3.4 The Second Derivative Test

1.

$$f(x) = \frac{x^2}{4} + \frac{4}{x} = \frac{x^2}{4} + 4x^{-1}$$
$$f'(x) = \frac{2x}{4} - 4x^{-2} = \frac{x}{2} - \frac{4}{x^2}$$
$$f''(x) = \frac{1}{2} + 8x^{-3} = \frac{1}{2} + \frac{8}{x^3}$$

Critical values:

$$\frac{x}{2} - \frac{4}{x^2} = 0$$
$$x^3 - 8 = 0$$
$$x^3 = 8$$
$$x = 2$$
$$f(2) = \frac{4}{4} + \frac{4}{2} = 1 + 2 = 3$$

Note that $f'(x)$ is undefined for $x = 0$.

Applying the Second Derivative Test:

$f''(2) = \frac{1}{2} + \frac{8}{x^3} = \frac{1}{2} + \frac{8}{8} > 0, f''(2)$, is undefined for $x = 0$.

There is a relative minimum at $x = 2$. The relative minimum of the graph is at $(2,3)$.

2. a.

$$f'(x) = 2x + a$$
$$2(1) + a = 0$$
$$2 + a = 0$$
$$a = -2$$
$$f(1) = 1 - 2(1) + b$$
$$3 = 1 - 2 + b$$
$$3 + 1 = b$$
$$4 = b$$

$$f(x) = x^2 - 2x + 4$$
$$f'(x) = -2x - 2$$
$$f''(x) = -2$$

Then $f(1) = 1 - 2 + 4 = 3$.

b. Applying the Second Derivative Test:

$$f''(1) = -2 < 0$$

The point $(1,3)$ is an absolute maximum of f.

3.

$$\begin{aligned} f(x) &= x^3 + x^2 \\ f'(x) &= 3x^2 + 2x \\ f''(x) &= 6x + 2 \end{aligned}$$

Find the critical values by solving $f'(x) = 0$.

$$\begin{aligned} f'(x) = 3x^2 + 2x &= 0 \\ x(3x+2) &= 0 \end{aligned}$$

$$\begin{aligned} x = 0 \text{ or } 3x + 2 &= 0 \\ 3x &= -2 \\ x &= -\frac{2}{3} \end{aligned}$$

Find where $f''(x) = 0$.

$$\begin{aligned} f''(x) &= 0 \\ 6x + 2 &= 0 \\ 6x &= -2 \\ x &= -\frac{1}{3} \end{aligned}$$

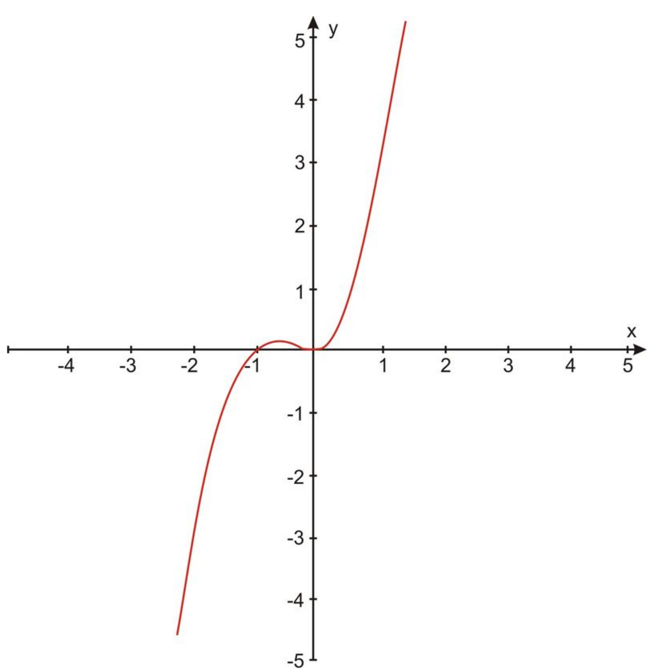
Find the function values for these special points.

$$\begin{aligned} f(x) &= x^3 + x^2 \\ f\left(-\frac{2}{3}\right) &= \left(-\frac{2}{3}\right)^3 + \left(-\frac{2}{3}\right)^2 = 0.15 \\ f\left(-\frac{1}{3}\right) &= \left(-\frac{1}{3}\right)^3 + \left(-\frac{1}{3}\right)^2 = 0.07 \\ f(0) &= 0 \end{aligned}$$

Divide the number line into the intervals using the values from $f'(x) = 0$ and $f''(x) = 0$ and make a table. Use a test point from each interval to check the signs of the first and second derivatives.

Interval	$\left(-\infty,-\frac{2}{3}\right)$	$\left(-\frac{2}{3},-\frac{1}{3}\right)$	$\left(-\frac{1}{3},0\right)$	$(0,+\infty)$
Text point $x=c$	$c=-1$	$c=-\frac{1}{2}$	$c=-\frac{1}{6}$	$c=1$
$f'(x)=3x^2+2x$	$3(-1)^2+2(-1)$	$3\left(-\frac{1}{2}\right)^2+2\left(-\frac{1}{2}\right)$	$3\left(-\frac{1}{6}\right)^2+2\left(-\frac{1}{6}\right)$	$3(1)^2+2(1)$
$f'(c)$	$=3-2>0$	$=\frac{3}{4}-1<0$	$=\frac{3}{36}-\frac{2}{6}<0$	$=3+2>0$
sign of $f'(x)$	$f'(x)>0$	$f'(x)<0$	$f'(x)<0$	$f'(x)>0$
$f''(x)=6x+2$	$6(-1)+2$	$6\left(\frac{-1}{2}\right)+2$	$6\left(\frac{-1}{6}\right)+2$	$6(1)+2$
$f''(c)$	$=-6+2<0$	$=-3+2<0$	$=-1+2>0$	$=6+2>0$
Sign of $f''(x)$	$f''(x)<0$	$f''(x)<0$	$f''(x)>0$	$f''(x)>0$
Shape of Graph	Increasing, Concave down	Decreasing, Concave down	Decreasing, Concave up	Increasing concave up

There is a relative maximum at $x=-\frac{2}{3}$ located at $\left(-\frac{2}{3},0.15\right)$. There is a relative minimum at $x=0$ located at $(0,0)$. There is a point of inflection at $\left(-\frac{1}{3},0.07\right)$.



4.

$$
\begin{aligned}
f(x) &= \frac{x^2+3}{x} \\
f'(x) &= \frac{x(2x)-\left(x^2+3\right)1}{x^2} \\
&= \frac{2x^2-x^2-3}{x^2} \\
&= \frac{x^2-3}{x^2} \\
f''(x) &= \frac{x^2(2x)-\left(x^2-3\right)(2x)}{x^4} \\
&= \frac{2x^3-2x^3+6x}{x^4} \\
&= \frac{6x}{x^4} \\
&= \frac{6}{x^3}
\end{aligned}
$$

$$
\begin{aligned}
f'(x) = \frac{x^2-3}{x^2} &= 0 \\
x^2-3 &= 0 \\
x^2 &= 3 \\
x &= \pm\sqrt{3}
\end{aligned}
$$

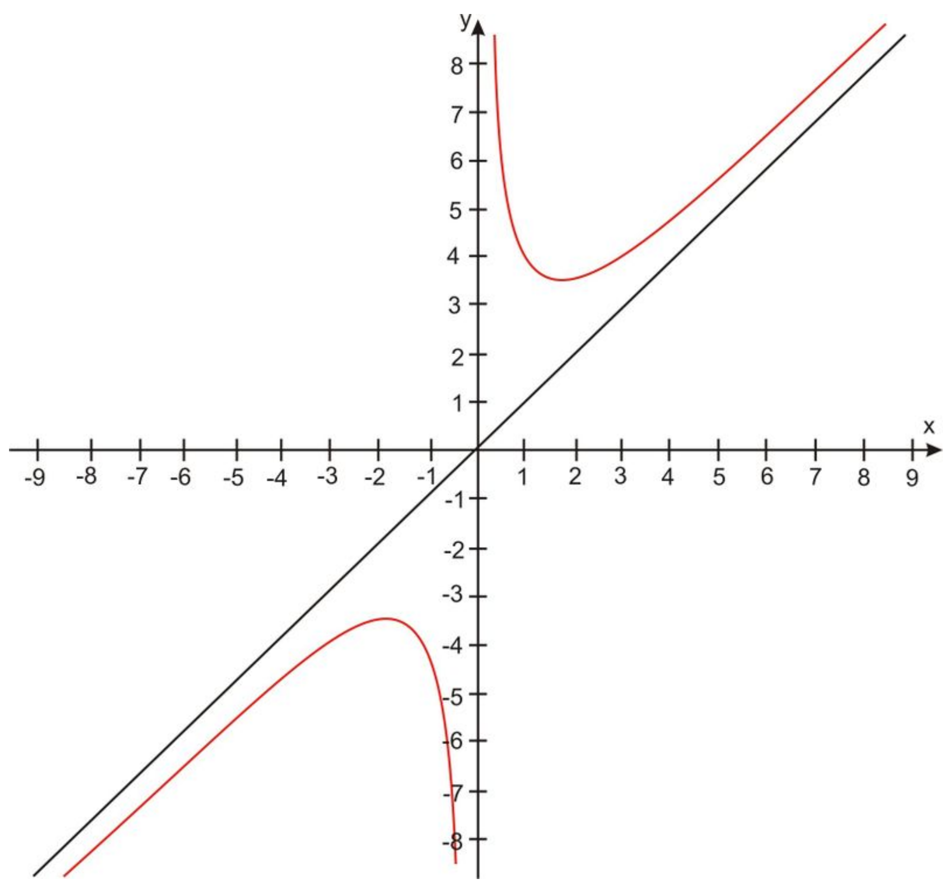
$f'(x)$ is undefined at $x = 0$.
$f''(x) = \frac{6}{x^3}$ is undefined for $x = 0$.
Find the function values:

$$
\begin{aligned}
f(x) &= \frac{x^2+3}{x} \\
f\left(-\sqrt{3}\right) &= -\frac{6}{\sqrt{3}} = -\frac{6\sqrt{3}}{3} = -2\sqrt{3} \\
f\left(\sqrt{3}\right) &= \frac{6}{\sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}
\end{aligned}
$$

Make the table of intervals.

Interval	$\left(-\infty,-\sqrt{3}\right)$	$\left(-\sqrt{3},0\right)$	$\left(0,\sqrt{3}\right)$	$\left(\sqrt{3},\infty\right)$
Text point $x=c$	$c=-2$	$c=-1$	$c=1$	$c=2$
$f'(x)=\frac{x^2-3}{x^2}$	$\frac{x^2-3}{x^2}$	$\frac{(-1)^2-3}{(-1)^2}$	$\frac{1^2-3}{1^2}$	$\frac{(2)^2-3}{(2)^2}$
$f'(c)$	$=\frac{(-2)^2-3}{(-2)^2}>0$	$=\frac{1-3}{1}<0$	$=-2<0$	$=\frac{1}{4}>0$
sign of $f'(x)$	$f'(x)>0$	$f'(x)<0$	$f'(x)<0$	$f'(x)>0$
$f''(x)=\frac{6}{x^3}$	$\frac{6}{(-2)^3}$	$\frac{6}{(-1)^3}$	$\frac{6}{(1)^3}$	$\frac{6}{(2)^3}$
$f''(c)$	$=\frac{6}{-8}<0$	$=\frac{6}{-1}<0$	$=\frac{6}{1}>0$	$=\frac{6}{8}>0$
Sign of $f''(x)$	$f''(x)<0$	$f''(x)<0$	$f''(x)>0$	$f''(x)>0$
Shape of Graph	Increasing, Concave down	Decreasing, Concave down	Decreasing, Concave up	Increasing concave up

The function has a relative maximum at $x=-\sqrt{3}$ located at $\left(-\sqrt{3},-2\sqrt{3}\right)$. There is a relative minimum at $x=\sqrt{3}$ located at $\left(\sqrt{3},2\sqrt{3}\right)$. There are no inflection points.



5.

$$f(x) = x^3 - 12x$$
$$f'(x) = 3x^2 - 12x$$
$$f''(x) = 6x$$

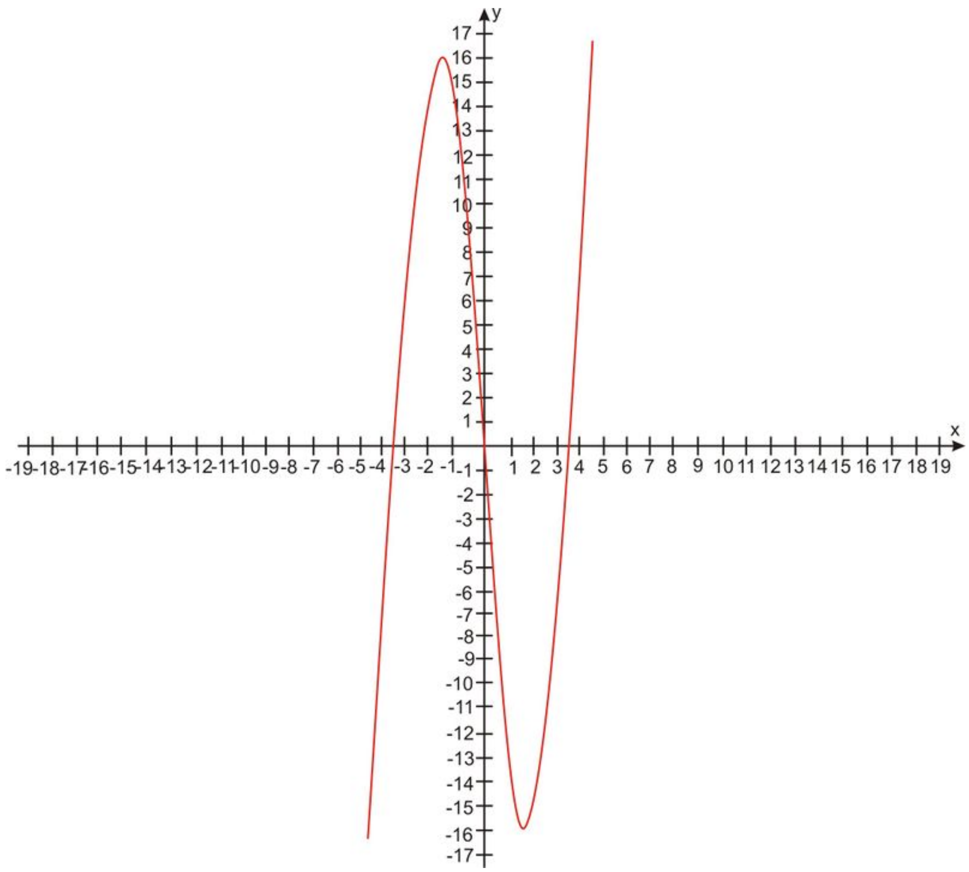
$$f'(x) = 3x^2 - 12 = 0$$
$$3\left(x^2-4\right) = 0$$
$$3(x-2)(x+2) = 0$$
$$(x-2) = 0 \text{ or } x+2 = 0$$
$$x = 2 \text{ or } x = -2$$

$$f''(x) = 0$$
$$6x = 0$$
$$x = 0$$

$$f(2) = 8 - 24 = -16$$
$$f(-2) = -8 + 24 = 16$$
$$f(0) = 0$$

Interval	$(-\infty,-2)$	$(-2,0)$	$(0,2)$	$(2,\infty)$
Text point $x=c$	$c=-3$	$c=-1$	$c=1$	$c=3$
$f'(x) = 3x^2-12$	$3x^2-12$	$3x^2-12$	$3x^2-12$	$3x^2-12$
$f'(c)$	$=3(-3)^2-12>0$	$=3(-1)^2-12<0$	$=3(1)^2-12<0$	$=3(3)^2-12>0$
sign of $f'(x)$	$f'(x)>0$	$f'(x)<0$	$f'(x)<0$	$f'(x)>0$
$f''(x)=6x$	$6x$	$6x$	$6x$	$6x$
$f''(c)$	$=6(-3)<0$	$=6(-1)<0$	$=6(1)>0$	$=6(3)<0$
Sign of $f''(x)$	$f''(x)<0$	$f''(x)<0$	$f''(x)>0$	$f''(x)>0$
Shape of Graph	Increasing, Concave down	Decreasing, Concave down	Decreasing, Concave up	Increasing concave up

The function has a relative maximum at $x=-2$ located at $(-2,16)$. The relative minimum is located at $(2,16)$. There is a point of inflection at $(0,0)$.



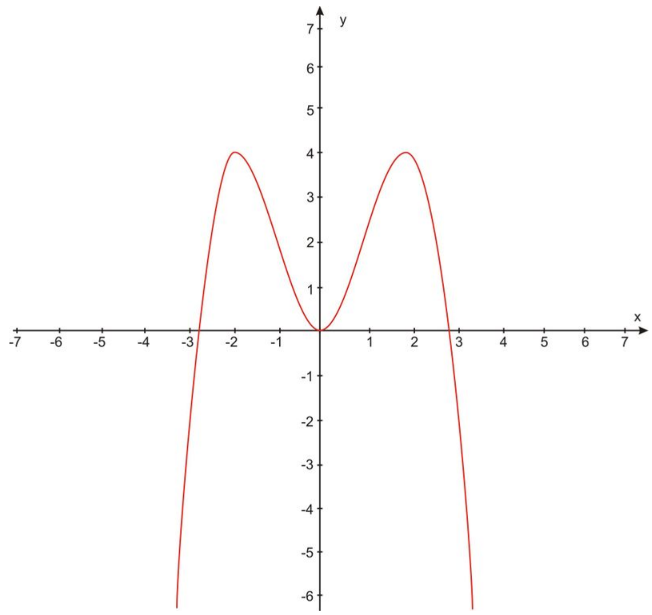
6.

$$
\begin{aligned}
f(x) &= -\frac{1}{4}x^4 + 2x^2 \\
f'(x) &= -x^3 + 4x \\
f''(x) &= -3x^2 + 4 \\
f'(x) &= -x^3 + 4x = 0 \\
-x\left(x^2 - 4\right) &= 0 \\
-x(x-2)(x+2) &= 0 \\
x = 0 \text{ or } x - 2 &= 0 \text{ or } x + 2 = 0 \\
x = 0 \text{ or } x &= 2 \text{ or } x = -2 \\
f''(x) &= 0 \\
-3x^2 + 4 &= 0 \\
x^2 &= \frac{-4}{-3} = \frac{4}{3} \\
x &= \pm\frac{2}{\sqrt{3}} = \pm\frac{2\sqrt{3}}{3}
\end{aligned}
$$

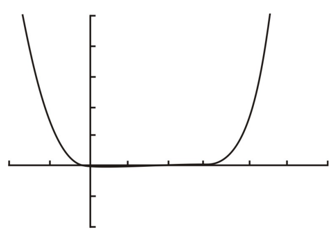
$$
\begin{aligned}
f(2) &= 4\\
f(-2) &= 4\\
f(0) &= 0\\
f\left(-\frac{2\sqrt{3}}{3}\right) &= -\frac{1}{4}\left(-\frac{2\sqrt{3}}{3}\right)^4 + 2\left(-\frac{2\sqrt{3}}{3}\right)^2\\
&= -\frac{1}{4}\left(\frac{16\times 9}{81}\right) + 2\left(\frac{12}{9}\right)\\
&= -\frac{4}{9} + \frac{24}{9} = \frac{20}{9}\\
f\left(\frac{2\sqrt{3}}{3}\right) &= -\frac{1}{4}\left(\frac{2\sqrt{3}}{3}\right)^4 + 2\left(\frac{2\sqrt{3}}{3}\right)^2\\
&= -\frac{1}{4}\left(\frac{16\times 9}{81}\right) + 2\left(\frac{12}{9}\right)\\
&= -\frac{4}{9} + \frac{24}{9} = \frac{20}{9}
\end{aligned}
$$

Interval	$(-\infty, -2)$	$\left(-2, -\frac{2\sqrt{3}}{3}\right)$	$\left(-\frac{2\sqrt{3}}{3}, 0\right)$	$\left(0, \frac{2\sqrt{3}}{3}\right)$	$\left(\frac{2\sqrt{3}}{3}, 2\right)$	$(2, \infty)$
Text point $x = c$	$c = -3$	$c = -1.5$	$c = -1$	$c = 1$	$c = 1.5$	$c = 3$
$f'(x) = -x^3 + 4x$	$-(-3)^3 + 4(-3)$	$-(-1.5)^3 + 4(-1.5)$	$-(-1)^3 + 4(-1)$	$-(1)^3 + 4(1)$	$-(1.5)^3 + 4(1.5)$	$-(3)^3 + 4(3)$
$f'(c)$	$= 27 - 12 > 0$	$= -2.625 < 0$	$= 1 - 4 < 0$	$= -1 + 4 > 0$	$= 2.625 > 0$	$= -27 + 12 < 0$
sign of $f'(x)$	$f'(x) > 0$	$f'(x) < 0$	$f'(x) < 0$	$f'(x) > 0$	$f'(x) > 0$	$f'(x) < 0$
$f''(x) = -3x^2 + 4$	$-3(-3)^2 + 4$	$-3(-1.5)^2 + 4$	$-3(-1)^2 + 4$	$-3(1)^2 + 4$	$-3(1.5)^2 + 4$	$-3(3)^2 + 4$
$f''(c)$	$= -27 + 4 < 0$	$= -6.75 + 4 < 0$	$= -3 + 4 > 0$	$= -3 + 4 > 0$	$= -6.75 + 4 < 0$	$= -27 + 4 < 0$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) < 0$	$f''(x) > 0$	$f''(x) > 0$	$f''(x) < 0$	$f''(x) < 0$
Shape of Graph	Increasing, Concave down	Decreasing, Concave down	Decreasing, Concave up	Increasing concave up	Increasing, Concave down	Decreasing Concave down

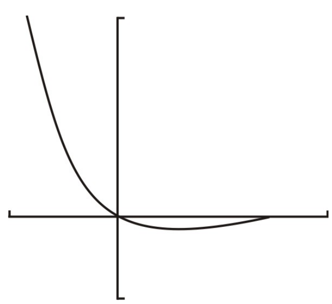
The relative maximums are at $x = 2$ and $x = -2$. They are located at $(-2, 4)$ and $(2, 4)$. The relative minimum is located at $(0, 0)$. There are two inflection points at $\left(-\frac{2\sqrt{3}}{3}, \frac{20}{9}\right)$ and $\left(\frac{2\sqrt{3}}{3}, \frac{20}{9}\right)$.



7.a. General Graph

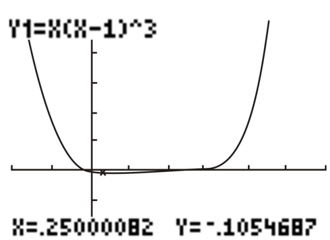


Close up of the interval $\left(0, \frac{1}{2}\right)$



You may need to zoom in even more on the graph. The graph is concave up in the interval.

b.



The relative minimum is at $(0.25, -0.10)$.

8. False

Find the first and second derivatives.

$$f(x) = x^4 + 4x^3$$
$$f'(x) = 4x^3 + 12x^2$$
$$f''(x) = 12x^2 + 24x$$

The critical values are:

$$4x^3 + 12x^2 = 0$$
$$4x^2(x+3) = 0$$
$$x = 0 \text{ or } x = -3$$

The possible inflection points are:

$$12x^2 + 24x = 0$$
$$12x(x+2) = 0$$
$$x = 0 \text{ or } x = -2$$

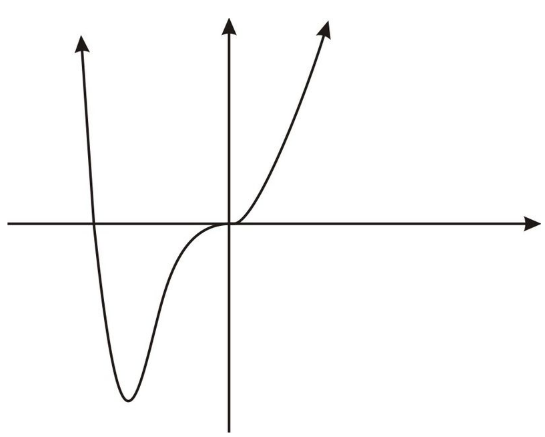
Check the possible inflection points.

Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, \infty)$
sign of $f''(x)$	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Shape of Graph	Concave up	Concave down	Concave up

Both $x = 0$ and $x = -2$ are inflection points.

$x = -3$ is a relative minimum because $f''(-3) > 0$.

9. One example is $f(x) = x^2 + \frac{1}{x-1}$. It has exactly relative minimum. Look at the graph.



10. One example is $f(x) = \sqrt{x}$ on the interval $(0, \infty)$.