3.3 The First Derivative Test

1. The function is increasing on $(0,3)$. The function is decreasing on $(3,6)$. The function is constant on $(6,+\infty)$.

2. The function is increasing on $(-\infty,0)$ and $(3,7)$. The function is decreasing on $(0,3)$.

3. a. Draw tangent lines to the graph to help you solve these problems.

$f'(-3)$ is positive. $f'(-3) > 0$.

b. $f'(1)$ is negative. $f'(1) < 0$.

c. $f'(3) = 0$

d. $f'(4)$ is positive. $f'(4) > 0$.

4.

$$f(x) = x^2 - \frac{1}{x} = 2x - x^{-1}$$
$$f'(x) = 2x + x^{-2} = 2x + \frac{1}{x^2}$$

Find the critical values.

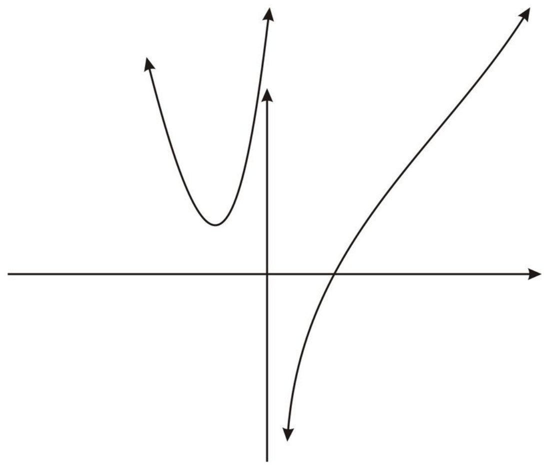
$$2x + \frac{1}{x^2} = 0$$
$$\frac{2x^3+1}{x^2} = 0$$
$$2x^3+1 = 0$$
$$2x^3 = -1$$
$$x^3 = -\frac{1}{2}$$
$$x = \sqrt[3]{-0.5} = -\sqrt[3]{0.5} = -0.79$$
$$f\left(-\sqrt[3]{0.5}\right) = 1.89$$

$f'(x)$ is undefined for $x = 0$.

Set up the intervals and make a table. Find test points to substitute into the derivative and check the sign of the derivative.

Interval	$\left(-\infty, -\sqrt[3]{0.5}\right)$	$\left(-\sqrt[3]{0.5}, 0\right)$	$(0,+\infty)$
Test point $x = c$	$c = -1$	$c = -0.1$	$c = 1$
$f'(c)$	$-2+1 = -2$	$-0.2 + \frac{1}{(-0.1)^2} = 99.8$	$2+1 = 3$
sign of $f'x$	$f'(x) < 0$	$f'(x) > 0$	$f'(x) > 0$

By the First Derivative Test, f' changes from negative to positive at $x = -\sqrt[3]{0.5}$. The critical value of $x = -\sqrt[3]{0.5}$ is a local minimum. The function decreases on $\left(-\infty, -\sqrt[3]{0.5}\right)$ and on $(0, +\infty)$. The function increases on $\left(-\sqrt[3]{0.5}, 0\right)$.



5.

$$f(x) = \left(x^2 - 1\right)^5$$
$$f'(x) = 5\left(x^2 - 1\right)^4 (2x)$$

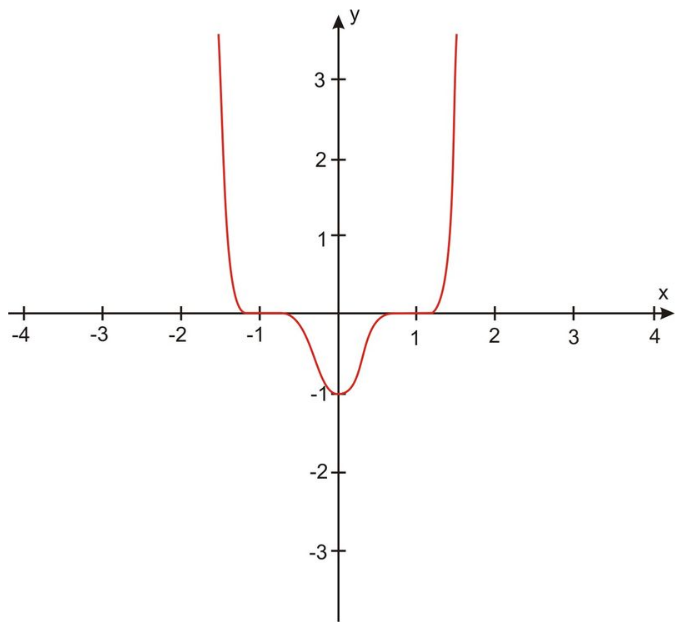
Find the critical values.

$$5\left(x^2 - 1\right)^4 (2x) = 0$$
$$\left(x^2 - 1\right)^4 = 0 \text{ or } 2x = 0$$
$$x^2 - 1 = 0 \text{ or } x = 0$$
$$x = 1 \text{ or } x = -1 \text{ or } x = 0$$
$$f(1) = 0$$
$$f(-1) = 0$$
$$f(0) = -1$$

Interval	$(-\infty, -1)$	$(-1, 0)$
Test point $x = c$	$c = -2$	$c = -0.5$
$f'(x) = 5\left(x^2 - 1\right)^4 (2x)$	$5\left((-2)^2 - 1\right)^4 (2 \times (-2))$	$5\left((-0.5)^2 - 1\right)^4 (2 \times (-0.5))$
$f'(c)$	$= 5(3)^4(-4)$	$= 5(-0.25 - 1)^4(-1)$
Sign of $f'(x)$	$f'(x) < 0$	$f'(x) < 0$

Interval	$(0,1)$	$(1,+\infty)$
Test point $x=c$	$c=0.5$	$c=2$
$f'(x)=5\left(x^2-1\right)^4(2x)$	$5\left((0.5)^2-1\right)^4(2\times(0.5))$	$5\left(2^2-1\right)^4(2\times 2)$
$f'(c)$	$=5(0.25-1)^4(1)$	$=5(4-1)^4(4)$
Sign of $f'(x)$	$f'(x)>0$	

By the First Derivative Test, there is an absolute minimum at $x=0$. The function is decreasing on $(-\infty,-1)$ and on $(-1,0)$. The function is increasing on $(0,1)$ and on $(1,+\infty)$.



6.

$$f(x)=\left(x^2-1\right)^4$$
$$f'(x)=4\left(x^2-1\right)^3(2x)$$

Find the critical values.

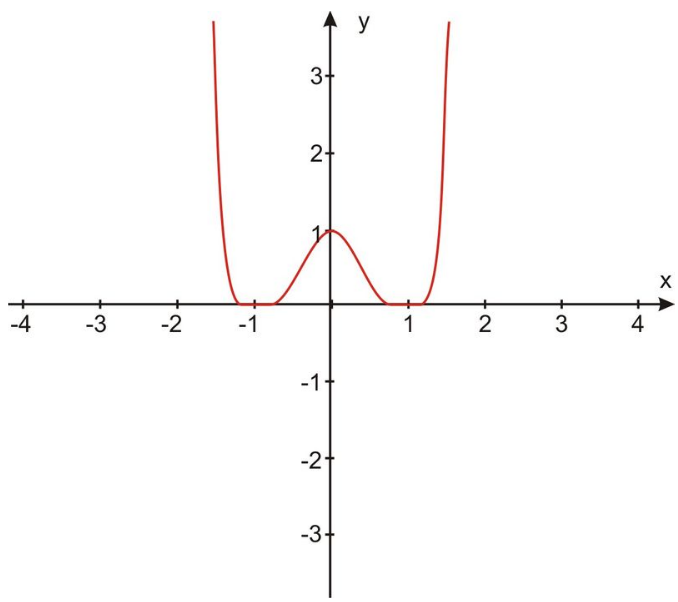
$$4\left(x^2-1\right)^3(2x)=0$$
$$\left(x^2-1\right)^3=0 \text{ or } 2x=0$$
$$x^2-1=0 \text{ or } x=0$$
$$x=1 \text{ or } x=-1 \text{ or } x=0$$

$$f(1)=0$$
$$f(-1)=0$$
$$f(0)=1$$

Interval	$(-\infty,-1)$	$(-1,0)$
Test point $x=c$	$c=-2$	$c=-0.5$
$f'(x)=4\left(x^2-1\right)^3(2x)$	$4\left((-2)^2-1\right)^3(2\times(-2))$	$4\left((-0.5)^2-1\right)^3(2\times(-0.5))$
$f'(c)$	$=4(3)^3(-4)$	$=4(-1.25)^3(-1)$
Sign of $f'(x)$	$f'(x)<0$	$f'(x)<0$

Interval	$(0,1)$	$(1,+\infty)$
Test point $x=c$	$c=0.5$	$c=2$
$f'(x)=4\left(x^2-1\right)^3(2x)$	$4\left((0.5)^2-1\right)^3(2\times(0.5))$	$4\left(2^2-1\right)^3(2\times 2)$
$f'(c)$	$=4(-1.25)^3(1)$	$=5(4-1)^3(4)$
Sign of $f'(x)$	$f'(x)<0$	$f'(x)>0$

By the First Derivative Test, there are absolute minimum at $x=-1$ and $x=1$. The function is decreasing on $(-\infty,-1)$ and on $(0,1)$. The function is increasing on $(-1,0)$ and on $(1,+\infty)$.



7. a.

$$f(x)=-x^2-4x-1$$
$$f'(x)=-2x-4$$

Find the critical values.

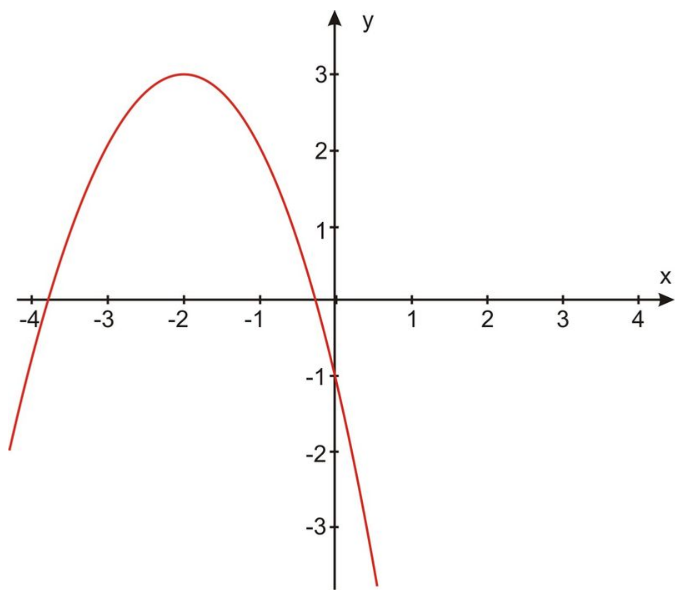
$$-2x-4=0$$
$$-2x=4$$
$$x=-2$$
$$f(-2)=-4-4(-2)-1=-4+8-1=3$$

Interval	$(-\infty,-2)$	$(-2,+\infty)$
Test point $x=c$	$c=-3$	$c=0$
$f'(x)=-2x-4$	$-2(-3)-4$	$-2(0)-4$
$f'(c)$	$=6-4=2$	$=0-4=-4$
Sign of $f'(x)$	$f'(x)>0$	$f'(x)<0$

The function is increasing on $(-\infty,-2)$. The function is decreasing on $(-2,+\infty)$.

b. By the First Derivative Test, there are absolute maximum at $x=-2$ with $f(-2)=3$.

c.



8. a.

$$f(x)=x^3+3x^2-9x+1$$
$$f'(x)=3x^2+3x-9$$

Find the critical values.

$$3x^2+3x-9=0$$
$$3\left(x^2+x-3\right)=0$$
$$x^2+x-3=0$$
$$(x+3)(x-1)=0$$

$$x+3=0 \text{ or } x-1=0$$
$$x=-3 \text{ or } x=1$$

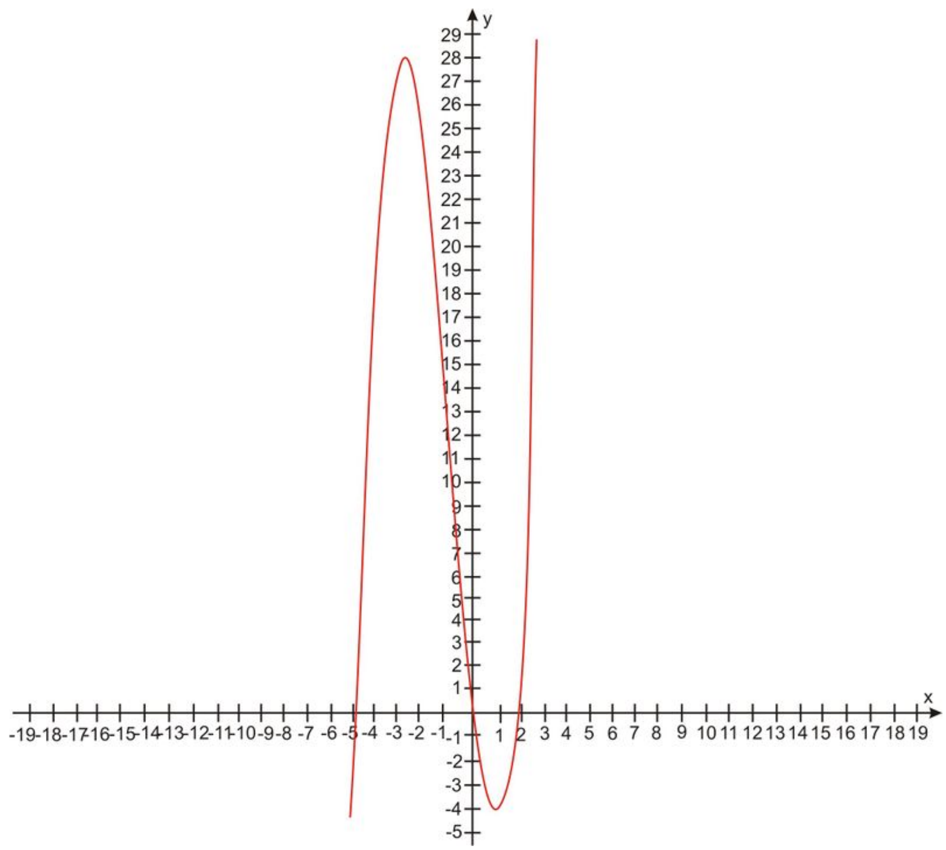
$$f(-3)=3^3+3(3)^2-9(3)+1=28$$
$$f(1)=1+3-9+1=-4$$

Interval	$(-\infty, -3)$	$(-3, 1)$	$(1, +\infty)$
Test point $x = c$	$c = -4$	$c = 0$	$c = 3$
$f'(x) = 3x^2 + 3x - 9$	$3(-4)^2 + 3(4) - 9$	$3(0)^2 + 3(0) - 9$	$3(3)^2 + 3(3) - 9$
$f'(c)$	$= 48 + 12 - 9 = 51$	$= -9$	$= 27 + 9 - 9 = 27$
Sign of $f'(x)$	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$

The function is increasing on $(-\infty, -3)$ and on $(3, +\infty)$. The function is decreasing on $(-3, 1)$.

b. There is a relative maximum at $x = -3$ with $f(-3) = 28$. There is a relative minimum at $x = 1$ with $f(1) = -4$.

c.



9.

$$f(x) = x^{\frac{2}{3}}(x-5) = x^{\frac{5}{3}} - 5x^{\frac{2}{3}}$$
$$f'(x) = \frac{5}{3}x^{\frac{2}{3}} - \frac{10}{3}x^{-\frac{1}{3}}$$

Find the critical values.

$$\frac{5}{3}x^{\frac{2}{3}} - \frac{10}{3}x^{-\frac{1}{3}} = 0$$
$$x^{\frac{1}{3}}\left(\frac{5}{3}x^{\frac{2}{3}} - \frac{10}{3}x^{-\frac{1}{3}}\right) = 0$$
$$\frac{5}{3}x - \frac{10}{3} = 0$$
$$\frac{5}{3}x = \frac{10}{3}$$
$$x = 2$$

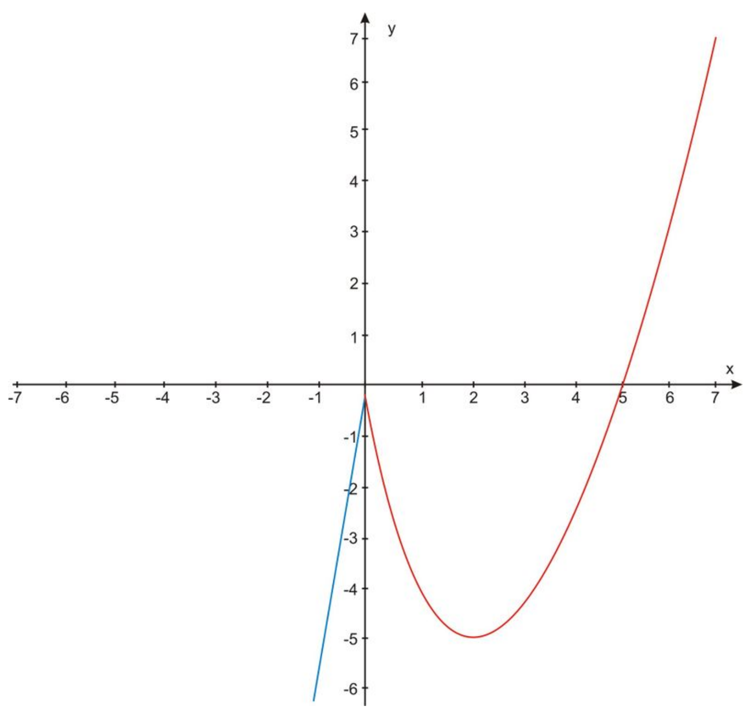
$f'(x) = \frac{5}{3}x^{\frac{2}{3}} - \frac{10}{3}x^{-\frac{1}{3}}$ is undefined for $x = 0$.

Interval	$(-\infty, 0)$	$(0, 2)$	$(2, +\infty)$
Test point$x = c$	$c = -1$	$c = 1$	$c = 3$
$f'(x) = \sqrt[3]{x^2} + (x-5)\left(\frac{2}{3\sqrt[3]{x}}\right)$	$\sqrt[3]{(-1)^2} + (-1-5)\left(\frac{2}{3\sqrt[3]{-1}}\right)$	$\sqrt[3]{(1)^2} + (1-5)\left(\frac{2}{3\sqrt[3]{1}}\right)$	$\sqrt[3]{(3)^2} + (3-5)\left(\frac{2}{3\sqrt[3]{3}}\right)$
$f'(c)$	$= 1 + 6\left(\frac{2}{3}\right)$	$= 1 - 4\left(\frac{2}{3}\right)$	$= \sqrt[3]{(3)^2} - 2\left(\frac{2}{3\sqrt[3]{3}}\right) = 5.3$
sign of $f'(x)$	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$

The function is increasing on $(-\infty, 0)$ and on $(2, +\infty)$. The function is decreasing on $(0, 2)$.

b. There is a relative maximum at $x = 0$ with $f(0) = 0$. There is a relative minimum at $x = 2$ with $f(2) = 2^{\frac{2}{3}}(2-5) = -3\left(2^{\frac{2}{3}}\right)$.

c.



10.

$$f(x) = 2x\sqrt{x^2+1} = 2x\left(x^2+1\right)^{\frac{1}{2}}$$
$$f'(x) = 2x\left[\frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}}(2x)\right] + \left(x^2+1\right)^{\frac{1}{2}}(2)$$
$$= \frac{x^2}{2\sqrt{x^2+1}} + 2\sqrt{x^2+1}$$

Find the critical values.

There are no solutions for $f'(x) = 0$ and $f'(x)$ is defined for all x. That means the function is increasing or decreasing everywhere. We can check for increasing or decreasing on the entire interval $(-\infty, \infty)$.

Interval	$(-\infty, \infty)$
Test point $x = c$	$c = 0$
$f'(x) = \frac{x^2}{2\sqrt{x^2+1}} + 2\sqrt{x^2+1}$	$\frac{0^2}{2\sqrt{0^2+1}} + 2\sqrt{0^2+1}$
$f'(c)$	$= 0 + 2$
Sign of $f'(x)$	$f'(x) > 0$

The function is increasing on $(-\infty, \infty)$.

b. There are no maximums or minimums.

c.