental (or, indeed, irrational)(same for e).

This is a biggie, since (let's face it) most numbers are transcendental, but we only really have a couple of examples of them in common circulation.

PROOF: This fact follows from the Lindemann-Weierstrass Theorem, a MUCH more general result, or the Gelfond-Schneider Theorem, a MUCH MUCH more general result (which also settles that pesky question about the rationality of $\sqrt{2}^{\sqrt{2}}$). As you may have guessed, I do not know the proofs of these theorems…

2. That there is no general formula for solving polynomial equations of degree 5*.*

Of course, we have algorithms for finding roots of quintic equations to arbitrary precision — we just can't always express the roots exactly using arithmetic operations, radicals, and so on.

This is a great one to bring up when discussing the quadratic formula – it's so natural to ask about equations with higher degree, and the expectation from students is that, while there probably are "quadratic-type formulas" for any polynomial, they are going to be complicated. It's a bit of a shock to find that degree 4 is as high as you can go!

PROOF: For this one, you need a paradigm-shifting change in perspective on polynomial equations, Galois Theory (which I have never studied, blah blah blah…)

3. That e^{-x^2} can't be integrated.

*Of course, the function **can** be integrated — else our whole theory of continuous probability would crumble — but once again, it's the existence of a nice expression for that integral, in terms of other elementary functions, that fails.*

This is a great problem to give your Calculus II students, after they've spent a month or so mastering the different techniques of integration. After struggling with the problem for a while, they will be ready (expectant!) for a "neat trick" that solves the problem — and the fact that NO such trick exists is a shocker! This is also a nice lead-in to a discussion about the relative difficulty of differentiation vs. integration (ironic, since many more functions are integrable than differentiable).

PROOF: There is a nice (and perhaps unexpected — but perhaps not, see below) connection between this and the previous problem — the proof comes out of Differential Galois Theory, the analogue of Galois theory for differential fields.

Negativity is *hard*

The examples above are all negative results, in the sense they they show "something is NOT" — either not in some easily defined class or not expressible by some means (in example 1, "transcendental" is just a shorthand for "NOT algebraic"). Somehow, showing something is NOT seems to be much harder than showing something IS. Why so hard? It must have something to do with the difference between producing an example illustrating an idea, and (on the other hand) clearly delineating the boundaries of the idea. The former is specific, and the latter is global. The former requires you to provide some kind of construction, the latter requires you to show that no such construction will work. In many cases, the negative result requires some fundamentally new perspective — and the consequences of the resulting proof extend far beyond the motivating problem!

There are many many more examples of this kind of thing, across all different areas of mathematics: in number theory, we have Wiles' proof of Fermat's Last Theorem (here's a nice blog-in-progress detailing the proof for the interested amateur), and in set theory the independence of the Continuum Hypothesis (and development of forcing). Power to the negative!

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3 Responses to *My monumental ignorance: proofs I wish I knew, and the challenge of negativity*

ihsan magazine *says:*
June 4, 2018 at 3:35 am

The very last thing you desire would be to plan your perfect vacation only to find out things thanks all 🙂

James Tomson *says:*
September 22, 2012 at 12:53 pm

Has any one looked at http://www.coolissues.com/mathematics/Wile'sproofofFLT.html?

Anonymous *says:*
March 18, 2012 at 1:00 pm

Wile's proof of FLT disproves Pythagorean theorem.
See http://www.coolissues.com/mathematics/Wile'sproofofFLT.html

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