Each basic rule of integration that you have studied so far was derived from a corresponding differentiation rule. Even though you have learned all the necessary tools for differentiating exponential, logarithmic, trigonometric, and algebraic functions, your set of tools for integrating these functions is not yet complete. In this chapter we will explore different ways of integrating functions and develop several integration techniques that will greatly expand the set of integrals to which the basic integration formulas can be applied. Before we do that, let us review the basic integration formulas that you are already familiar with from previous chapters.

1. The Power Rule $(n \neq -1)$:
$\int x^n dx = \frac{x^{n+1}}{n+1} + C.$
2. The General Power Rule $(n \neq -1)$:
$\int u^n \frac{du}{dx} dx = \int u^n du = \frac{u^{n+1}}{n+1} + C.$
3. The Simple Exponential Rule:

$\int e^x dx = e^x + C.$
4. The General Exponential Rule:

$\int e^u \frac{du}{dx} dx = \int e^u du = e^u + C.$
5. The Simple Log Rule:

$\int \frac{1}{x} dx = \ln|x| + C.$
6. The General Log Rule:

$\int \frac{du/dx}{u} dx = \int \frac{1}{u} du = \ln|u| + C.$
It is important that you remember the above rules because we will be using them extensively to solve more complicated integration problems. The skill that you need to develop is to determine which of these basic rules is needed to solve an integration problem.

Learning Objectives

A student will be able to:

- Compute by hand the integrals of a wide variety of functions by using the technique of $u-$substitution.
- Apply the $u-$substitution technique to definite integrals.
- Apply the $u-$substitution technique to trig functions.

Probably one of the most powerful techniques of integration is *integration by substitution.* In this technique, you choose part of the integrand to be equal to a variable we will call u and then write the entire integrand in terms of u. The difficulty of the technique is deciding which term in the integrand will be best for substitution by u. However, with practice, you will develop a skill for choosing the right term.

Recall from Chapter 2 that if u is a differentiable function of x and if n is a real number and $n\neq-1$, then the Chain Rule tells us that

$$\frac{d}{dx}[u^n]=nu^{n-1}\frac{du}{dx}.$$

The reverse of this formula is the integration formula,

$$\int u^n du=\frac{u^{n+1}}{n+1}+C, n\neq-1.$$

Sometimes it is not easy to integrate directly. For example, look at this integral:

$$\int(5x-2)^2dx.$$

One way to integrate is to first expand the integrand and then integrate term by term.

$$\int(5x-2)^2dx=\int(25x^2-20x+4)dx=25\int x^2dx-20\int xdx+\int 4dx=\frac{25}{3}x^3-10x^2+4x+C.$$

That is easy enough. However, what if the integral was

$$\int(5x-2)^{15}dx?$$

Would you still expand the integrand and then integrate term by term? That would be impractical and time-consuming. A better way of doing this is to change the variables. Changing variables can often turn a difficult integral, such as the one above, into one that is easy to integrate. The method of doing this is called ***integration by substitution,*** or for short, the u***-substitution method***. The examples below will show you how the method is used.

Example 1:

Evaluate $\int(x+1)^5dx$.

Solution:

Let $u=x+1$. Then $du=d(x+1)=1dx=dx$. Substituting for $u$and du we get

$$\int(x+1)^5dx=\int u^5du.$$

Integrating using the power rule,

$$=\frac{u^6}{6}+C.$$

Since $u=x+1$, substituting back,

$$=\frac{(x+1)^6}{6}+C.$$

Example 2:

Evaluate $\int\sqrt{4x+3}\,dx$.

Solution:

Let $u=4x+3$. Then $du=4dx$. Solving for dx,

$$dx=du/4.$$

Substituting,

$$=\int u^{1/2}\cdot\frac{1}{4}dx=\frac{1}{4}\int u^{1/2}dx=\frac{1}{4}\frac{u^{3/2}}{3/2}+C.$$

Simplifying,

$=16u^{3/2}+C=16(4x+3)^{3/2}+C.$

Trigonometric Integrands

We can apply the change of variable technique to trigonometric functions as long as u is a differentiable function of x. Before we show how, recall the basic trigonometric integrals:

$\int \cos u du \int \sin u du \int \sec^2 u du \int \csc^2 u du \int (\sec u)(\tan u) du \int (\csc u)(\cot u) du = \sin u + C, = -\cos u + C, = \tan u + C, = -\cot u + C, = \sec u + C, = -\csc u + C.$

Example 3:

Evaluate $\int \cos(3x+2) dx$.

Solution:

The argument of the cosine function is $3x+2$. So we let $u=3x+2$.Then $du=3dx$, or $dx=du/3$.

Substituting,

$\int \cos(3x+2) dx = \int \cos u \cdot 13 dx = 13 \int \cos u dx.$

Integrating,

$=13 \sin u + C = 13 \sin(3x+2) + C.$

Example 4:

This example requires us to use trigonometric identities before we substitute. Evaluate

$\int 1 \cos^2 3x dx.$

Solution:

Since $sec 3x = 1 cos 3x$, the integral becomes

$\int 1 \cos^2 3x dx = \int \sec^2 3x dx.$

Substituting for the argument of the secant, $u=3x$, then $du=3dx$, or $dx=du/3$. Thus our integral becomes,

$\int \sec^2 u . 13 du = 13 \int \sec^2 u du = 13 \tan u + C = 13 \tan(3x) + C.$

Some integrations of trigonometric functions involve the logarithmic functions as a solution, as shown in the following example.

Example 5:

Evaluate $\int \tan x dx$.

Solution:

As you may have guessed, this is not a straightforward integration. We need to make use of trigonometric identities to simplify it. Since $\tan x=\sin x/\cos x$,

$\int \tan x\,dx=\int \frac{\sin x}{\cos x}dx$.

Now make a change of variable x. Choose $u=\cos x$. Then $du=-\sin x\,dx$, or $dx=-du/\sin x$. Substituting,

$\int \frac{\sin x}{\cos x}dx=\int \frac{\sin x}{u}\left(-\frac{du}{\sin x}\right)=-\int \frac{du}{u}$.

This integral should look obvious to you. The integrand is the derivative of the natural logarithm $\ln u$.

$=-\ln|u|+C=-\ln|\cos x|+C$.

Another way of writing it, since $-\ln|u|=\ln\frac{1}{|u|}$, is

$=\ln\left|\frac{1}{\cos x}\right|+C=\ln|\sec x|+C$.

Using Substitution on Definite Integrals

Example 6:

Evaluate $\int_1^3 x\sqrt{2x-1}\,dx$.

Solution:

Let $u=2x-1$. Then $du=2dx$, or $dx=du/2$. Before we substitute, we need to determine the new limits of integration in terms of the $u$variable. To do so, we simply substitute the limits of integration into $u=2x-1$:

Lower limit: For $x=1, u=2(1)-1=1$.

Upper limit: For $x=3, u=2(3)-1=5$.

We now substitute u and the associated limits into the integral:

$\int_1^5 x\sqrt{u}\frac{du}{2}$.

As you may notice, the variable x is still hanging there. To write it in terms of u, since $u=2x-1$, solving for x, we get, $x=(u+1)/(2)$.Substituting back into the integral,

$=\int_1^5 \frac{u+1}{2}\sqrt{u}\frac{du}{2}=\frac{1}{4}\int_1^5 (u+1)\sqrt{u}\,du=\frac{1}{4}\int_1^5 (u+1)u^{1/2}du=\frac{1}{4}\int_1^5 (u^{3/2}+u^{1/2})du=\frac{1}{4}\left[\frac{2u^{5/2}}{5}+\frac{2u^{3/2}}{3}\right]_1^5$.

Applying the Fundamental Theorem of Calculus by inserting the limits of integration and calculating,

$=\frac{1}{4}\left(\left[\frac{2(5)^{3/2}}{3}+\frac{2(5)^{1/2}}{1}\right]-\left[\frac{2(1)^{3/2}}{3}+\frac{2(1)^{1/2}}{1}\right]\right)$.

Calculating and simplifying, we get

$=\frac{4\sqrt{5}-2}{3}$.

We could have chosen $u=\sqrt{2x-1}$ instead. You may want to try to solve the integral with this substitution. It might be easier and less tedious.

Example 7:

Let’s try the substitution method of definite integrals with a trigonometric integrand.

Evaluate $\int_0^{\pi/4} \tan x \sec^2 x\,dx$.

Solution:

Try $u=\tan x$. Then $du=\sec^2 x\,dx, dx=du/\sec^2 x$.
Lower limit: For $x=0, u=\tan 0=0$.
Upper limit: For $x=\pi/4, u=\tan\frac{\pi}{4}=1$.
Thus

$$\int_0^{\pi/4} \tan x \sec^2 x\,dx=\int_0^1 u\,du=\left[\frac{u^2}{2}\right]_0^1=\frac{1}{2}-0=\frac{1}{2}.$$

Review Questions

In the following exercises, evaluate the integrals.

1. $\int 3(x-8)^2 dx$
2. $\int \sqrt{2+x}\,dx$
3. $\int \frac{1}{\sqrt{2+x}}dx$
4. $\int \frac{x^2}{x+1}dx$
5. $\int \frac{e^{-x}}{e^{-x}+2}dx$
6. $\int 3\sqrt{t}+5t\,dt$
7. $\int \frac{2}{\sqrt{3x-1}}dx$
8. $\int \sin x \cos x\,dx$
9. $\int \cos x\sqrt{1-\cos^2 x}\,dx$
10. $\int \sin^5 x \cos x\,dx$
11. $\int x^3 \cos 4x^4 dx$
12. $\int \sec^2(2x+4)dx$
13. $\int_0^2 x e^{x^2} dx$
14. $\int_0^{\sqrt{\pi}} x \sin x^2 dx$
15. $\int_0^1 x(x+5)^4 dx$