4.8 Numerical Integration

1. $\triangle x = \frac{1-0}{8} = \frac{1}{8}$

$$\int_0^1 x^2 e^{-x} dx$$
$$= \frac{1}{16}\left[f(0)+2f\left(\frac{1}{8}\right)+2f\left(\frac{2}{8}\right)+2f\left(\frac{3}{8}\right)+2f\left(\frac{4}{8}\right)+2f\left(\frac{5}{8}\right)+2f\left(\frac{6}{8}\right)+2f\left(\frac{7}{8}\right)+f(1)\right]$$
$$= \frac{1}{16}\left[0+2\left(\frac{1}{8}\right)^2 e^{-\frac{1}{8}}+2\left(\frac{2}{8}\right)^2 e^{-\frac{2}{8}}+2\left(\frac{3}{8}\right)^2 e^{-\frac{3}{8}}+2\left(\frac{4}{8}\right)^2 e^{-\frac{4}{8}}+2\left(\frac{5}{8}\right)^2 e^{-\frac{5}{8}}+2\left(\frac{6}{8}\right)^2 e^{-\frac{6}{8}}+2\left(\frac{7}{8}\right)^2 e^{-\frac{7}{8}}+(1)^2 e^{-1}\right]$$
$$= \frac{1}{16}[0.028+0.097+0.193+0.303+0.148+0.531+0.638+0.368]$$
$$\approx 0.16$$

2. $\triangle x = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2}$

$$\int_1^4 \ln\sqrt{x}dx = \frac{1}{4}\left[f(1)+2f\left(\frac{3}{2}\right)+2f(2)+2f\left(\frac{5}{2}\right)+2f(3)+2f\left(\frac{7}{2}\right)+f(4)\right]$$
$$= \frac{1}{4}\left[0+2\ln\left(\sqrt{\frac{3}{2}}\right)+2\ln\left(\sqrt{2}\right)+2\ln\left(\sqrt{\frac{5}{2}}\right)+2\ln\left(\sqrt{3}\right)+2\ln\left(\sqrt{\frac{7}{2}}\right)+\ln\left(\sqrt{4}\right)\right]$$
$$= \frac{1}{4}[0.405+0.693+0.916+1.099+1.253+0.693]$$
$$\approx 1.26$$

3. $\triangle x = \frac{1-0}{4} = \frac{1}{4}$

$$\int_0^1 \sqrt{1+x^4}dx = \frac{1}{8}\left[f(0)+2f\left(\frac{1}{4}\right)+2f\left(\frac{1}{2}\right)+2f\left(\frac{3}{4}\right)+f(1)\right]$$
$$= \frac{1}{8}\left[1+2\sqrt{1+\left(\frac{1}{4}\right)^4}+2\sqrt{1+\left(\frac{1}{2}\right)^4}+2\sqrt{1+\left(\frac{3}{4}\right)^4}+\sqrt{1+(1)^4}\right]$$
$$= \frac{1}{8}[1+2.004+2.061+2.295+1.414]$$
$$\approx 1.10$$

4. $\triangle x = \frac{3-1}{8} = \frac{2}{8} = \frac{1}{4}$

$$\begin{aligned}\int_1^3 \frac{1}{x}dx &= \frac{1}{8}\left[f(1)+2f\left(\frac{5}{4}\right)+2f\left(\frac{6}{4}\right)+2f\left(\frac{7}{4}\right)+2f(2)+2f\left(\frac{9}{4}\right)+2f\left(\frac{10}{4}\right)+2f\left(\frac{11}{4}\right)+f(3)\right]\\ &= \frac{1}{8}\left[1+2\left(\frac{4}{5}\right)+2\left(\frac{4}{6}\right)+2\left(\frac{4}{7}\right)+2\left(\frac{1}{2}\right)+2\left(\frac{4}{9}\right)+2\left(\frac{4}{10}\right)+2\left(\frac{4}{11}\right)+\frac{1}{3}\right]\\ &= \frac{1}{8}[1+1.6+1.333+1.143+1+0.889+0.8+0.727+0.667]\\ &\approx 1.10\end{aligned}$$

5. Find n such that $|\text{Error}_{\text{Trapezoidal}}| \le 0.001$. $|f''(x)| = |\frac{2}{x^3}| \le 2$ because the maximum value of the second derivative is 2 on the interval $[1,3]$.

$$\begin{aligned}\frac{2(3-1)^3}{12n^2} &\le 0.001\\ \frac{16}{12n^2} &\le 0.001\\ 16 &\le 0.012n^2\\ \frac{16}{0.012} &\le n^2\\ \sqrt{\frac{16}{0.012}} &\le n\\ 36.5 &\le n\end{aligned}$$

Thus, you can choose $n = 37$ in order for the Trapezoidal Estimate to be within 0.001 of the actual integral.

6. $\triangle x = \frac{1-0}{8} = \frac{1}{8}$

$$\begin{aligned}&\int_0^1 x^2e^{-x}dx\\ &= \frac{1}{24}\left[f(0)+4f\left(\frac{1}{8}\right)+2f\left(\frac{2}{8}\right)+4f\left(\frac{3}{8}\right)+2f\left(\frac{4}{8}\right)+4f\left(\frac{5}{8}\right)+2f\left(\frac{6}{8}\right)+4f\left(\frac{7}{8}\right)+f(1)\right]\\ &= \frac{1}{24}\left[0+4\left(\frac{1}{8}\right)^2e^{-\frac{1}{8}}+2\left(\frac{2}{8}\right)e^{-\frac{2}{8}}+4\left(\frac{3}{8}\right)^2e^{-\frac{3}{8}}+2\left(\frac{4}{8}\right)e^{-\frac{4}{8}}+4\left(\frac{5}{8}\right)^2e^{-\frac{5}{8}}+2\left(\frac{6}{8}\right)^2e^{-\frac{6}{8}}+4\left(\frac{7}{8}\right)^2e^{-\frac{7}{8}}+(1)^2e^{-1}\right]\\ &= \frac{1}{24}[0.055+0.097+0.387+0.303+0.836+0.531+1.276+0.368]\\ &\approx 0.16\end{aligned}$$

7. $\triangle x = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2}$

$$\begin{aligned}\int_1^4 \sqrt{x}\ln x\,dx &= \frac{1}{6}\left[f(1)+4f\left(\frac{3}{2}\right)+2f(2)+4f\left(\frac{5}{2}\right)+2f(3)+4f\left(\frac{7}{2}\right)+2f(4)\right]\\ &= \frac{1}{6}\left[0+4\sqrt{\frac{3}{4}}\ln\left(\frac{3}{2}\right)+2\sqrt{2}\ln(2)+4\sqrt{\frac{5}{2}}\ln\left(\frac{5}{2}\right)+2\sqrt{3}\ln(3)+4\sqrt{\frac{7}{2}}\ln\left(\frac{7}{2}\right)+\sqrt{4}\ln(4)\right]\\ &= \frac{1}{6}[0+1.986+1.961+5.795+3.805+9.374+2.773]\\ &\approx 4.28\end{aligned}$$

8. $\triangle x=\frac{2-0}{6}=\frac{2}{6}=\frac{1}{3}$

$$\begin{aligned}\int_0^2 \sqrt{x^4+1}dx &= \frac{1}{9}\left[f(0)+4f\left(\frac{1}{3}\right)+2f\left(\frac{2}{3}\right)+4f(1)+2f\left(\frac{4}{3}\right)4f\left(\frac{5}{3}\right)+f(2)\right]\\ &= \frac{1}{9}\left[1+4\sqrt{\left(\frac{1}{3}\right)^4+1}+2\sqrt{\left(\frac{2}{3}\right)^4+1}+4\sqrt{(1)^4+1}+2\sqrt{\left(\frac{4}{3}\right)^4+1}+4\sqrt{\left(\frac{5}{3}\right)^4+1}+1+\sqrt{(2)^4+1}\right]\\ &= \frac{1}{9}[1+4.025+2.189+5.657+4.080+11.809+4.123]\\ &\approx 3.65\end{aligned}$$

9. $\triangle x=\frac{1-0}{4}=\frac{1}{4}$

$$\begin{aligned}\int_0^1 \sqrt{1+x^4}dx &= \frac{1}{12}\left[f(0)+4f\left(\frac{1}{4}\right)+2f\left(\frac{2}{4}\right)+4f\left(\frac{3}{4}\right)+f(1)\right]\\ &= \frac{1}{12}\left[1+4\sqrt{1+\left(\frac{1}{4}\right)^4}+2\sqrt{1+\left(\frac{2}{4}\right)^4}+4\sqrt{1+\left(\frac{3}{4}\right)^4}+\sqrt{1+(1)^4}\right]\\ &= \frac{1}{12}[1+4.008+2.062+4.589+1.414]\\ &\approx 1.09\end{aligned}$$

10. The problem needs to be fixed. The fourth derivative of e is 0 and thus this problem cannot be solved as it is written.