1.8 Infinite Limits

1.

x	$f(x) = \dfrac{(x+2)^2}{(x-2)^2 - 1}$
3.01	1248.76
3.001	12498.75
3.0001	124998.75

$$\lim_{x \to 3^+} \frac{(x+2)^2}{(x-2)^2 - 1} = +\infty$$

2.

$$\begin{aligned}
\lim_{x \to \infty} \frac{(x+2)^2}{(x-2)^2 - 1} &= \lim_{x \to \infty} \frac{x^2+4x+4}{x^2-4x+4-1} \\
&= \lim_{x \to \infty} \frac{x^2+4x+4}{x^2-4x+3} \\
&= \lim_{x \to \infty} \frac{\frac{x^2}{x^2}+\frac{4x}{x^2}+\frac{4}{x^2}}{\frac{x^2}{x^2}-\frac{4x}{x^2}+\frac{3}{x^2}} \\
&= \lim_{x \to \infty} \frac{1+\frac{4}{x}+\frac{4}{x^2}}{1-\frac{4}{x}+\frac{3}{x^2}} \\
&= \frac{1+0+0}{1-0+0} \\
&= 1
\end{aligned}$$

3.

x	$f(x) = \dfrac{(x+2)^2}{(x-2)^2 - 1}$
1.01	-455.28
1.001	-4505.25
1.0001	$-45,005.25$

$$\lim_{x \to 1^+} \frac{(x+2)^2}{(x-2)^2 - 1} = -\infty$$

4. $\lim_{x \to \infty} \frac{2x-1}{x+1} = \lim_{x \to \infty} \frac{\frac{2x}{x}-\frac{1}{x}}{\frac{x}{x}+\frac{1}{x}} = \lim_{x \to \infty} \frac{2-\frac{1}{x}}{1+\frac{1}{x}} = \frac{2-0}{1+0} = 2$

5. $\lim_{x\to-\infty}\frac{x^5+3x^4+1}{x^3-1} = \lim_{x\to-\infty}\frac{\frac{x^5}{x^5}-\frac{3x^4}{x^5}+\frac{1}{x^5}}{\frac{x^3}{x^5}+\frac{1}{x^5}} = \lim_{x\to-\infty}\frac{1+\frac{3}{x^4}+\frac{1}{x^5}}{\frac{1}{x^2}+\frac{1}{x^5}}$

The limit of the denominator is 0, so this is an indeterminate form. We can argue the limit in this way: the numerator approaches 1 as x goes to $-\infty$. The denominator approaches 0 and is a positive quantity because $\frac{1}{x^2} > \frac{1}{x^5}$. The ratio $\frac{1+\frac{3}{x^4}+\frac{1}{x^5}}{\frac{1}{x^2}+\frac{1}{x^5}}$ is a positive quantity that increases without bound because 1 divided by a very small positive number is a large positive number.

Thus, $\lim_{x\to-\infty}\frac{x^5+3x^4+1}{x^3-1} = \infty$.

6.

$$\begin{aligned}
\lim_{x\to\infty}\frac{3x^4-2x^2+3x+1}{2x^4-2x^2} &= \lim_{x\to\infty}\frac{\frac{3x^4}{x^4}-\frac{2x^2}{x^4}+\frac{3x}{x^4}+\frac{1}{x^4}}{\frac{2x^4}{x^4}-\frac{2x^2}{x^4}} \\
&= \lim_{x\to-\infty}\frac{3-\frac{2}{x^2}+\frac{3}{x^3}+\frac{1}{x^4}}{2+\frac{2}{x^2}} \\
&= \frac{3-0+0+0}{2+0} = \frac{3}{2}
\end{aligned}$$

7.

$$\begin{aligned}
\lim_{x\to\infty}\frac{2x^3-x+3}{x^5-2x^3+2x-3} &= \lim_{x\to\infty}\frac{\frac{2x^3}{x^5}-\frac{x}{x^5}+\frac{3}{x^5}}{\frac{x^5}{x^5}-\frac{2x^3}{x^5}+\frac{2x}{x^5}-\frac{3}{x^5}} \\
&= \lim_{x\to\infty}\frac{\frac{2}{x^2}-\frac{1}{x^4}+\frac{3}{x^5}}{1-\frac{2}{x^2}+\frac{2}{x^4}-\frac{3}{x^5}} \\
&= \frac{0-0+0}{1-0+0-0} = 0
\end{aligned}$$

8. Zero: Set numerator $= 0$.

$$\begin{aligned}
(x+4)^2 &= 0 \\
x+4 &= 0 \\
x &= -4
\end{aligned}$$

Vertical asymptotes: Set denominator $= 0$.

$$\begin{aligned}
(x-4)^2-1 &= 0 \\
x^2-8x+16-1 &= 0 \\
x^2-8x+15 &= 0 \\
(x-5)(x-3) &= 0 \\
x = 5 \ \text{ or } \ x &= 3
\end{aligned}$$

The vertical asymptotes are $x = 5$ or $x = 3$.

End behavior:

$$\begin{aligned}\lim_{x\to\infty}\frac{(x+4)^2}{(x-4)^2-1} &= \lim_{x\to\infty}\frac{x^2+8x+16}{x^2-8x+15}\\ &= \lim_{x\to\infty}\frac{\frac{x^2}{x^2}+\frac{8x}{x^2}+\frac{16}{x^2}}{\frac{x^2}{x^2}-\frac{8x}{x^2}+\frac{15}{x^2}}\\ &= \lim_{x\to\infty}\frac{1+\frac{8}{x}+\frac{16}{x^2}}{1-\frac{8}{x}+\frac{15}{x^2}} = \frac{1+0+0}{1+0+0} = 1\end{aligned}$$

$$\begin{aligned}\lim_{x\to-\infty}\frac{(x+4)^2}{(x-4)^2-1} &= \lim_{x\to-\infty}\frac{x^2+8x+16}{x^2-8x+15}\\ &= \lim_{x\to-\infty}\frac{\frac{x^2}{x^2}+\frac{8x}{x^2}+\frac{16}{x^2}}{\frac{x^2}{x^2}-\frac{8x}{x^2}+\frac{15}{x^2}}\\ &= \lim_{x\to-\infty}\frac{1+\frac{8}{x}+\frac{16}{x^2}}{1-\frac{8}{x}+\frac{15}{x^2}} = \frac{1+0+0}{1+0+0} = 1\end{aligned}$$

9. Zero:

There are no vertical asymptotes.

$$\begin{aligned}\lim_{x\to\infty}\left(-3x^3-x^2+2x+2\right) &= -\infty\\ \lim_{x\to-\infty}\left(-3x^3-x^2+2x+2\right) &= \infty\end{aligned}$$

10. $\frac{2x^2-8}{x+2} = \frac{2\left(x^2-4\right)}{x+2} = \frac{2(x-2)(x+2)}{x+2} = 2\,(x-2)$

Zero:

$$\begin{aligned}2\,(x-2) &= 0\\ x-2 &= 0\\ x &= 2\end{aligned}$$

There are no vertical asymptotes. There is a discontinuity at $x=-2$.

$$\begin{aligned}\lim_{x\to\infty}\frac{2x^2-8}{x+2} &= \lim_{x\to\infty}\left[2\,(x-2)\right] = \infty\\ \lim_{x\to-\infty}\frac{2x^2-8}{x+2} &= \lim_{x\to-\infty}\left[2\,(x-2)\right] = -\infty\end{aligned}$$