
5.3 The Length of a Plane Curve

1.

$$\begin{aligned}f'(x) &= \frac{1}{3} \times \frac{3}{2} (x^2 + 2)^{\frac{1}{2}} (2x) = x \sqrt{x^2 + 2} \\L &= \int_0^3 \sqrt{1 + \left(x \sqrt{x^2 + 2}\right)^2} dx \\&= \int_0^3 \sqrt{1 + x^2(x^2 + 2)} dx \\&= \int_0^3 \sqrt{1 + 2x^2 + x^4} dx \\&= \int_0^3 \sqrt{(1 + x^2)^2} dx \\&= \int_0^3 (1 + x^2) dx \\&= \left(x + \frac{x^3}{3}\right) \Big|_0^3 \\&= 3 + 9 \\&= 12\end{aligned}$$

2.

$$\begin{aligned}
 y(x) &= \frac{1}{6}y^3 + \frac{1}{2}y^{-1} \\
 y'(x) &= \frac{1}{2}y^2 - \frac{1}{2}y^{-2} \\
 L &= \int_1^2 \sqrt{1 + \left(\frac{1}{2}y^2 - \frac{1}{2y^2}\right)^2} dy \\
 &= \int_1^2 \sqrt{1 + \frac{1}{4}y^4 - \frac{1}{2} + \frac{1}{2}y^{-4}} dy \\
 &= \int_1^2 \sqrt{\frac{1}{2} + \frac{1}{4}y^4 + \frac{1}{4y^4}} dy \\
 &= \int_1^2 \sqrt{\left(\frac{1}{2}y^2 + \frac{1}{2y^2}\right)^2} dy \\
 &= \int_1^2 \left(\frac{1}{2}y^2 + \frac{1}{2y^2}\right) dy \\
 &= \left(\frac{y^3}{6} - \frac{1}{2y}\right) \Big|_1^2 \\
 &= \frac{8}{6} - \frac{1}{4} - \frac{1}{6} + \frac{1}{2} \\
 &= \frac{17}{12}
 \end{aligned}$$

$$3. x = \int_0^y \sqrt{\sec^4 t - 1} dt$$

By the Fundamental Theorem of Calculus, $\frac{dx}{dy} = \sqrt{\sec^4 y - 1}$.

$$\begin{aligned}
 L &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{1 + \left(\sqrt{\sec^4 y - 1} \right)^2} dy \\
 &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{1 + \sec^4 y - 1} dy \\
 &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\sec^4 y} dy \\
 &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 y dy \\
 &= \tan y \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
 &= \tan\left(\frac{\pi}{4}\right) - \tan\left(-\frac{\pi}{4}\right) \\
 &= 1 - (-1) \\
 &= 2
 \end{aligned}$$

4.

$$\begin{aligned}
 x^{\frac{2}{3}} + y^{\frac{2}{3}} &= 1 \\
 y &= \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}} \\
 \frac{dy}{dx} &= x^{-\frac{1}{3}} \left(1 - x^{\frac{2}{3}}\right)^{\frac{1}{2}}
 \end{aligned}$$

Using symmetry,

$$\begin{aligned}
 L &= 4 \int_0^1 \left[1 + \left(x^{-\frac{1}{3}} \left(1 - x^{\frac{2}{3}}\right)^{\frac{1}{2}} \right)^2 \right]^{\frac{1}{2}} dx \\
 &= 4 \int_0^1 \left[1 + x^{-\frac{2}{3}} \left(1 - x^{\frac{2}{3}}\right) \right]^{\frac{1}{2}} dx \\
 &= 4 \int_0^1 \left(x^{-\frac{2}{3}} \right)^{\frac{1}{2}} dx \\
 &= 4 \int_0^1 \left(x^{-\frac{1}{3}} \right) dx \\
 &= 4 \times \frac{3}{2} x^{\frac{2}{3}} \Big|_0^1 \\
 &= 6
 \end{aligned}$$

5. $y = kx^2$

Let $x = S$.

Then

$y = h$ and

$$\begin{aligned}h &= kS^2 \\ \frac{h}{S^2} &= k \\ \frac{dy}{dx} &= 2kx \\ &= 2\left(\frac{h}{S^2}\right)x\end{aligned}$$

Then

$$\begin{aligned}L &= \int_{-S}^S \sqrt{1 + (2kx)^2} dx \\ &= 2 \int_0^S \sqrt{1 + 4\frac{h^2}{S^4}x^2} dx\end{aligned}$$