3.7 Optimization

1.

$$f(x) = 2x^2 - 6x + 6$$
$$f'(x) = 4x - 6$$

Find the critical values.

$$4x - 6 = 0$$
$$x = \frac{6}{4} = \frac{3}{2}$$

Find the function values of the critical value and the endpoints.

$$f(0) = 6$$
$$f(5) = 50 - 30 + 6 = 26$$
$$f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 6 = \frac{3}{2}$$

There is an absolute minimum at $\left(\frac{3}{2}, \frac{3}{2}\right)$. There is an absolute maximum at $(5, 26)$.

2.

$$f(x) = x^3 + 3x^2$$
$$f'(x) = 3x^2 + 6x$$

Find the critical values.

$$3x^2 + 6x = 0$$
$$3x(x+2) = 0$$
$$x = 0 \text{ or } x = -2$$

Find the function values of the critical value and the endpoints.

$$f(0) = 0$$
$$f(-2) = 6$$
$$f(3) = 27 + 27 = 54$$

There is an absolute minimum at $(0, 0)$. There is an absolute maximum at $(3, 54)$.

3.

$$f(x) = 3x^{\frac{2}{3}} - 6x + 6$$
$$f'(x) = 2x^{-\frac{1}{3}} - 6$$

Find the critical values.

$$2x^{-\frac{1}{3}} - 6 = 0$$
$$2x^{-\frac{1}{3}} = 6$$
$$x^{-\frac{1}{3}} = 3$$
$$x^{-1} = 27$$
$$x = \frac{1}{27}$$

Find the function values of the critical value and the endpoints.

$$f\left(\frac{1}{27}\right) = 3\left(\sqrt[3]{27}\right)^2 - 6\left(\frac{1}{27}\right) + 6 = 27 - \frac{6}{27} + 6 = 32.8$$
$$f(1) = 3$$
$$f(8) = 12 - 48 + 6 = -30$$

There is an absolute minimum at $(1,3)$. There is an absolute maximum at $(8,-30)$.

4.

$$f(x) = x^4 - x^3$$
$$f'(x) = 4x^3 - 3x^2$$

Find the critical values.

$$4x^3 - 3x^2 = 0$$
$$x^2(4x-3) = 0$$
$$x^2 = 0 \text{ or } 4x - 3 = 0$$
$$x = 0 \text{ or } x = \frac{3}{4}$$

Find the function values of the critical value and the endpoints.

$$f(0) = 0$$
$$f\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)^4 - \left(\frac{3}{4}\right)^3 = -0.105$$
$$f(-2) = 24$$
$$f(2) = 8$$

There is an absolute minimum at $\left(\frac{3}{4},-0.105\right)$. There is an absolute maximum at $(-2,-24)$.

5. The primary equation is $P=2x+2y$.

The secondary equation is $xy=2000$.

Both feasible domains require $x>0$ and $y>0$.

Solve the secondary equation for y : $y=\frac{2000}{x}$.

Substitute the secondary equation into the primary equation and simplify:

$$\begin{aligned} P &= 2x+2y \\ &= 2x+2\left(\frac{2000}{x}\right) \\ &= 2x+\frac{4000}{x} \end{aligned}$$

Find the critical value:

$$\begin{aligned} P &= 2x+\frac{2000}{x} \\ \frac{dP}{dx} &= 2+\frac{(-1)4000}{x^2} && = 2-\frac{4000}{x^2} \\ 2-\frac{4000}{x^2} &= 0 \\ 2x^2-4000 &= 0 \\ x^2-2000 &= 0 \\ x^2 &= 2000 \\ x &= 20\sqrt{5} \\ f\left(20\sqrt{5}\right) &= \frac{2000}{20\sqrt{5}} = 20\sqrt{5} \end{aligned}$$

Use the second derivative to find if $20\sqrt{5}$ is an absolute maximum or minimum.

$$\begin{aligned} \frac{d^2P}{dx^2} &= \frac{8000}{x^3} \\ \left.\frac{d^2P}{dx^2}\right|_{x=20\sqrt{5}} &= \frac{8000}{\left(20\sqrt{5}\right)^3} > 0 \end{aligned}$$

$20\sqrt{5}$ is an absolute minimum. The dimensions of the rectangle with a perimeter as small as possible is $20\sqrt{5}$ by $20\sqrt{5}$.

6. The primary equation is $S=x+y$.

The secondary equation is $xy=50$.

The feasible domain is $x,y>0$ or $x,y<0$

Solve $xy=50$ for y.

$$y = \frac{50}{x}$$

Substitute $y = \frac{50}{x}$ into the primary equation.

$$S = x + \frac{50}{x}$$
$$\frac{dS}{dx} = 1 - \frac{50}{x^2}$$

Find the critical values.

$$1 - \frac{50}{x^2} = 0$$
$$x^2 - 50 = 0$$
$$x = \sqrt{50} = \pm 5\sqrt{2}$$

$$x = 5\sqrt{2}, \text{ than } y = \frac{50}{5\sqrt{2}} = 5\sqrt{2}$$
$$x = -5\sqrt{2}, \text{ than } y = -\frac{50}{5\sqrt{2}} = -5\sqrt{2}$$

Use the second derivative test to determine which critical value is the absolute minimum.

$$\frac{d^2S}{dx^2} = 2\left(\frac{50}{x^3}\right)$$
$$\left.\frac{d^2S}{dx^2}\right|_{x=5\sqrt{2}} = \frac{100}{\left(5\sqrt{2}\right)^3} > 0$$
$$\left.\frac{d^2S}{dx^2}\right|_{x=-5\sqrt{2}} = \frac{100}{\left(-5\sqrt{2}\right)^3} < 0$$

Thus, $5\sqrt{2}$ is an absolute minimum. The numbers $5\sqrt{2}$ and $5\sqrt{2}$ give the absolute minimum of the sum S.

7. The domain is $[0,45]$.

$$s(t) = -0.025t^2 + t + 15$$
$$s'(t) = -0.050t + 1$$
$$-0.050t = -1$$
$$t = 20$$
$$s(20) = 25$$
$$s''(t) = -0.05 < 0$$

Find the function values of the endpoints and the critical values.

$$s(0) = 15$$
$$s(45) = 9.375$$
$$s(20) = 25$$

$t = 20$ is an absolute maximum.

At $t = 20$ feet, the basketball will reach a height of 25 feet.

8. $h(t) = -\frac{1}{3}t^3 + 4t^3 + 25t + 4$

a. Find the first derivative.

$$h'(t) = -t^2 + 8t + 25$$
$$-t^2 + 8t + 25 = 0$$
$$t^2 - 8t - 25 = 0$$

Use the quadratic formula to find the critical values.

$$t = \frac{8 \pm \sqrt{64 - 4(1)(-25)}}{2}$$
$$t = 4 \pm \sqrt{41}$$

Since $t \geq 0$, we consider the critical value $t = 4 + \sqrt{41}$.

$$h''(t) = -2t + 8$$
$$h''\left(4 + \sqrt{41}\right) = -2\left(4 + \sqrt{41}\right) + 8 = -2\sqrt{41} < 0$$

$t = 4 + \sqrt{41}$ is a maximum.

b. $h\left(4 + \sqrt{41}\right) = 321.7$ ft

c. The rocket hits the ground when $h(t) = 0$. Use a graphing calculator to find the zero of the function after $h\left(4 + \sqrt{41}\right) = 321.7$. The rocket hits the ground when $t \approx 16.6$ seconds.

9. Let A be a given perimeter. The primary equation is $A = xy$.

The secondary equation is $P = 2x + 2y$.

Both feasible domains require $x > 0$ and $y > 0$.

Solve the secondary equation for y:

$$P = 2x + 2y$$
$$P - 2x = 2y.$$
$$\frac{P}{2} - x = y$$

Substitute the secondary equation into the primary equation and simplify:

$$\begin{aligned} A &= xy \\ &= x\left(\frac{P}{2} - x\right) \\ &= \frac{Px}{2} - x^2 \end{aligned}$$

Find the critical value:

$$\begin{aligned} A &= \frac{Px}{2} - x^2 \\ \frac{dA}{dx} &= \frac{P}{2} - 2x \\ \frac{P}{2} - 2x &= 0 \\ -2x &= -\frac{P}{2} \\ x &= \frac{P}{4} \end{aligned}$$

Find the corresponding y.

$$\begin{aligned} P &= 2x + 2y \\ P &= 2\left(\frac{P}{4}\right) + 2y \\ P - \frac{P}{2} &= 2y \\ \frac{P}{2} &= 2y \\ \frac{P}{4} &= y \end{aligned}$$

Use the second derivative to find if $\frac{P}{4}$ is an absolute maximum or minimum.

$$\begin{aligned} \frac{d^2A}{dx^2} &= -2 \\ \frac{d^2P}{dx^2}\bigg|_{x=\frac{P}{4}} &= -2 < 0 \end{aligned}$$

$\frac{P}{2}$ is an absolute maximum. The dimensions of the rectangle with a area as large as possible is $\frac{P}{2}$ by $\frac{P}{2}$.

10. Let P be a given perimeter. The primary equation is $P = 2x + 2y$.

The secondary equation is $A = xy$.

Both feasible domains require $x > 0$ and $y > 0$.

Solve the secondary equation for $y: \; y = \frac{A}{x}$.

Substitute the secondary equation into the primary equation and simplify:

$$\begin{aligned} P &= 2x+2y \\ &= 2x+2\left(\frac{A}{x}\right) \\ &= 2x+\frac{2A}{x} \end{aligned}$$

Find the critical value:

$$\begin{aligned} P &= 2x+\frac{2A}{x} \\ \frac{dP}{dx} &= 2+\frac{2A}{x^2} = 2-\frac{2A}{x^2} \\ 2-\frac{2A}{x^2} &= 0 \\ 2x^2-2A &= 0 \\ x^2-A &= 0 \\ x^2 &= 2000 \\ x &= \sqrt{A} \end{aligned}$$

Find the corresponding y.

$$\begin{aligned} A = xy = \sqrt{A}y \\ \sqrt{A} = y \end{aligned}$$

Use the second derivative to find if $\sqrt{A}$ is an absolute maximum or minimum.

$$\begin{aligned} \frac{d^2P}{dx^2} &= \frac{4A}{x^3} \\ \frac{d^2P}{dx^2}\Big|_{x=\sqrt{A}} &= \frac{4A}{\left(\sqrt{A}\right)^3} > 0 \end{aligned}$$

$\sqrt{A}$ is an absolute minimum. The dimensions of the rectangle with a perimeter as small as possible is $\sqrt{A}$ by $\sqrt{A}$.