7.3 Integration by Partial Fractions

1.

$$\frac{1}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1}$$
$$1 = A(x-1) + B(x+1)$$

Let $x = 1$.

$$1 = A(1-1) + B(x+1)$$
$$1 = B(2)$$
$$\frac{1}{2} = B$$

Let $x = -1$.

$$1 = A(x-1) + B(-1+1)$$
$$1 = A(-2)$$
$$-\frac{1}{2} = A$$

$$\begin{aligned}\int \frac{1}{x^2-1}dx &= \int \left(\frac{A}{x+1} + \frac{B}{x-1}\right)dx \\ &= \int \left(\frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1}\right)dx \\ &= -\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| \\ &= \frac{1}{2}\left|\frac{x-1}{x+1}\right| + C\end{aligned}$$

2.

$$\frac{x}{x^2-2x-3} = \frac{A}{x-3} + \frac{B}{x+1}$$
$$x = A(x+1) + B(x-3)$$

Let $x = 3$.

$$3 = A(3+1) + B(3-3)$$
$$3 = A(4)$$
$$\frac{3}{4} = A$$

Let $x = -1$.

$$\begin{aligned} x &= A(x+1) + B(-1-3) \\ -1 &= B(-4) \\ \frac{1}{4} &= B \end{aligned}$$

$$\begin{aligned} \int \frac{x}{x^2-2x-3} dx &= \int \frac{\frac{3}{4}}{x-3} dx + \int \frac{\frac{1}{4}}{x+1} dx \\ &= \frac{3}{4} \ln|x-3| + \frac{1}{4} \ln|x+1| + C \end{aligned}$$

3.

$$\begin{aligned} \frac{1}{x^3+x^2-2x} &= \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-1} \\ 1 &= A(x+2)(x-1) + Bx(x-1) + Cx(x+2) \end{aligned}$$

Let $x = 0$.

$$\begin{aligned} 1 &= A(2)(-1) \\ -\frac{1}{2} &= A \end{aligned}$$

Let $x = -2$.

$$\begin{aligned} 1 &= B(-2)(-3) \\ \frac{1}{6} &= B \end{aligned}$$

Let $x = 1$.

$$\begin{aligned} 1 &= C(1)(3) \\ \frac{1}{3} &= C \end{aligned}$$

$$\begin{aligned} \int \frac{1}{x^3+x^2-2x} dx &= -\int \frac{\frac{1}{2}}{x} dx + \int \frac{\frac{1}{6}}{x+2} dx + \int \frac{\frac{1}{3}}{x-1} dx \\ &= -\frac{1}{2} \ln|x| + \frac{1}{6} \ln|x+2| + \frac{1}{3} \ln|x-1| + C \end{aligned}$$

4.

$$\begin{aligned}\int \frac{x^3}{x^2+4}dx &= \int x\,dx - \int \frac{4x}{x^2+4}dx \\ &= \frac{x^2}{2} - 2\ln|x^2+4| + C\end{aligned}$$

5.

$$\begin{aligned}\int_0^1 \frac{\phi}{1+\phi}d\phi &= \int_0^1 \left(1 - \frac{1}{1+\phi}\right)d\phi \\ &= [\phi - \ln|1+\phi|]_0^1 \\ &= 1 - \ln(2)\end{aligned}$$

6.

$$\begin{aligned}\frac{x-1}{x^2(x+1)} &= \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x+1} \\ x-1 &= A(x+1) + Bx(x+1) + Cx^2\end{aligned}$$

Let $x = -1$.

$$\begin{aligned}-2 &= C(1) \\ -2 &= C\end{aligned}$$

Let $x = 0$.

$$\begin{aligned}-1 &= A(1) \\ -1 &= A\end{aligned}$$

Pick any other value of x to find B.

$$\begin{aligned}x-1 &= A(x+1) + Bx(x+1) + Cx^2 \\ 2-1 &= -(2+1) + B(6) - (2)(4) \\ 1 &= -3 + 6B - 8 \\ 12 &= 6B \\ 2 &= B\end{aligned}$$

$$\begin{aligned}
\int_1^5 \frac{x-1}{x^2(x+1)}dx &= -\int_1^5 \frac{1}{x^2}dx + \int_1^5 \frac{2}{x}dx - \int_1^5 \frac{2}{x+1}dx \\
&= -(-x^{-1})|_1^5 + 2\ln|x||_1^5 - 2\ln|x+1||_1^5 \\
&= \frac{1}{5} - 1 + 2\ln 5 + 0 - 2(\ln 6 - \ln 2) \\
&= -\frac{4}{5} + 2\ln 5 - 2\ln 3 \\
&= -\frac{4}{5} + 2\ln\frac{5}{3}
\end{aligned}$$

7. Let $u = \sin\theta$. Then $du = \cos\theta$.

$$\begin{aligned}
\frac{1}{u^2+4u-5} &= \frac{A}{u+5} + \frac{B}{u-1} \\
1 &= A(u-1) + B(u+5)
\end{aligned}$$

Let $u = 1$.

$$\begin{aligned}
1 &= B(1+5) \\
1 &= 6B \\
\frac{1}{6} &= B
\end{aligned}$$

Let $u = -5$.

$$\begin{aligned}
1 &= A(-5-1) \\
1 &= -6A \\
-\frac{1}{6} &= A
\end{aligned}$$

Then

$$\begin{aligned}
\int \frac{1}{u^2+4u-5}du &= -\int \frac{\frac{1}{6}}{u+5}du + \int \frac{\frac{1}{6}}{u-1}du \\
&= -\frac{1}{6}(\ln|u+5| - \ln|u-1|) + C \\
&= -\frac{1}{6}(\ln|\sin\theta+5| - \ln|\sin\theta-1|) + C \\
&= \frac{1}{6}\ln\left|\frac{\sin\theta-1}{\sin\theta+5}\right| + C
\end{aligned}$$

8. Let $u = e^\theta$ and $du = e^\theta d\theta$.

$$\int \frac{3e^{\theta}}{e^{2\theta}-1}d\theta = \int \frac{3}{u^2-1}du$$

Then

$$\begin{aligned}\frac{3}{u^2-1} &= \frac{A}{u+1}+\frac{B}{u-1}\\ 3 &= A(u-1)+B(u+1)\end{aligned}$$

Let $u = 1$.

$$\begin{aligned}3 &= B(1+1)\\ 3 &= 2B\\ \frac{3}{2} &= B\end{aligned}$$

Let $u = -1$.

$$\begin{aligned}3 &= A(-2)\\ 3 &= -2A\\ -\frac{3}{2} &= A\end{aligned}$$

$$\begin{aligned}\int \frac{3}{u^2-1}du &= \int \frac{-\frac{3}{2}}{u+1}du + \int \frac{\frac{3}{2}}{u-1}du\\ &= -\frac{3}{2}\ln|u+1| + \frac{3}{2}\ln|u-1| + C\\ &= -\frac{3}{2}\ln|e^{\theta}+1| + \frac{3}{2}\ln|e^{\theta}-1| + C\end{aligned}$$

9.

$$\begin{aligned}\int_{-\ln 3}^{\ln 4} \frac{1}{2+e^x}dx &= \int_{-\ln 3}^{\ln 4} \frac{1}{e^x(2e^{-x}+1)}dx\\ &= \int_{-\ln 3}^{\ln 4} \frac{e^{-x}}{2e^{-x}+1}dx\end{aligned}$$

Let $u = e^{-x}$. Then $du = -e^{-x}dx$.

$$\begin{aligned}
\int_{-\ln 3}^{\ln 4} \frac{e^{-x}}{2e^{-x}+1}dx &= -\int_{3}^{\frac{1}{4}} \frac{du}{2u+1} \\
&= -\frac{1}{2}\ln|2u+1|\Big|_{3}^{\frac{1}{4}} \\
&= -\frac{1}{2}\left(\ln\frac{3}{2} - \ln 7\right) \\
&= \frac{1}{2}\left(\ln 7 - \ln\frac{3}{2}\right) \\
&= \frac{1}{2}\ln\frac{14}{3}
\end{aligned}$$

10.

$$\begin{aligned}
\frac{1}{a^2-x^2} &= \frac{A}{a-x} + \frac{B}{a+x} \\
1 &= A(a+x) + B(a-x)
\end{aligned}$$

Let $x = b$.

$$\begin{aligned}
1 &= A(a+a) \\
\frac{1}{2a} &= A
\end{aligned}$$

Let $x = -a$.

$$\begin{aligned}
1 &= B(a-(-a)) \\
1 &= 2aB \\
\frac{1}{2a} &= B
\end{aligned}$$

$$\begin{aligned}
\int \frac{1}{a^2-x^2}dx &= \int \frac{\frac{1}{2a}}{a-x}dx + \int \frac{\frac{1}{2a}}{a+x}dx \\
&= -\frac{1}{2a}\ln|a-x| + \frac{1}{2a}\ln|a+x| \\
&= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C
\end{aligned}$$