3.1 Related Rates

1. a. Answers will vary.

b. Answers will vary.

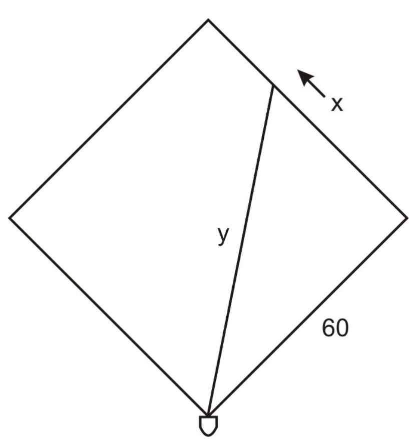
2.

$$\begin{aligned} 4x^2+16y^2 &= 32 \\ 4\left(2x\frac{dx}{dt}\right)+16\left(2y\frac{dy}{dt}\right) &= 0 \\ 8x\frac{dx}{dt}+32y\frac{dy}{dt} &= 0 \end{aligned}$$

Substitute $(2,1)$ and $\frac{dx}{dt}=3$ into the last equation and solve for $\frac{dy}{dx}$.

$$\begin{aligned} 8x\frac{dx}{dt}+32y\frac{dy}{dt} &= 0 \\ 16(1)(3)+32\frac{dy}{dt} &= 0 \\ 48+32\frac{dy}{dt} &= 0 \\ 32\frac{dy}{dt} &= -48 \\ \frac{dy}{dt} &= \frac{-48}{32} = \frac{3 \text{ ft}}{2 \text{ sec}} \end{aligned}$$

3. Draw a diagram of the situation. The runner is $\frac{2}{3}(60)=40$ ft from first base. The player's rate is $\frac{dx}{dt}=\frac{18 \text{ ft}}{\text{sec}}$.



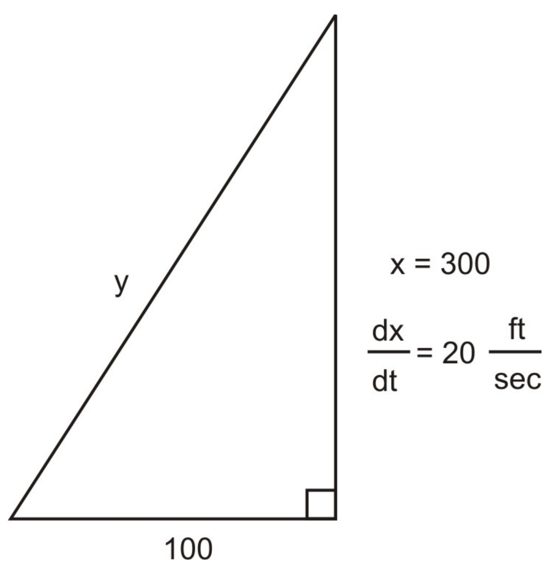
The variable y represents the distance between the runner and home plate. The variable x represents the distance traveled by the runner. The rate at which the distance between the runner and home plate is changing is $\frac{dy}{dt}$. The runner is $\frac{2}{3}(60)=40$ ft from first base. The player's rate of change is $\frac{dx}{dt}=\frac{18 \text{ ft}}{\text{sec}}$. The diagram shows that a right triangle is formed with x, the side of the diamond, and y. Use the Pythagorean Theorem to solve for y.

$$\begin{aligned}60^2+40^2&=y^2\\3600+1600&=y^2\\5200&=y^2\\\sqrt{5200}&=y\end{aligned}$$

Now, differentiate $60^2+x^2=y^2$ with respect to time t and substitute the known values to find $\frac{dy}{dt}$.

$$\begin{aligned}60^2+x^2&=y^2\\0+2x\frac{dx}{dt}&=2y\frac{dy}{dt}\\2x\frac{dx}{dt}&=2y\frac{dy}{dt}\\2(40)\frac{18\text{ ft}}{\text{sec}}&=2\left(\sqrt{5200}\right)\frac{dy}{dt}\\\frac{1440\text{ ft}}{\text{sec}}&=2\left(\sqrt{5200}\right)\frac{dy}{dt}\\\frac{1440\text{ ft}}{2\left(\sqrt{5200}\right)\text{sec}}&=\frac{dy}{dt}\\\frac{720\text{ ft}}{\sqrt{5200}\,\text{sec}}&=\frac{dy}{dt}\\\frac{9.98\text{ ft}}{\text{sec}}&\approx\frac{dy}{dt}\end{aligned}$$

4. Draw a diagram of the situation. The balloon was 300 ft from the ground. The balloon's rate of change was $\frac{dx}{dt}=\frac{20\text{ ft}}{\text{sec}}$.



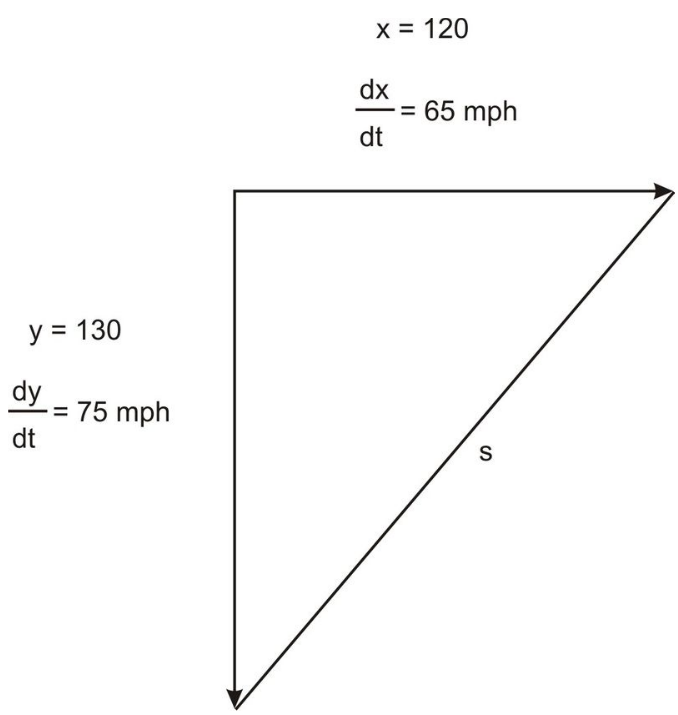
The variable y represents the distance between Mr. Smith's place and the balloon's place. The variable x represents the height of the balloon. The rate at which the distance between Mr. Smith's place and the balloon's place was changing is $\frac{dy}{dt}$ .The diagram shows that a right triangle is formed with x, the height of the balloon, and y. Use the Pythagorean Theorem to solve for y.

$$\begin{aligned} 300^2+100^2 &= y^2 \\ 90,000+10,000 &= y^2 \\ 100,000 &= y^2 \\ \sqrt{100,000} &= y \end{aligned}$$

Now, differentiate $100^2+x^2=y^2$ with respect to time t and substitute the known values to find $\frac{dy}{dt}$.

$$\begin{aligned} 100^2+x^2 &= y^2 \\ 0+2x\frac{dx}{dt} &= 2y\frac{dy}{dt} \\ 2x\frac{dx}{dt} &= 2y\frac{dy}{dt} \\ 2(300)\frac{20 \text{ ft}}{\text{sec}} &= 2\left(\sqrt{100,000}\right)\frac{dy}{dt} \\ \frac{12,000 \text{ ft}}{\text{sec}} &= 2\left(\sqrt{100,000}\right)\frac{dy}{dt} \\ \frac{12,000 \text{ ft}}{2\left(\sqrt{100,000}\right)\text{sec}} &= \frac{dy}{dt} \\ \frac{6,000 \text{ ft}}{\sqrt{100,000}\,\text{sec}} &= \frac{dy}{dt} \\ \frac{18.97 \text{ ft}}{\text{sec}} &\approx \frac{dy}{dt} \end{aligned}$$

5. Draw a diagram of the situation. Let x represent the distance traveled by the first train. The rate of change of the first train was $\frac{dx}{dt}=\frac{65 \text{ mi}}{\text{hr}}$. Let y represent the distance traveled by the second train. The rate of change of the second train was $\frac{dy}{dt}=\frac{75 \text{ mi}}{\text{hr}}$. At 3 PM, the distance $y=130$ mi and the distance $x=120$ mi. Let s represent the distance between the two trains.



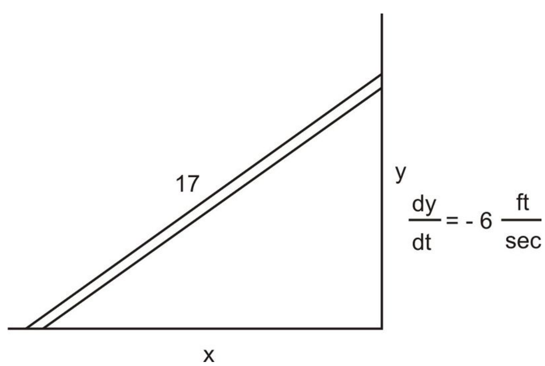
Use the Pythagorean Theorem to solve for s.

$$\begin{aligned}
120^2+130^2 &= s^2\\
14,400+16,900 &= s^2\\
31,300 &= s^2\\
\sqrt{31,300} &= s
\end{aligned}$$

Now, differentiate $y^2+x^2=s^2$ with respect to time t and substitute the known values to find $\frac{ds}{dt}$.

$$\begin{aligned}
y^2+x^2 &= s^2\\
2y\frac{dy}{dt}+2x\frac{dx}{dt} &= 2s\frac{ds}{dt}\\
y\frac{dy}{dt}+x\frac{dx}{dt} &= s\frac{ds}{dt}\\
(130)\,\frac{75\text{ mi}}{\text{hr}}+(120)\,\frac{65\text{ mi}}{\text{hr}} &= \sqrt{31,300}\frac{ds}{dt}\\
\frac{9,750\text{ mi}}{\text{hr}}+\frac{7,800\text{ mi}}{\text{hr}} &= \sqrt{31,300}\frac{ds}{dt}\\
\frac{17,750\text{ mi}}{\sqrt{31,300}\text{ hr}} &= \frac{ds}{dt}\\
\frac{99.20\text{ mi}}{\text{hr}} &\approx \frac{ds}{dt}
\end{aligned}$$

6. Draw a diagram of the situation. Let x represent the distance on the ground between the bottom of the ladder and the wall. Let y represent the height of the ladder against the wall. The rate of change of the ladder is $\frac{dy}{dt}=-\frac{6\text{ ft}}{\text{sec}}$. The distance between the bottom of the ladder and the wall is 17 ft.



Use the Pythagorean Theorem to solve for x when $y=8$.

$$\begin{aligned}
x^2+8^2 &= 17^2\\
x^2+64 &= 289\\
x^2 &= 225\\
x &= 15
\end{aligned}$$

Now, differentiate $17^2+x^2=y^2$ with respect to time t and substitute the known values to find $\frac{dx}{dt}$.

$$
\begin{aligned}
x^2+y^2 &= 17^2 \\
2x\frac{dx}{dt}+2y\frac{dx}{dt} &= 0 \\
x\frac{dx}{dt}+y\frac{dx}{dt} &= 0 \\
(15)\,\frac{dx}{dt}+(8)\,\frac{-6\text{ ft}}{\text{sec}} &= 0 \\
(15)\,\frac{dx}{dt}-\frac{48\text{ ft}}{\text{sec}} &= 0 \\
(15)\,\frac{dx}{dt} &= \frac{48\text{ ft}}{\text{sec}} \\
\frac{dx}{dt} &= \frac{48\text{ ft}}{(15)\text{ sec}} \\
\frac{dx}{dt} &\approx \frac{16\text{ ft}}{5\text{ sec}}
\end{aligned}
$$

7. $A = lw$ where ℓ represents the length of the rectangle, w represents the width, and A represents the area of the rectangle. Then $\frac{dl}{dt} = \frac{6\text{ ft}}{\text{min}}$ and $\frac{dw}{dt} = \frac{2\text{ ft}}{\text{min}}$. Differentiate the equation $A = lw$ with respect to time t.

$$
\begin{aligned}
A &= lw \\
\frac{dA}{dt} &= \frac{dl}{dt}w+l\frac{dw}{dt} \\
&= \frac{6\text{ ft}}{\text{min}}(15)+(25)\,\frac{2\text{ ft}}{\text{min}} \\
&= \frac{90\text{ ft}}{\text{min}}+\frac{50\text{ ft}}{\text{min}} \\
&= \frac{140\text{ ft}}{\text{min}}
\end{aligned}
$$

8. When $\frac{dp}{dt} = \frac{-10}{\text{week}}$, find $\frac{dx}{dt}$.

9. Let $s =$ length of one side of the cube. Then volume $V = s^3$.

$$
\begin{aligned}
V &= s^3 \\
\frac{dV}{dt} &= 3s^2\frac{ds}{dt} \\
\frac{dV}{dt} &= 3\,(6\text{ in.})^2\,\frac{1\text{ in.}}{4\text{ min}} \\
&= \frac{27\text{ in.}^3}{4\text{ min}}
\end{aligned}
$$

10. a. $A = \pi r^2$

Solve for r when $A = 36\pi\text{ in.}^2$

$$\begin{aligned}
36\pi &= \pi r^2 \\
36 &= r^2 \\
6 &= r \\
A &= \pi r^2 \\
\frac{dA}{dt} &= 2\pi r \frac{dr}{dt} \\
\frac{24 \text{ in.}}{\text{min}} &= 2\pi(6)\frac{dr}{dt} \\
\frac{24 \text{ in.}}{(12\pi)\text{ min}} &= \frac{dr}{dt} \\
\frac{2 \text{ in.}}{\pi \text{ min}} &= \frac{dr}{dt}
\end{aligned}$$

b.

$$\begin{aligned}
C &= 2\pi r \\
\frac{dC}{dt} &= 2\pi \frac{dr}{dt} \\
&= 2\pi\left(\frac{2 \text{ in.}}{\pi \text{ min}}\right) \\
&= \frac{4 \text{ in.}}{\text{min}}
\end{aligned}$$