Learning Objectives

A student will be able to:

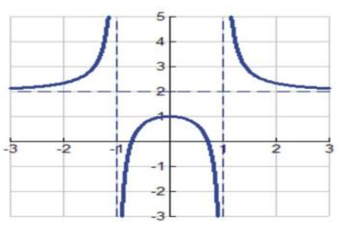
- Examine end behavior of functions on infinite intervals.
- Determine horizontal asymptotes.
- Examine indeterminate forms of limits of rational functions.
- Apply L'Hôpital's Rule to find limits.
- Examine infinite limits at infinity.

Introduction

In this lesson we will return to the topics of infinite limits and end behavior of functions and introduce a new method that we can use to determine limits that have indeterminate forms.

Examine End Behavior of Functions on Infinite Intervals

Suppose we are trying to analyze the end behavior of rational functions. Let's say we looked at some rational functions such as $f(x)=2x^2-1x^2-1$and showed that $\lim_{x\to+\infty}f(x)=2$ and $\lim_{x\to-\infty}f(x)=2$. We required an analysis of the end behavior of f since computing the limit by direct substitution yielded the indeterminate form $\infty\infty$. Our approach to compute the infinite limit was to look at actual values of the function $f(x)$as x approached $\pm\infty$. We interpreted the result graphically as the function having a horizontal asymptote at $f(x)=2$.



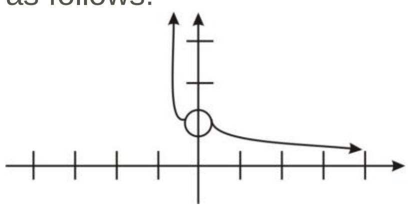
We were then able to find infinite limits of more complicated rational functions such as $\lim_{x\to\infty}3x^4-2x^2+3x+12x^4-2x^2+x-3=32$ using the fact that $\lim_{x\to\infty}1x^p=0,p>0$. Similarly, we used such an approach to compute limits whenever direct substitution resulted in the indeterminate form 00, such as $\lim_{x\to1}x^2-1x-1=2$.
Now let's consider other functions of the form $(f(x)/g(x))$ where we get the indeterminate forms 00 and $\infty\infty$ and determine an appropriate analytical method for computing the limits.
Example 1:

Consider the function $f(x)=\ln(x+1)x$ and suppose we wish to find $\lim_{x\to 0}\ln(x+1)x$ and $\lim_{x\to\infty}\ln(x+1)x$ We note the following:

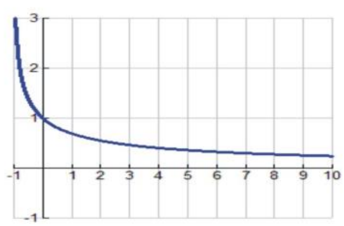
1. Direct substitution leads to the indeterminate forms 00 and $\infty\infty$.
2. The function in the numerator is not a polynomial function, so we cannot use our previous methods such as applying $\lim_{x\to\infty}1x_p=0$.

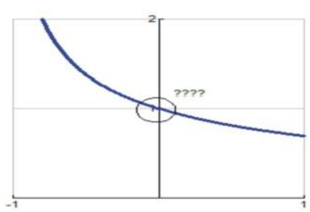
Let's examine both the graph and values of the function for appropriate $x$values, to see if they cluster around particular y values. Here is a sketch of the graph and a table of extreme values.

We first note that domain of the function is $(-1,0)\cup(0,+\infty)$ and is indicated in the graph as follows:



So, $\lim_{x\to 0}\ln(x+1)x$ appears to approach the value 1 as the following table suggests. Note: Please see Differentiation and Integration of Logarithmic and Exponential Functions in Chapter 6 for more on derivatives of Logarithmic functions.





$x-0.1-0.00100.0010.1\ln(x+1)/x1.053611.0005\text{undef}0.99950.953102$

So we infer that $\lim_{x\to 0}\ln(x+1)x=1$.

For the infinite limit, $\lim_{x\to\infty}\ln(x+1)x=1$, the inference of the limit is not as obvious. The function appears to approach the value 0 but does so very slowly, as the following table suggests.

$x1050100100010000\ln(x+1)/x0.239790.0786370.0461510.0069090.000921$

This unpredictable situation will apply to many other functions of the form. Hence we need another method that will provide a different tool for analyzing functions of the form $f(x)g(x)$.

L'Hôpital's Rule: Let functions f and g be differentiable at every number other than c in some interval, with $g'(x)\neq 0$ if $x\neq c$. If $\lim_{x\to c}f(x)=\lim_{x\to c}g(x)=0$, or if $\lim_{x\to c}f(x)=\pm\infty$ and $\lim_{x\to c}g(x)=\pm\infty$,then:

1. $\lim_{x\to c}f(x)g(x)=\lim_{x\to c}f'(x)g'(x)$ as long as this latter limit exists or is infinite.
2. If f and g are differentiable at every number x greater than some number a, with $g'(x)\neq 0$, then $\lim_{x\to\infty}f(x)g(x)=\lim_{x\to\infty}f'(x)g'(x)$ as long as this latter limit exists or is infinite.

Let's look at applying the rule to some examples.

Example 2:

We will start by reconsidering the previous example, $f(x)=\ln(x+1)x$,and verify the following limits using L'Hôpital's Rule:

$$\lim_{x\to 0}\ln(x+1)x\lim_{x\to\infty}\ln(x+1)x=1.=0.$$

Solution:

Since $\lim_{x\to 0}\ln(x+1)=\lim_{x\to 0}x=0$, L'Hôpital's Rule applies and we have

$$\lim_{x\to 0}\ln(x+1)x=\lim_{x\to 0\,1x+1}1=11=1.$$

Likewise,

$$\lim_{x\to\infty}\ln(x+1)x=\lim_{x\to\infty\,1x+1}1=01=0.$$

Now let's look at some more examples.

Example 3:

Evaluate $\lim_{x\to 0}e^x-1x$.

Solution:

Since $\lim_{x\to 0}(e^x-1)=\lim_{x\to 0}x=0$, L'Hôpital's Rule applies and we have

$$\lim_{x\to 0}e^x-1x=\lim_{x\to 0}e^x1=11=1.$$

Example 4:

Evaluate $\lim_{x\to+\infty}x^2e^x$

Solution:

Since $\lim_{x\to+\infty}x^2=\lim_{x\to+\infty}e^x=+\infty$, L'Hôpital's Rule applies and we have

$$\lim_{x\to+\infty}x^2e^x=\lim_{x\to+\infty}2xe^x.$$

Here we observe that we still have the indeterminate form $\infty\infty$. So we apply L'Hôpital's Rule again to find the limit as follows:

$$\lim_{x\to+\infty}\frac{x^2}{e^x}=\lim_{x\to+\infty}\frac{2x}{e^x}=\lim_{x\to+\infty}\frac{2}{e^x}=0$$

L'Hôpital's Rule can be used repeatedly on functions like this. It is often useful because polynomial functions can be reduced to a constant.

Let’s look at an example with trigonometric functions.

Example 5:

Evaluate $\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

Solution:

Since $\lim_{x\to 0}(1-\cos x)=\lim_{x\to 0}x^2=0$, L’Hôpital’s Rule applies and we have

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}.$$

Lesson Summary

1. We learned to examine end behavior of functions on infinite intervals.
2. We determined horizontal asymptotes of rational functions.
3. We examined indeterminate forms of limits of rational functions.
4. We applied L’Hôpital’s Rule to find limits of rational functions.

Review Questions

1. Use your graphing calculator to estimate $\lim_{x\to+\infty}[x[\ln(x+3)-\ln(x)]]$.
2. Use your graphing calculator to estimate $\lim_{x\to+\infty}x\ln(1+2e^{x})$.

In problems #3–10, use L’Hôpital’s Rule to compute the limits, if they exist.

3. $\lim_{x\to 3}\frac{x^2-9}{x-3}$
4. $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$
5. $\lim_{x\to+\infty}\frac{\ln(x)}{\sqrt{x}}$
6. $\lim_{x\to+\infty}x^2e^{-2x}$
7. $\lim_{x\to 0}(1-x)^{\frac{1}{x}}$
8. $\lim_{x\to 0}\frac{e^x-1-x}{x^2}$
9. $\lim_{x\to-\infty}\frac{e^x-1-x}{x^2}$
10. $\lim_{x\to\infty}x^{-\frac{1}{4}}\ln(x)$