2.6 Implicit Differentiation

1.

$$\begin{aligned}
x^2+y^2 &= 500\\
\frac{d}{dx}\left[x^2+y^2\right] &= \frac{d}{dx}[500]\\
\frac{d}{dx}\left[x^2\right]+\frac{d}{dx}\left[y^2\right] &= \frac{d}{dx}[500]\\
2x+2y\frac{dy}{dx} &= 0\\
2y\frac{dy}{dx} &= -2x\\
\frac{dy}{dx} &= \frac{-2x}{2y}\\
\frac{dy}{dx} &= -\frac{x}{y}
\end{aligned}$$

2.

$$\begin{aligned}
x^2y+3xy-2 &= 1\\
\frac{d}{dx}\left[x^2y+3xy-2\right] &= \frac{d}{dx}[1]\\
\frac{d}{dx}\left[x^2y\right]+\frac{d}{dx}[3xy]-\frac{d}{dx}[2] &= \frac{d}{dx}[1]\\
y\frac{dy}{dx}\left[x^2\right]+x^2\frac{d}{dx}[y]+y\frac{dy}{dx}[3x]+3x\frac{d}{dx}[y]-0 &= 0\\
y(2x)+x^2\frac{dy}{dx}+y(3)+3x\frac{dy}{dx} &= 0\\
2xy+x^2\frac{dy}{dx}+3y+3x\frac{dy}{dx} &= 0\\
x^2\frac{dy}{dx}+3x\frac{dy}{dx} &= -2xy-3y\\
\left(x^2+3x\right)\frac{dy}{dx} &= -2xy-3y\\
\frac{dy}{dx} &= \frac{-2xy-3y}{x^2+3x}\\
\frac{dy}{dx} &= \frac{-y(2x+3)}{x(x+3)}
\end{aligned}$$

3.

$$\begin{aligned}
\frac{1}{x}+\frac{1}{y} &= \frac{1}{2}\\
x^{-1}+y^{-1} &= \frac{1}{2}\\
\frac{d}{dx}\left[x^{-1}+y^{-1}\right] &= \frac{d}{dx}\left[\frac{1}{2}\right]\\
\frac{d}{dx}\left[x^{-1}\right]+\frac{d}{dx}\left[y^{-1}\right] &= 0\\
-x^{-2}-y^{-2}\frac{dy}{dx} &= 0\\
-y^{-2}\frac{dy}{dx} &= x^{-2}\\
\frac{dy}{dx} &= \frac{x^{-2}}{-y^{-2}}\\
\frac{dy}{dx} &= -\frac{y^2}{x^2}
\end{aligned}$$

4.

$$\begin{aligned}
\sqrt{x}-\sqrt{y} &= \sqrt{3}\\
x^{\frac{1}{2}}-y^{\frac{1}{2}} &= \sqrt{3}\\
\frac{d}{dx}\left[x^{\frac{1}{2}}-y^{\frac{1}{2}}\right] &= \frac{d}{dx}\left[\sqrt{3}\right]\\
\frac{d}{dx}\left[x^{\frac{1}{2}}\right]-\frac{d}{dx}\left[y^{\frac{1}{2}}\right] &= 0\\
\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} &= 0\\
-\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} &= -\frac{1}{2}x^{-\frac{1}{2}}\\
\frac{dy}{dx} &= \frac{-\frac{1}{2}x^{-\frac{1}{2}}}{-\frac{1}{2}y^{-\frac{1}{2}}}\\
\frac{dy}{dx} &= \frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}\\
\frac{dy}{dx} &= \frac{\sqrt{y}}{\sqrt{x}} = \sqrt{\frac{y}{x}}
\end{aligned}$$

5.

$$
\begin{aligned}
\sin\left(25xy^2\right) &= x \\
\frac{d}{dx}\left[\sin\left(25xy^2\right)\right] &= \frac{d}{dx}[x] \\
\cos\left(25xy^2\right)\left(y^2(25)+25x\left(2y\frac{dy}{dx}\right)\right) &= 1 \\
\cos\left(25xy^2\right)\left(25y^2+50xy\frac{dy}{dx}\right) &= 1 \\
25y^2\cos\left(25xy^2\right)+50xy\cos\left(25xy^2\right)\frac{dy}{dx} &= 1 \\
50xy\cos\left(25xy^2\right)\frac{dy}{dx} &= 1-25y^2\cos\left(25xy^2\right) \\
\frac{dy}{dx} &= \frac{1-25y^2\cos\left(25xy^2\right)}{50xy\cos\left(25xy^2\right)}
\end{aligned}
$$

6.

$$
\begin{aligned}
\tan^3\left(x^2-y^2\right) &= \tan\left(\frac{\pi}{4}\right) \\
\frac{d}{dx}\left[\tan^3\left(x^2-y^2\right)\right] &= \frac{dy}{dx}\left[\tan\left(\frac{\pi}{4}\right)\right] \\
3\tan^2\left(x^2-y^2\right)\sec^2\left(x^2-y^2\right)\left(2x-2y\frac{dy}{dx}\right) &= 0 \\
3\tan^2\left(x^2-y^2\right)\sec^2\left(x^2-y^2\right)(2x)-3\tan^2\left(x^2-y^2\right)\sec^2\left(x^2-y^2\right)\left(2y\frac{dy}{dx}\right) &= 0 \\
-3\tan^2\left(x^2-y^2\right)\sec^2\left(x^2-y^2\right)\left(2y\frac{dy}{dx}\right) &= -3\tan^2\left(x^2-y^2\right)\sec^2\left(x^2-y^2\right)(2x) \\
\frac{dy}{dx} &= \frac{-3\tan^2\left(x^2-y^2\right)\sec^2\left(x^2-y^2\right)(2x)}{-3\tan^2\left(x^2-y^2\right)\sec^2\left(x^2-y^2\right)(2y)} \\
\frac{dy}{dx} &= \frac{x}{y}
\end{aligned}
$$

7.

$$\begin{aligned}
x^2y - y^2x &= -1 \\
\frac{d}{dx}\left[x^2y - y^2x\right] &= \frac{d}{dx}[-1] \\
\frac{d}{dx}\left[x^2y\right] - \frac{d}{dx}\left[y^2x\right] &= 0 \\
y\frac{dy}{dx}\left[x^2\right] + x^2\frac{d}{dx}[y] - \left(x\frac{dy}{dx}\left[y^2\right] + y^2\frac{d}{dx}[x]\right) &= 0 \\
y(2x) + x^2\left(\frac{dy}{dx}\right) - x^2(2y)\left(\frac{dy}{dx}\right) - y^2(1) &= 0 \\
2xy + x^2\left(\frac{dy}{dx}\right) - 2x^2y\left(\frac{dy}{dx}\right) - y^2 &= 0 \\
x^2\left(\frac{dy}{dx}\right) - 2x^2y\left(\frac{dy}{dx}\right) &= y^2 - 2xy \\
\left(x^2 - 2x^2y\right)\frac{dy}{dx} &= y^2 - 2xy \\
\frac{dy}{dx} &= \frac{y^2 - 2xy}{x^2 - 2x^2y}
\end{aligned}$$

$$\begin{aligned}
m &= \frac{y^2 - 2xy}{x^2 - 2x^2y} \\
&= \frac{1-2}{1-2} \\
&= \frac{-1}{-1} = 1
\end{aligned}$$

8.

$$\begin{aligned}
\sin(xy) &= y \\
\frac{d}{dx}[\sin(xy)] &= \frac{d}{dx}[y] \\
\cos(xy)\left(y\frac{d}{dx}[x] - x\frac{d}{dx}[y]\right) &= \frac{dy}{dx} \\
\cos(xy)\left(y(1) - x\frac{dy}{dx}\right) &= \frac{dy}{dx} \\
y\cos(xy) - x\cos x\frac{dy}{dx} &= \frac{dy}{dx} \\
y\cos(xy) &= \frac{dy}{dx} - x\cos x\frac{dy}{dx} \\
y\cos(xy) &= \frac{dy}{dx}(1 - x\cos x) \\
\frac{y\cos(xy)}{1 - x\cos x} &= \frac{dy}{dx}
\end{aligned}$$

$$\begin{aligned}
m &= \frac{y\cos(xy)}{1-x\cos x}\\
&= \frac{1\cos(\pi\times 1)}{1-\pi\cos\pi}\\
&= \frac{1(-1)}{1+\pi}\\
&= -\frac{1}{1+\pi}
\end{aligned}$$

9.

$$\begin{aligned}
x^3y^3 &= 5\\
\frac{d}{dx}\left[x^3y^3\right] &= \frac{d}{dx}[5]\\
y^3\left(\frac{d}{dx}\left[x^3\right]\right)+x^3\left(\frac{d}{dx}\left[y^3\right]\right) &= 0\\
y^3\left(3x^2\right)+x^3\left(3y^2\frac{dy}{dx}\right) &= 0\\
3x^2y^3+3x^3y^2\frac{dy}{dx} &= 0\\
3x^3y^2\frac{dy}{dx} &= -3x^2y^3\\
\frac{dy}{dx} &= \frac{-3x^2y^3}{3x^3y^2}\\
\frac{dy}{dx} &= -\frac{y}{x}
\end{aligned}$$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left[-\frac{y}{x}\right]\\
&= -\frac{x\frac{d}{dx}[y]-y\frac{d}{dx}[x]}{x^2}\\
&= -\frac{x\frac{dy}{dx}-y(1)}{x^2}\\
&= -\frac{x\frac{dy}{dx}-y}{x^2}\\
&= -\frac{x\frac{dy}{dx}-y}{x^2}\\
&= -\frac{x\left(-\frac{y}{x}\right)-y}{x^2}\\
&= -\frac{-y-y}{x^2}\\
&= -\frac{-2y}{x^2}\\
&= \frac{2y}{x^2}
\end{aligned}$$

10.

$$\begin{aligned} y^2 &= kx \\ 2y\frac{dy}{dx} &= k \\ \frac{dy}{dx} &= \frac{k}{2y} \end{aligned}$$

Then slope $m = \frac{k}{2y_0}$ at (x_0, y_0). Note that $y_0^2 = kx_0$.

The equation of the tangent line is

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - y_0 &= \frac{k}{2y_0}(x - x_0) \\ 2y_0y - 2y_0^2 &= k(x - x_0) \\ 2y_0y - 2kx_0 &= kx - kx_0 \\ 2y_0y &= kx - kx_0 + 2kx_0 \\ 2y_0y &= kx + kx_0 \\ 2y_0y &= k(x + x_0) \\ y_0y &= \frac{1}{2}k(x + x_0) \end{aligned}$$