1.4 The Calculus

1. a. The formula for slope of the tangent line for $f(x) = x^2$ at $x = 3$ is $m = \frac{x^2-3^2}{x-3} = \frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3} = x+3$.

$P(x,x^2)$	$m = x+1$
$P(2.9, 8.41)$	$2.9+3 = 5.9$
$P(2.95, 8.7025)$	$2.95+3 = 5.95$
$P(2.975, 8.850625)$	$2.975+3 = 5.975$
$P(2.995, 8.970025)$	$2.995+3 = 5.995$
$P(2.999, 8.994001)$	$2.999+3 = 5.999$

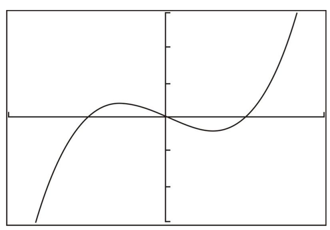
b. The sequence of slopes are approaching $m = 6$.

2. a. Draw tangent lines on the graph of $f(x) = x^2$. The slope of the tangent line is negative for $x < 0$.

b. The slope of the tangent line is 0 for $x = 0$.

c. One example is a polynomial function such as $p(x) = x^3 - 4x$.

3. a.

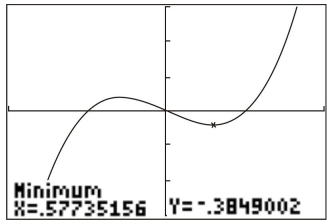


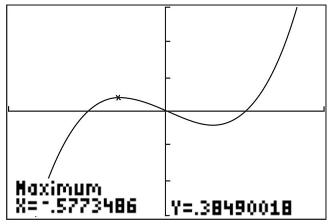
The formula for slope of the tangent line for $f(x) = x^3 - x$ at $x = 2$ is $m = \frac{x^3-x-(2^3-2)}{x-2} = \frac{x^3-x-(8-2)}{x-2} = \frac{x^3-x-6}{x-2}$.

x	$m = \frac{x^3-x-6}{x-2}$
2.1	$\frac{2.1^3-2.1-6}{2.1-2} = \frac{9.261-2.1-6}{0.1} = 11.61$
2.05	11.3025
2.005	11.030025
2.001	11.006001
2.0001	11.0006

The sequence of values are approaching $m = 11$.

b. The tangent lines have slopes of 0 at maximum or minimum points.



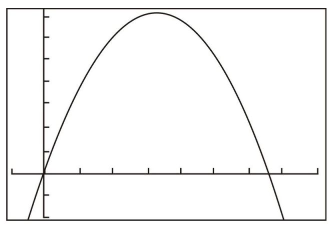


The tangent lines have slopes of 0 when $x = 0.57$ and $x = -0.57$.

c. Using the graph, the tangent line appear to have positive slope for $x < -0.57$ and $x > 0.57$.

d. Using the graph, the tangent line appear to have negative slope for $-0.57 < x < 0.57$.

4. a. Use the calculator to generate the graph of $C(x)$.



b. The function will be maximized for $x = -\frac{b}{2a} = -\frac{200}{2(-0.3)} = \frac{200}{0.6} = 333.33$

c. To estimate the slope of the tangent line at $x = 200$, use the slope formula for the points $(200, C(200))$ and $(200.01, C(200.01))$:

$$\frac{C(200.01) - C(200)}{200.01 - 200} = \frac{28,850.80 - 28,850}{0.01} = \frac{0.80}{0.01} = 80.$$

To estimate the slope of the tangent line at $x = 300$, use the slope formula for the points $(300, C(300))$ and $(300.01, C(300.01))$:

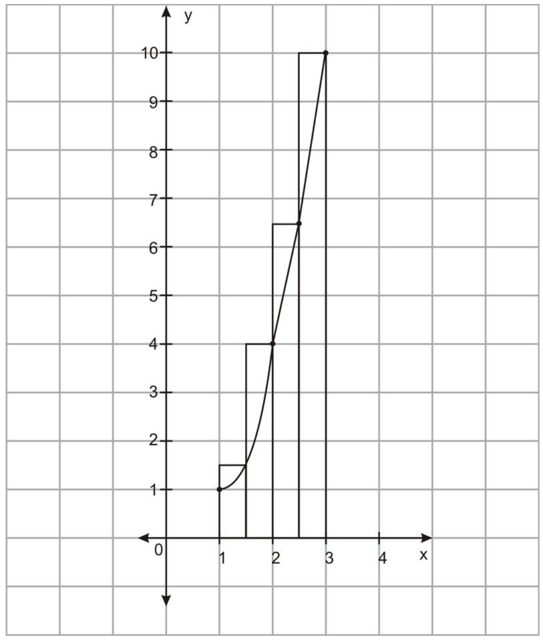
$$\frac{C(300.01) - C(300)}{300.01 - 300} = \frac{33,850.20 - 33,850}{0.01} = \frac{0.20}{0.01} = 20.$$

To estimate the slope of the tangent line at $x = 400$, use the slope formula for the points $(400, C(400))$ and $(400.01, C(400.01))$:

$$\frac{C(400.01) - C(400)}{400.01 - 400} = \frac{32,849.60 - 32,850}{0.01} = \frac{-0.4}{0.01} = -40.$$

d. The marginal cost is positive for $x < 333.33$.

5. a. Divide the area under the curve from $x = 1$ to $x = 3$ in four equal rectangles.



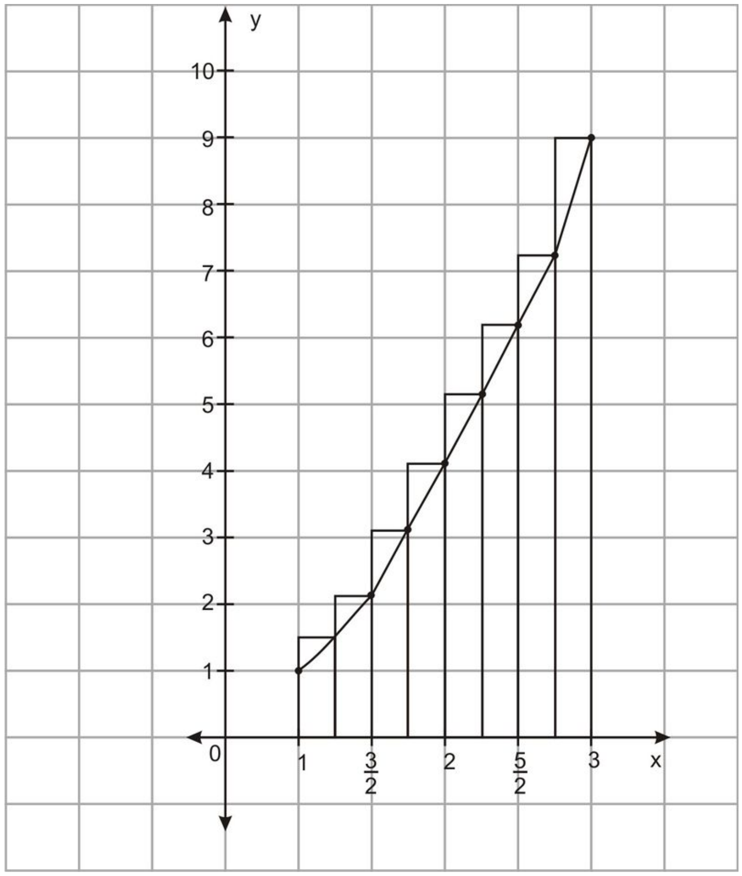
Each width is equal to $\frac{1}{2}$. Call the rectangles R_1 to R_4.

The areas are:

$$R_1 = \frac{1}{2} \times f\left(\frac{3}{2}\right) = \frac{1}{2} \times \frac{9}{4} = \frac{9}{8}$$
$$R_2 = \frac{1}{2} \times f(2) = \frac{1}{2} \times 4 = 2$$
$$R_3 = \frac{1}{2} \times f\left(\frac{5}{2}\right) = \frac{1}{2} \times \frac{25}{4} = \frac{25}{8}$$
$$R_4 = \frac{1}{2} \times f(3) = \frac{1}{2} \times 9 = \frac{9}{2}$$

The approximation under the curve is $R_1 + R_2 + R_3 + R_4 = \frac{9}{8} + 2 + \frac{25}{8} + \frac{9}{2} = 10.75$.

b. Divide the area under the curve from $x = 1$ to $x = 3$ in eight equal rectangles.



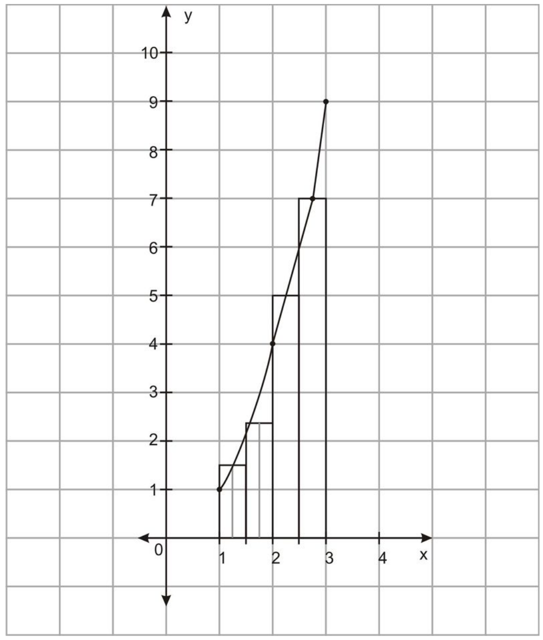
Each width is equal to $\frac{1}{4}$. Call the rectangles R_1 to R_8.

The areas are:

$$R_1 = \frac{1}{4} \times f\left(\frac{5}{4}\right) = \frac{1}{4} \times \frac{25}{16} = \frac{25}{64}$$
$$R_2 = \frac{1}{4} \times f\left(\frac{3}{2}\right) = \frac{1}{4} \times \frac{9}{4} = \frac{9}{16}$$
$$R_3 = \frac{1}{4} \times f\left(\frac{7}{4}\right) = \frac{1}{4} \times \frac{49}{16} = \frac{49}{64}$$
$$R_4 = \frac{1}{4} \times f(2) = \frac{1}{4} \times 4 = 1$$
$$R_5 = \frac{1}{4} \times f\left(\frac{9}{4}\right) = \frac{1}{4} \times \frac{81}{16} = \frac{81}{64}$$
$$R_6 = \frac{1}{4} \times f\left(\frac{5}{2}\right) = \frac{1}{4} \times \frac{25}{4} = \frac{25}{16}$$
$$R_7 = \frac{1}{4} \times f\left(\frac{11}{4}\right) = \frac{1}{4} \times \frac{121}{16} = \frac{121}{64}$$
$$R_8 = \frac{1}{4} \times f(3) = \frac{1}{4} \times 9 = \frac{9}{4}$$

The approximation of the area under the curve is $R_1 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7 + R_8 = 9.6875$.

c.



Each width is equal to $\frac{1}{2}$. Call the rectangles R_1 to R_4.

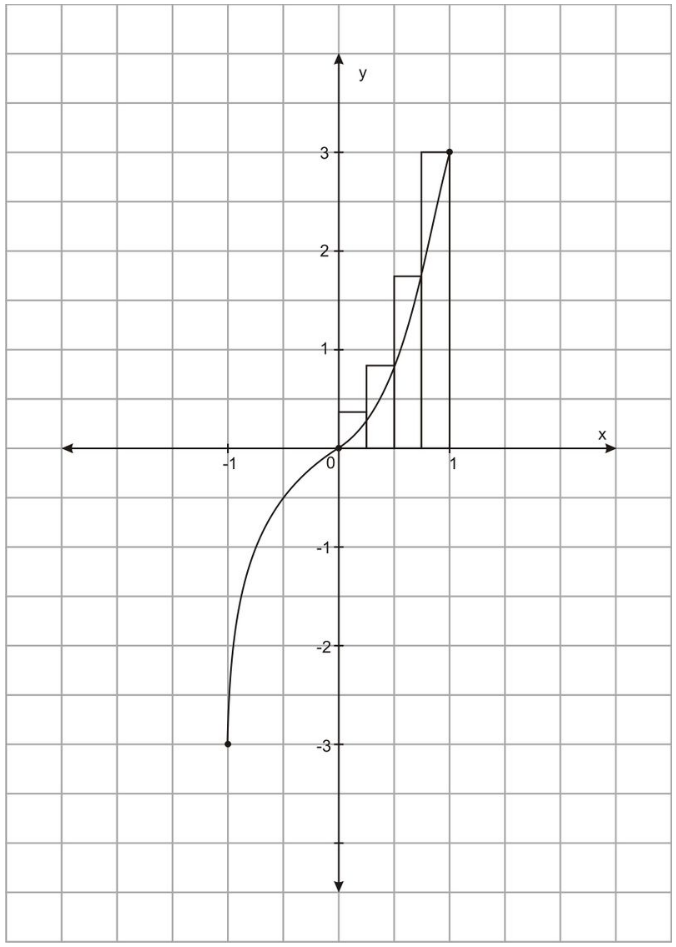
The midpoint of R_1 is $\frac{1+\frac{3}{2}}{2} = \frac{\frac{5}{2}}{2} = \frac{5}{4}$. The midpoint of R_2 is $\frac{\frac{3}{2}+2}{2} = \frac{\frac{7}{2}}{2} = \frac{7}{4}$. The midpoint of R_3 is $\frac{2+\frac{5}{2}}{2} = \frac{\frac{9}{2}}{2} = \frac{9}{4}$. The midpoint of R_4 is $\frac{\frac{5}{2}+3}{2} = \frac{\frac{11}{2}}{2} = \frac{11}{4}$.

The areas are:

$$R_1 = \frac{1}{2} \times f\left(\frac{5}{4}\right) = \frac{1}{2} \times \frac{25}{16} = \frac{25}{32}$$
$$R_2 = \frac{1}{2} \times f\left(\frac{7}{4}\right) = \frac{1}{2} \times \frac{49}{16} = \frac{49}{32}$$
$$R_3 = \frac{1}{2} \times f\left(\frac{9}{4}\right) = \frac{1}{2} \times \frac{81}{16} = \frac{81}{32}$$
$$R_4 = \frac{1}{2} \times f\left(\frac{11}{4}\right) = \frac{1}{2} \times \frac{121}{16} = \frac{121}{32}$$

The approximation of the area under the curve is $R_1 + R_2 + R_3 + R_4 = \frac{25}{32} + \frac{49}{32} + \frac{81}{32} + \frac{121}{32} = 8.625$.

6. a.



Each width is equal to $\frac{1}{4}$. Call the rectangles R_1 to R_4.

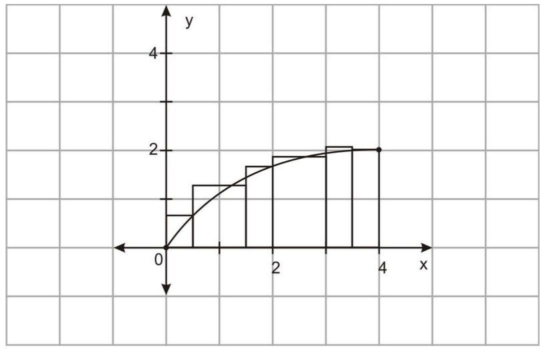
The areas are:

$$R_1 = \frac{1}{4} \times f\left(\frac{1}{4}\right) = \frac{1}{4} \times \left[\left(-\frac{1}{4}\right)^3 + 4\left(\frac{1}{4}\right)\right] = \frac{1}{4} \times \left(-\frac{1}{64} + 1\right) = \frac{63}{256}$$
$$R_2 = \frac{1}{4} \times f\left(\frac{1}{2}\right) = \frac{1}{4} \times \left[\left(-\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right)\right] = \frac{1}{4} \times \left(-\frac{1}{8} + 2\right) = \frac{15}{32}$$
$$R_3 = \frac{1}{4} \times f\left(\frac{3}{4}\right) = \frac{1}{4} \times \left[\left(-\frac{3}{4}\right)^3 + 4\left(\frac{3}{4}\right)\right] = \frac{1}{4} \times \left(-\frac{37}{64} + 3\right) = \frac{155}{256}$$
$$R_4 = \frac{1}{4} \times f(1) = \frac{1}{4} \times \left[(-1)^3 + 4(1)\right] = \frac{1}{4} \times (-1 + 4) = \frac{3}{4}$$

The approximation of the area under the curve is $R_1 + R_2 + R_3 + R_4 = \frac{63}{256} + \frac{15}{32} + \frac{155}{256} + \frac{3}{4} = 2.07$.

b. The area from $x = -1$ to $x = 0$ is below the $x-$axis. We are not finding area under a curve but the are between the curve and the $x-$axis. The area from $x = -1$ to $x = 0$ is below the $x-$axis is symmetric to the area under the curve from $x = 0$ to $x = 1$.

7. The length of the interval is $4 - 0 = 4$. Divide 4 by 6 to get that the length of each sub-interval is of length $\frac{4}{6} = \frac{2}{3}$.



Call the rectangles R_1 to R_6.

The areas are:

$$R_1 = \frac{1}{6} \times f\left(\frac{2}{3}\right) = \frac{1}{6} \times \sqrt{\frac{2}{3}} = 0.136$$
$$R_2 = \frac{1}{6} \times f\left(\frac{4}{3}\right) = \frac{1}{6} \times \sqrt{\frac{4}{3}} = 0.192$$
$$R_3 = \frac{1}{6} \times f(2) = \frac{1}{6} \times \sqrt{2} = 0.236$$
$$R_4 = \frac{1}{6} \times f\left(\frac{8}{3}\right) = \frac{1}{6} \times \sqrt{\frac{8}{3}} = 0.272$$
$$R_3 = \frac{1}{6} \times f\left(\frac{10}{3}\right) = \frac{1}{6} \times \sqrt{\frac{10}{3}} = 0.304$$
$$R_3 = \frac{1}{6} \times f(4) = \frac{1}{6} \times 2 = 0.333$$

The approximation of the area under the curve is $R_1 + R_2 + R_3 + R_4 + R_5 + R_6 = 0.136 + 0.192 + 0.236 + 0.272 + 0.304 + 0.333 = 1.473$.

8. The average velocity of a falling object from $t = a$ to $t = b$ is given by $\frac{s(b)-s(a)}{b-a} = \frac{4.9b^2-4.9a^2}{b-a}$. Simplifying, we get the average velocity is $\frac{4.9\left(b^2-a^2\right)}{b-a} = \frac{4.9(b-a)(b+a)}{b-a} = 4.9(b+a)$.

Make a sequence of values of x that get closer to $b = 4$ and find the average velocity between each x and 4.

x	average velocity $= 4.9(4+x)$
3.9	$4.9(4+3.9) = 4.9(7.9) = 38.71$
3.95	$4.9(4+3.95) = 4.9(7.95) = 38.955$
3.99	$4.9(4+3.99) = 4.9(7.99) = 39.151$
3.999	$4.9(4+3.999) = 4.9(7.999) = 39.1951$
3.9999	$4.9(4+3.9999) = 4.9(7.9999) = 39.19951$

The velocity of the ball after 4 seconds is 39.2 m/sec.