7.6 Improper Integrals

Review Questions

1. a. This is improper because there is an infinite discontinuity between the limits of integration at $x = 3$.

b. This is not improper.

c. This is improper because there is an infinite discontinuity at $x = 0$.

d. This is improper because the integral has an infinite interval of integration.

e. This is not improper.

2. $\int\limits_{l}^{\infty} \frac{1}{x^{2.001}} dx = \lim_{l\to\infty} \frac{-x^{1.001}}{1.001}\Big|_{1}^{l} = \frac{1}{1.001}$

3.

$$\begin{aligned}\int\limits_{-\infty}^{-2}\left[\frac{1}{x-1}-\frac{1}{x+1}\right]dx &= \lim_{l\to-\infty}\left[\ln|x-1|-\ln|x+1|\right]_{l}^{-2}\\ &= \ln 3 - 0 - \lim_{l\to-\infty}\left[\ln|l-1|-\ln|l+1|\right]\\ &= \ln 3 - \ln\left|\frac{l-1}{l+1}\right|\\ &= \ln 3 - \ln\left|\frac{l\left(1-\frac{1}{l}\right)}{l\left(1+\frac{1}{l}\right)}\right|\\ &= \ln 3 - 0\\ &= \ln 3\end{aligned}$$

4.

$$\begin{aligned}\int\limits_{-\infty}^{0} e^{5x}dx &= \lim_{l\to-\infty}\left(\frac{1}{5}e^{5x}\right)\Big|_{l}^{0}\\ &= \lim_{l\to-\infty}\left(\frac{1}{5}\times l-\frac{1}{5}e^{5l}\right)\\ &= \frac{1}{5}-0\\ &= \frac{1}{5}\end{aligned}$$

5. The integral is divergent.

6. $\int\limits_{\frac{-\pi}{2}}^{\frac{-\pi}{2}} \tan x = 0$ (Look at the symmetry of the graph on the interval.)

7.

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x|_0^1$$
$$= \sin^{-1} 1 - \sin^{-1} 0$$
$$= \frac{\pi}{2} - 0$$
$$= \frac{\pi}{2}$$

8. a.

$$V = \pi \int_0^\infty (e^{-x})^2 dx$$
$$= \pi \int_0^\infty e^{-2x} dx$$
$$= \pi \left(\frac{-e^{-2x}}{2}\right)_0^\infty$$
$$= \pi \left(0 + \frac{1}{2}\right)$$
$$= \frac{\pi}{2}$$

b.

$$A = 2\pi \int_0^\infty f(x) \sqrt{1 + [f'(x)]^2} dx$$
$$= 2\pi \int_0^\infty e^{-x} \sqrt{1 + [-e^{-x}]^2} dx$$
$$= 2\pi \int_0^\infty e^{-x} \sqrt{1 + e^{-2x}} dx$$

Let $e^{-x} = \tan u$. Then $-e^{-x}dx = \sec^2 u\, du$.

$$
\begin{aligned}
A &= 2\pi \int_0^{\infty} e^{-x} \sqrt{1+e^{-2x}} dx \\
&= 2\pi \int_{\infty}^{0} \sec^2 u \sqrt{1+\tan^2 u}\, du \\
&= 2\pi \int_{\infty}^{0} \sec^3 u\, du \\
&= 2\pi \left(\frac{1}{2} \sec u \tan u + \frac{1}{2} \ln|\sec u + \tan u| \right) \\
&= 2\pi \left((1+e^{-2x})^{\frac{1}{2}} e^{-x} + \ln \left| (1+e^{-2x})^{\frac{1}{2}} + e^{-x} \right|_{\infty}^{0} \right) \\
&= \pi[2 + \ln(\sqrt{2}+1)]
\end{aligned}
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