7.2 Integration by Parts

1. Let $u = 3x$ and $dv = e^x dx$.

Then $du = 3\ dx$ and $v = e^x$.

$$\begin{aligned}\int 3xe^x dx &= uv - \int v\ du \\ &= 3xe^x - 3\int e^x dx \\ &= 3xe^x - 3e^x + C\end{aligned}$$

2. Let $u = x^2$ and $dv = e^{-x}\ dx$.

Then $du = 2x\ dx$ and $v = -e^{-x}$.

$$\begin{aligned}\int x^2 e^{-x} dx &= uv - \int v\ du \\ &= -x^2 e^{-x} + \int 2xe^{-x} dx\end{aligned}$$

Let $u = 2x$ and $dv = e^{-x} dx$.

Then $du = 2\ dx$ and $v = -e^{-x}$.

$$\begin{aligned}\int x^2 e^{-x} dx &= -x^2 e^{-x} + \int 2xe^{-x} dx \\ &= -x^2 e^{-x} + (uv - \int v\ du) \\ &= -x^2 e^{-x} + \left(-2xe^{-x} - \int -2xe^{-x} dx\right) \\ &= -x^2 e^{-x} - 2xe^{-x} + \left(-\left[-2(-e^{-x})\right]\right) + C \\ &= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C\end{aligned}$$

3. Let $u = \ln(3x+2)$ and $dv = 1\ dx$. Then $du = \frac{3}{3x+2} dx$ and $v = x$.

$$\begin{aligned}\int \ln(3x+2) dx &= uv - \int v\ du \\ &= x\ln|3x+2| - \int x\frac{3}{3x+2} dx \\ &= x\ln|3x+2| - \int \frac{3x}{3x+2} dx\end{aligned}$$

Let $u = 3x+2$ and $x = \frac{u-2}{3}$. Then $du = 3\ dx$.

$$\begin{aligned}\int \ln(3x+2)dx &= x\ln|3x+2| - \int \frac{3x}{3x+2}dx \\ &= x\ln|3x+2| - \left(\int \frac{u-2}{3u}du\right) \\ &= x\ln|3x+2| - \left(\int \left(\frac{1}{3} - \frac{2}{3u}\right)du\right) \\ &= x\ln|3x+2| - \frac{1}{3}u + \frac{2}{3}\ln|u| \\ &= x\ln|3x+2| - \frac{1}{3}(3x+2) + \frac{2}{3}\ln|3x+2|\end{aligned}$$

4. Let $u = \sin^{-1} x$ and $dv = 1\ dx$. Then $du = \frac{1}{\sqrt{1-x^2}}dx$ and $v = x$.

$$\begin{aligned}\int \sin^{-1} x dx &= uv - \int\ v\ du \\ &= x\sin^{-1} x - \int \frac{1}{\sqrt{1-x^2}}dx\end{aligned}$$

Let $u = 1 - x^2$. Then $du = -2x\ dx$.

$$\begin{aligned}\int \sin^{-1} x dx &= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}dx \\ &= x\sin^{-1} x + \frac{1}{2}\int u^{\frac{-1}{2}}du \\ &= x\sin^{-1} x + \frac{1}{2}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C \\ &= x\sin^{-1} x + \sqrt{1-x^2} + C\end{aligned}$$

5. $\int \sec^3 x\ dx = \int \sec^2 x \sec x\ dx$

Let $u = \sec\ x$ and $dv = \sec^2 x\ dx$. Then $du = \tan\ x\ dx$ and $v = \tan x$.

$$\begin{aligned}\int \sec^3 x\ dx &= \int \sec^2 x \sec x\ dx \\ &= uv - \int\ v\ du \\ &= (\sec x)(\tan x) - \int (\tan^2\ x)\sec x\ dx \\ &= (\sec x)(\tan x) - \int (\sec^2 x - 1)\sec x\ dx \\ &= (\sec x)(\tan x) - \int \sec^3 x\ dx + \int \sec x\ dx \\ 2\int \sec^3 x\ dx &= (\sec x)(\tan x) + \int \sec x\ dx \\ &= (\sec x)(\tan x) + \ln|\sec x + \tan x| \\ \int \sec^3 x\ dx &= \frac{1}{2}(\sec x)(\tan x) + \frac{1}{2}\ln|\sec x + \tan x| + C\end{aligned}$$

6. Let $u = \ln(3x)$ and $dv = 2x\,dx$. Then $du = \frac{3}{3x}dx = \frac{1}{x}dx$ and $v = x^2$.

$$\begin{aligned}\int 2x\ln(3x)dx &= uv - \int v\,du\\ &= x^2\ln(3x) - \int \frac{x^2}{x}dx\\ &= x^2\ln(3x) - \int x\,dx\\ &= x^2\ln(3x) - \frac{x^2}{2}\\ &= x^2\ln(3x) - \frac{x^2}{2} + C\end{aligned}$$

7. Let $u = \ln x$. Then $du = \frac{1}{x}dx$.

$$\begin{aligned}\int \frac{(\ln x)^2}{x}dx &= \int u^2\,du\\ &= \frac{u^3}{3} + C\\ &= \frac{(\ln x)^2}{3} + C\end{aligned}$$

8. Let $u = 5x - 2$. Then $du = 5\,dx$ and $x = \frac{u+2}{5}$.

$$\begin{aligned}\int x\sqrt{5x-2}dx &= \frac{1}{5}\int \left(\frac{u+2}{5}\right)\sqrt{u}\,du\\ &= \frac{1}{5}\int \frac{u^{\frac{3}{2}}}{5} + \frac{2u^{\frac{1}{2}}}{5}du\\ &= \frac{1}{5}\left(\frac{u^{\frac{5}{2}}}{5\left(\frac{5}{2}\right)} + \frac{2u^{\frac{3}{2}}}{5\left(\frac{3}{2}\right)}\right)\\ &= \frac{2(5x-2)^{\frac{5}{2}}}{125} + \frac{2(5x-2)^{\frac{5}{2}}}{75} + C\end{aligned}$$

Let $u = x$ and $dv = \sqrt{5x-2}dx$.Then $du = dx$ and $v = \frac{1}{5}\frac{(5x-2)^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{15}(5z-2)^{\frac{3}{2}}$.

$$\begin{aligned}\int x\sqrt{5x-2}dx &= \frac{2}{15}x(5x-2)^{\frac{3}{2}} - \int \frac{2}{15}(5x-2)^{\frac{3}{2}}dx\\ &= \frac{2}{15}x(5x-2)^{\frac{3}{2}} - \frac{2}{15}\times\frac{1}{5}\frac{(5x-2)^{\frac{5}{2}}}{\frac{5}{2}}\\ &= \frac{2}{15}x(5x-2)^{\frac{3}{2}} - \frac{4}{375}(5x-2)^{\frac{5}{2}} + C\end{aligned}$$

9. Let $u = x^2$ and $dv = e^{5x}dx$.

TABLE 7.1:

Alternate signs	u and its derivatives	dv and its antiderivatives
$+$	x^2	e^{5x}
$-$	$2x$	$\frac{1}{5}e^{5x}$
$+$	2	$\frac{1}{25}e^{5x}$
$-$	0	$\frac{1}{125}e^{5x}$

$$\int x^2e^{5x}dx = \frac{1}{5}x^2e^{5x} - \frac{2}{25}xe^{5x} + \frac{2}{125}e^{5x} + C$$

10. Let $u = x^2$ and $dv = e^x\ dx$.

TABLE 7.2:

Alternate signs	u and its derivatives	dv and its antiderivatives
$+$	x^2	e^x
$-$	$2x$	e^x
$+$	2	e^x
$-$	0	e^x

$$\begin{aligned}\int_0^1 x^2e^xdx &= (x^2e^x - 2xe^x + 2e^x)|_0^1 \\ &= e - 2e + 2e - (2) \\ &= e - 2\end{aligned}$$

11. Let $u = \ln(x+1)$ and $dv = 1\ dx$. Then $du = \frac{1}{x+1}dx$ and $v = x$.

$$\begin{aligned}\int_1^3 \ln(x+1)dx &= x\ln(x+1)|_1^3 - \int_1^3 \frac{x}{x+1}dx \\ &= x\ln(x+1)|_1^3 - \int_1^3 \left(1 - \frac{1}{x+1}\right)dx \\ &= [x\ln(x+1) - x + \ln(x+1)]_1^3 \\ &= 3\ln4 - 3 + \ln4 - (\ln2 - 1 + \ln2) \\ &= 4\ln4 - 2 - 2\ln2 \\ &= 8\ln2 - 2\ln2 - 2 \\ &= 6\ln2 - 2\end{aligned}$$