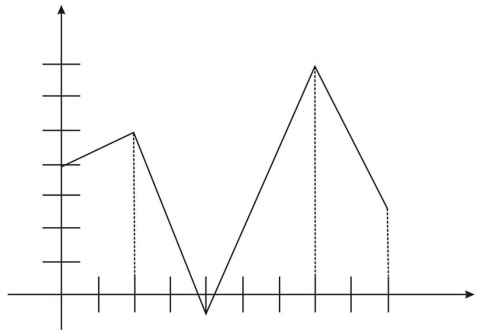
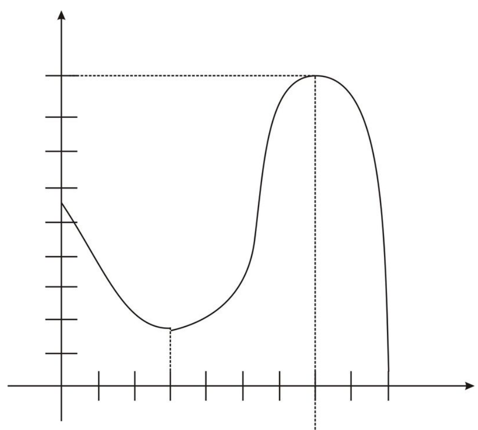
3.2 Extrema and the Mean Value Theorem

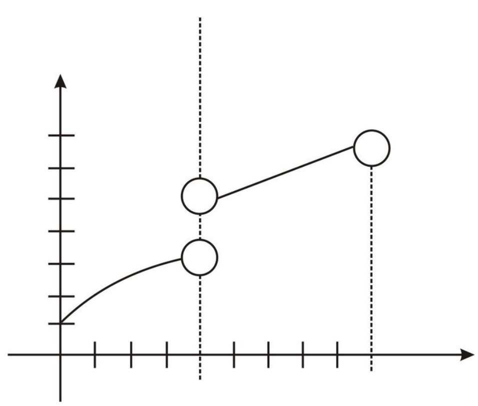
1. The absolute maximum is at $x = 7$. The absolute minimum is at $x = 4$. There is a relative maximum at $x = 2$. The extreme values of f are $f(7) = 7$ and $f(4) = -1$.



2. The absolute maximum is at $x = 7$. The absolute minimum is at $x = 9$. There is a relative minimum at $x = 3$. The extreme values of f are $f(7) = 9$ and $f(9) = 0$.



3. The absolute minimum is at $x = 0$. There is no maximum because the function is not continuous on the closed interval. The extreme value of f is $f(0) = 1$.



4.

$$f(x) = -x^2 - 6x + 4$$
$$f'(x) = -2x - 6$$

Find the critical values of f.

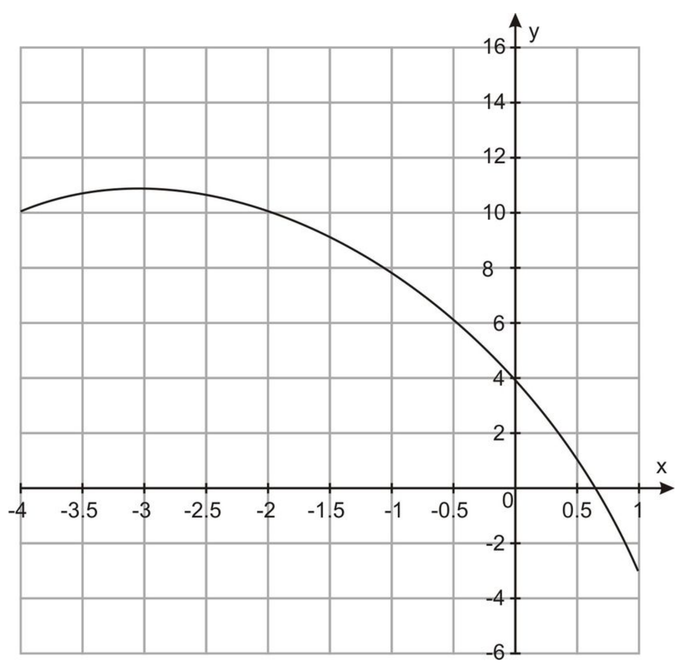
Solve $f'(x) = 0$.

$$-2x - 6 = 0$$
$$-2x = 6$$
$$x = -3$$
$$f(-3) = -(-3)^2 - 6(-3) + 4 = -9 + 18 + 4 = 13$$

Check the endpoints:

$$x = -4$$
$$f(-4) = -4^2 - 6(-4) + 4 = 12$$
$$x = 1$$
$$f(1) = -1^2 - 6(1) + 4 = -3$$

Compare function values to find the maximum and minimum. The absolute maximum is at $x = -3$ because $f(-3)$ is the greatest value. The absolute minimum is at $x = 1$ because $f(1)$ is the smallest value. The extrema are $f(-13) = 13$ and $f(1) = -3$.



5.

$$f(x) = x^3 - x^4$$
$$f'(x) = 3x^2 - 4x^3$$

Find the critical values of f.

Solve $f'(x) = 0$.

$$3x^2 - 4x^3 = 0$$
$$x^2(3-4x) = 0$$

$$x^2 = 0 \quad \text{or} \quad 3-4x = 0$$
$$x = 0 \quad \text{or} \quad -4x = -3$$
$$x = 0 \quad \text{or} \quad x = \frac{3}{4}$$

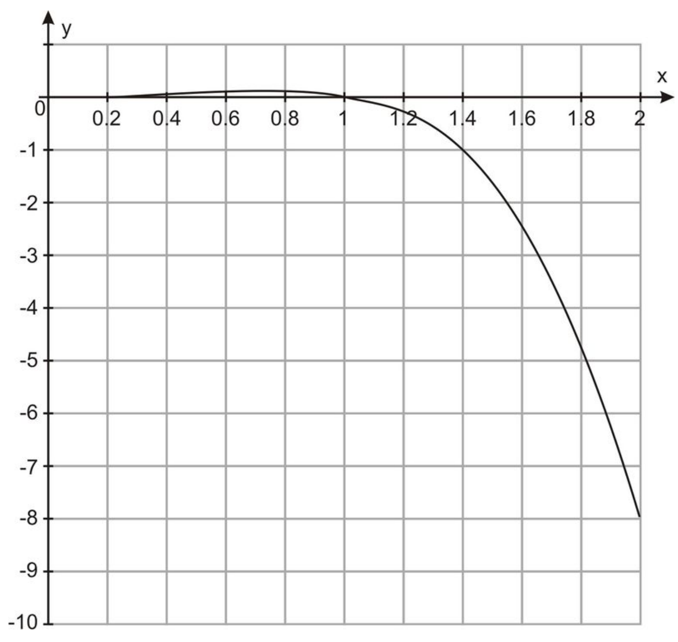
Find the function values: $f(0) = 0$ and $f\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)^3 - \left(\frac{3}{4}\right)^4 = \frac{27}{64} - \frac{81}{256} = \frac{108}{256} - \frac{81}{256} = \frac{27}{256} \approx 0.1055$.

Find function values of the endpoints:

0 is one endpoint and was already checked.

$$f(2) = 8 - 16 = -8$$

The absolute maximum at $x = \frac{3}{4}$. The absolute minimum at $x = 2$. The extrema are $f\left(\frac{3}{4}\right) \approx 0.1055$ and $f(2) = -8$.



6.

$$f(x) = -x^2 + \frac{4}{x^2} = -x^2 + 4x^{-2}$$
$$f'(x) = -2x - 8x^{-3} = -2x - \frac{8}{x^3}$$

Find the critical values of f.

Solve $f'(x) = 0$.

$$\begin{aligned}
-2x - \frac{8}{x^3} &= 0 \\
-2x^4 - 8 &= 0 \\
x^4 - 4 &= 0 \\
\left(x^2 - \sqrt{4}\right)\left(x^2 + \sqrt{4}\right) &= 0 \\
\left(x^2 - 2\right)\left(x^2 + 2\right) &= 0
\end{aligned}$$

$$\begin{aligned}
x^2 - 2 &= 0 \\
x - \sqrt{2} &= 0 \qquad \text{or} \quad x + \sqrt{2} = 0 \\
x &= \sqrt{2} \quad \text{or} \quad x = -\sqrt{2} \\
x^2 + 2 &= 0 \\
x^2 &= -2
\end{aligned}$$

There are no real number solutions for $x^2 + 2 = 0$.

Since the variable is in the denominator of one term of $f'(x)$, set that denominator equal to 0.

$$\begin{aligned}
x^3 &= 0 \\
x &= 0
\end{aligned}$$

$f'(x)$ is undefined for $x = 0$. This is another critical value.

Find the function values of the critical values in the interval $[-2, 0]$ and of the endpoints.

$$f\left(-\sqrt{2}\right) = -2 + \frac{4}{2} = -2 + 2 = 0$$

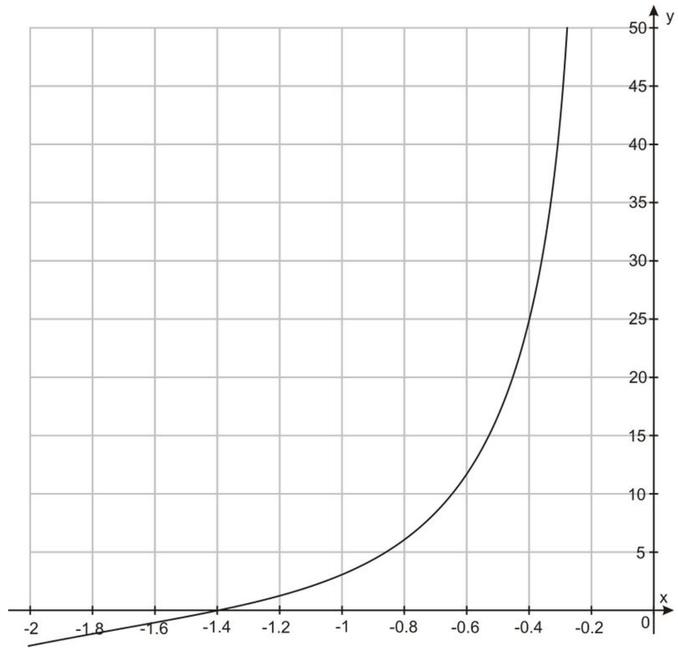
$f(0)$ is undefined.

Find function values of the endpoints:

0 is one endpoint and $f(0)$ was already found to be undefined.

$$f(-2) = -4 + \frac{4}{4} = -3.$$

The absolute minimum is at $x = -\sqrt{2}$. The extrema is $f\left(-\sqrt{2}\right) = 0$. There is no absolute maximum as $f(x)$ approaches ∞ as x approaches 0.



7. $f(x) = 3x^2 - 12x$ is continuous and differentiable because it is a polynomial.

Solve $3x^3 - 12x = 0$:

$$\begin{aligned} 3x^3 - 12x &= 0 \\ 3x\left(x^2 - 4\right) &= 0 \\ 3x = 0 \text{ or } x^2 - 4 &= 0 \\ x = 0 \text{ or } x = 2 \text{ or } x &= -2 \end{aligned}$$

On $[-2,0], f(-2) = f(0) = 0$. On $[0,2], f(0) = f(2) = 0$

$$f'(x) = 9x^2 - 12$$

Set $f'(x) = 0$ and solve for the critical values.

$$\begin{aligned} 9x^2 - 12 &= 0 \\ 3\left(3x^2 - 4\right) &= 0 \\ 3x^2 - 4 &= 0 \end{aligned}$$

$$x = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \quad \text{or} \quad x = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

By Rolle's Theorem, there is at least one critical value in $(-2,0)$. That value is $c = -\frac{2\sqrt{3}}{3}$. There is at least one critical value in $(0,2)$. That value is $c = \frac{2\sqrt{3}}{3}$.

8.

$$f(x) = x^2 - \frac{2}{x-1} = x^2 - 2(x-1)^{-1}$$
$$f'(x) = 2x + 2(x-1)^{-2} = 2x - \frac{2}{(x-1)^2}$$

On the interval $[-1,0], f(-1) = 1 - \frac{2}{-2} = 2$ and $f(0) = 2$. By Rolle's Theorem, there is a critical value in $(-1,0)$. Solve

$$\begin{aligned} f'(x) &= 0 \\ 2x - \frac{2}{(x-1)^2} &= 0 \\ 2x(x-1)^2 - 2 &= 0 \\ 2x\left(x^2 - 2x + 1\right) - 2 &= 0 \\ x\left(x^2 - 2x + 1\right) - 1 &= 0 \\ x^3 - 2x^2 + x - 1 &= 0 \end{aligned}$$

[I cannot continue to solve the problem algebraically as written.]

9. $f(x)$ is continuous on $[1,2]$.

$f'(x) = \frac{x(1)-(x+2)1}{x^2} = \frac{x-(x+2)}{x^2} = \frac{x-x-2}{x^2} = -\frac{2}{x^2}$ has the interval $(1,2)$ in its domain.

There is a number c such that $f(2) - f(1) = (2-1)f'(c)$.

$$\begin{aligned} f(2) &= \frac{4}{2} = 2 \\ f(1) &= \frac{3}{1} = 3 \\ f(2) - f(1) &= (2-1)f'(c) \\ 2 - 3 &= 1f'(c) \\ -1 &= f'(c) \end{aligned}$$

Solve for c.

$$\begin{aligned} -\frac{2}{c^2} &= -1 \\ -2 &= -c^2 \\ 2 &= c^2 \\ \pm\sqrt{2} &= c \end{aligned}$$

The value of c is $\sqrt{2}$.

10. On $[0,r], f(0) = 0$. Also, $f(r) = 0$ because r is a root of f. Note that $f'(x) = 3x^2 + 2a_1x + a_2$ is the derivative of $f(x)$. Then by Rolle's Theorem, $f'(x) = 3x^2 + 2a_1x + a^2$ has a root in the interval $(0,r)$. Thus, $f'(x) = 3x^2 + 2a_1x + a_2$ has a positive root that is less than r because there is a root in $(0,r)$.