Learning Objectives

A student will be able to:

- Find the limit of basic functions.
- Use properties of limits to find limits of polynomial, rational and radical functions.
- Find limits of composite functions.
- Find limits of trigonometric functions.
- Use the Squeeze Theorem to find limits.

Introduction

In this lesson we will continue our discussion of limits and focus on ways to evaluate limits. We will observe the limits of a few basic functions and then introduce a set of laws for working with limits. We will conclude the lesson with a theorem that will allow us to use an indirect method to find the limit of a function.

Direct Substitution and Basic Limits

Let's begin with some observations about limits of basic functions. Consider the following limit problems:

$\lim_{x\to 2} 5, \lim_{x\to 4} x.$

These are examples of limits of basic constant and linear functions, $f(x)=c$ and $f(x)=mx+b$.

We note that each of these functions are defined for all real numbers. If we apply our techniques for finding the limits we see that

$\lim_{x\to 2} 5 \lim_{x\to 4} x = 5, = 4,$

and observe that for each the limit equals the value of the function at the $x-$value of interest:

$\lim_{x\to 2} 5 \lim_{x\to 4} x = f(5) = 5, = f(4) = 4.$

Hence $\lim_{x\to a} f(x) = f(a)$. This will also be true for some of our other basic functions, in particular all polynomial and radical functions, provided that the function is defined at $x=a$. For example, $\lim_{x\to 3} x^3 = f(3) = 27$ and $\lim_{x\to 4} \sqrt{x} = f(4) = 2$. The properties of functions that make these facts true will be discussed in Lesson 1.7. For now, we wish to use this idea for evaluating limits of basic functions. However, in order to evaluate limits of more complex function we will need some properties of limits, just as we needed laws for dealing with complex problems involving exponents. A simple example illustrates the need we have for such laws.

Example 1:

Evaluate $\lim_{x\to 2}(x^3+2x−−√)$. The problem here is that while we know that the limit of each individual function of the sum exists, $\lim_{x\to 2}x^3=8$ and $\lim_{x\to 2}2x−−√=2$, our basic limits above do not tell us what happens when we find the limit of a sum of functions. We will state a set of properties for dealing with such sophisticated functions.

Properties of Limits

Suppose that $\lim_{x\to a}f(x)$ and $\lim_{x\to a}g(x)$ both exist. Then

1. $\lim_{x\to a}[cf(x)]=c\lim_{x\to a}f(x)$ where c is a real number,
2. $\lim_{x\to a}[f(x)]^n=[\lim_{x\to a}f(x)]^n$ where n is a real number,
3. $\lim_{x\to a}[f(x)\pm g(x)]=\lim_{x\to a}f(x)\pm\lim_{x\to a}g(x)$,
4. $\lim_{x\to a}[f(x)\cdot g(x)]=\lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x)$,
5. $\lim_{x\to a}[f(x)g(x)]=\lim_{x\to a}f(x)\lim_{x\to a}g(x)$ provided that $\lim_{x\to a}g(x)\neq 0$.

With these properties we can evaluate a wide range of polynomial and radical functions. Recalling our example above, we see that

$$\lim_{x\to 2}(x^3+2x−−√)=\lim_{x\to 2}(x^3)+\lim_{x\to 2}(2x−−√)=8+2=10.$$

Find the following limit if it exists:

$$\lim_{x\to -4}(2x^2−−x−−−√).$$

Since the limit of each function within the parentheses exists, we can apply our properties and find

$$\lim_{x\to -4}(2x^2−−x−−−√)=\lim_{x\to -4}2x^2−\lim_{x\to -4}−x−−−√.$$

Observe that the second limit, $\lim_{x\to -4}−x−−−√$, is an application of Law #2 with $n=12$. So we have $\lim_{x\to -4}(2x^2−−x−−−√)=\lim_{x\to -4}2x^2−\lim_{x\to -4}−x−−−√=32−2=30$.

In most cases of sophisticated functions, we simplify the task by applying the Properties as indicated. We want to examine a few exceptions to these rules that will require additional analysis.

Strategies for Evaluating Limits of Rational Functions

Let’s recall our example

$$\lim_{x\to 1}x^2−1x−1.$$

We saw that the function did not have to be defined at a particular value for the limit to exist. In this example, the function was not defined for $x=1$. However we were able to evaluate the limit numerically by checking functional values around $x=1$ and found $\lim_{x\to 1}x^2−1x−1=2$.

Note that if we tried to evaluate by direct substitution, we would get the quantity $0/0$, which we refer to as an ***indeterminate form.*** In particular, Property #5 for finding limits does not apply since $\lim_{x\to 1}(x−1)=0$. Hence in order to evaluate the limit without using numerical or graphical techniques we make the following observation. The numerator of

the function can be factored, with one factor common to the denominator, and the fraction simplified as follows:
$x^2-1x-1=(x+1)(x-1)x-1=x+1.$
In making this simplification, we are indicating that the original function can be viewed as a linear function for x values close to but not equal to 1, that is,
$x^2-1x-1=x+1$ for $x\neq 1$. In terms of our limits, we can say
$\lim_{x\to 1}x^2-1x-1=\lim_{x\to 1}(x+1)=1+1=2.$

Example 2:

Find $\lim_{x\to 0}x^2+5xx.$
This is another case where direct substitution to evaluate the limit gives the indeterminate form $0/0$. Reducing the fraction as before gives:
$\lim_{x\to 0}x^2+5xx=\lim_{x\to 0}(x+5)=5.$

Example 3:

$\lim_{x\to 9}x--\sqrt{}-3x-9.$
In order to evaluate the limit, we need to recall that the difference of squares of real numbers can be factored as $x^2-y^2=(x+y)(x-y).$
We then rewrite and simplify the original function as follows:

$x--\sqrt{}-3x-9=x--\sqrt{}-3(x--\sqrt{}+3)(x--\sqrt{}-3)=1(x--\sqrt{}+3).$
Hence $\lim_{x\to 9}x--\sqrt{}-3x-9=\lim_{x\to 9}1x--\sqrt{}+3=16.$
You will solve similar examples in the homework where some clever applications of factoring to reduce fractions will enable you to evaluate the limit.

Limits of Composite Functions

While we can use the Properties to find limits of composite functions, composite functions will present some difficulties that we will fully discuss in the next Lesson. We can illustrate with the following examples, one where the limit exists and the other where the limit does not exist.

Example 4:

Consider $f(x)=1x+1$, $g(x)=x^2$. Find $\lim_{x\to -1}(f\circ g)(x)$.
We see that $(f\circ g)(x)=1x^2+1$ and note that property #5 does hold. Hence by direct substitution we have $\lim_{x\to -1}(f\circ g)(x)=1(-1)^2+1=12.$

Example 5:

Consider $f(x)=1x+1$, $g(x)=-1.$ Then we have that $f\ (g(x))$ is undefined and we get the indeterminate form $1/0$. Hence $\lim_{x\to -1}(f\circ g)(x)$ does not exist.

Limits of Trigonometric Functions

In evaluating limits of trigonometric functions we will look to rely more on numerical and graphical techniques due to the unique behavior of these functions. Let’s look at a couple of examples.

Example 6:

Find $\lim_{x \to 0} \sin(x)$.
We can find this limit by observing the graph of the sine function and using the **[CALC VALUE]** function of our calculator to show that $\lim_{x \to 0} \sin x = 0$.
While we could have found the limit by direct substitution, in general, when dealing with trigonometric functions, we will rely less on formal properties of limits for finding limits of trigonometric functions and more on our graphing and numerical techniques.

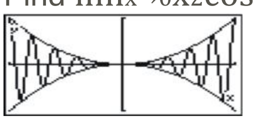
The following theorem provides us a way to evaluate limits of complex trigonometric expressions.

Squeeze Theorem

Suppose that $f(x) \le g(x) \le h(x)$ for x near a, and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$.
Then $\lim_{x \to a} g(x) = L$.
In other words, if we can find bounds for a function that have the same limit, then the limit of the function that they bound must have the same limit.

Example 7:

Find $\lim_{x \to 0} x^2 \cos(10\pi x)$.



From the graph we note that:

1. The function is bounded by the graphs of x^2 and $-x^2$
2. $\lim_{x \to 0} x^2 = \lim_{x \to 0} (-x^2) = 0$.

Hence the Squeeze Theorem applies and we conclude that $\lim_{x \to 0} x^2 \cos(10\pi x) = 0$.

Lesson Summary

1. We learned to find the limit of basic functions.
2. We learned to find the limit of polynomial, rational and radical functions.
3. We learned how to find limits of composite and trigonometric functions.
4. We used the Squeeze Theorem to find special limits.

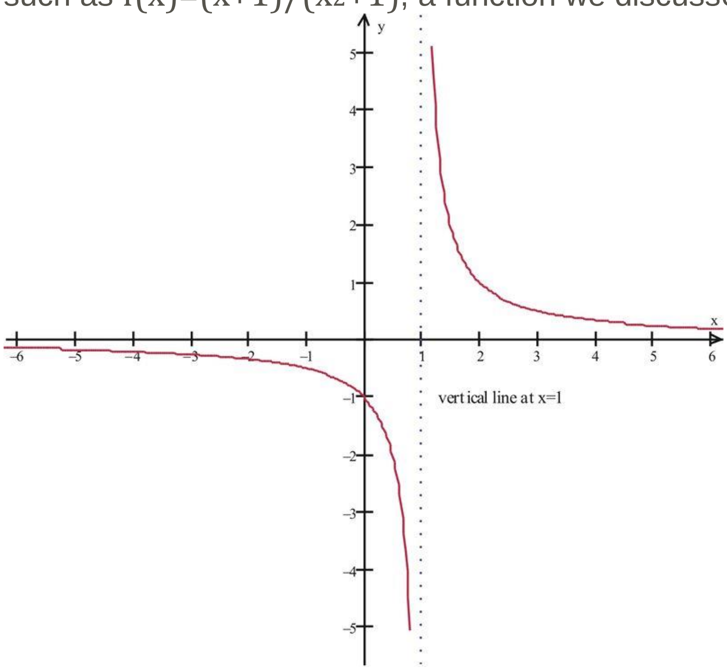
Learning Objectives

A student will be able to:

- Find infinite limits of functions.
- Analyze properties of infinite limits.
- Identify asymptotes of functions.
- Analyze end behavior of functions.

Introduction

In this lesson we will discuss infinite limits. In our discussion the notion of infinity is discussed in two contexts. First, we can discuss infinite limits in terms of the value a function as we increase x without bound. In this case we speak of the ***limit of*** $f(x)$ ***as*** x ***approaches*** ∞ and write $\lim_{x\to\infty} f(x)$. We could similarly refer to the ***limit of*** $f(x)$ ***as*** x ***approaches*** -∞ and write $\lim_{x\to-\infty} f(x)$.
The second context in which we speak of infinite limits involves situations where the function values increase without bound. For example, in the case of a rational function such as $f(x)=(x+1)/(x^2+1)$, a function we discussed in previous lessons:



At $x=1$, we have the situation where the graph grows without bound in both a positive and a negative direction. We say that we have a vertical asymptote at $x=1$, and this is indicated by the dotted line in the graph above.
In this example we note that $\lim_{x\to1} f(x)$ does not exist. But we could compute both one-sided limits as follows.
$\lim_{x\to1^-} f(x)=-\infty$ and $\lim_{x\to1^-} f(x)=+\infty$.

More formally, we define these as follows:

Definition:

The right-hand limit of the function $f(x)$ at $x=a$ is infinite, and we write $\lim_{x\to a^+} f(x)=\infty$, if for every positive number k, there exists an open interval $(a,a+\delta)$ contained in the domain of $f(x)$, such that $f(x)$ is in (k,∞) for every x in $(a,a+\delta)$.

The definition for negative infinite limits is similar.

Suppose we look at the function $f(x)=(x+1)/(x^2-1)$ and determine the infinite limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
We observe that as x increases in the positive direction, the function values tend to get smaller. The same is true if we decrease x in the negative direction. Some of these extreme values are indicated in the following table.

x	100	200	-100	-200
$f(x)$	$.0101$	$.0053$	$-.0099$	$-.005$

We observe that the values are getting closer to $f(x)=0$. Hence $\lim_{x\to\infty} f(x)=0$ and $\lim_{x\to-\infty} f(x)=0$.

Since our original function was roughly of the form $f(x)=\frac{1}{x}$, this enables us to determine limits for all other functions of the form $f(x)=\frac{1}{x^p}$ with $p>0$. Specifically, we are able to conclude that $\lim_{x\to\infty} \frac{1}{x^p}=0$. This shows how we can find infinite limits of functions by examining the ***end behavior*** of the function $f(x)=\frac{1}{x^p}, p>0$.

The following example shows how we can use this fact in evaluating limits of rational functions.

Example 1:

Find $\lim_{x\to\infty} \frac{2x^3-x^2+x-1}{x^6-x^5+3x^4-2x+1}$.
Solution:

Note that we have the indeterminate form, so Limit Property #5 does not hold. However, if we first divide both numerator and denominator by the quantity x^6, we will then have a function of the form
$\frac{f(x)}{g(x)}=\frac{\frac{2x^3}{x^6}-\frac{x^2}{x^6}+\frac{x}{x^6}-\frac{1}{x^6}}{\frac{x^6}{x^6}-\frac{x^5}{x^6}+\frac{3x^4}{x^6}-\frac{2x}{x^6}+\frac{1}{x^6}}=\frac{\frac{2}{x^3}-\frac{1}{x^4}+\frac{1}{x^5}-\frac{1}{x^6}}{1-\frac{1}{x}+\frac{3}{x^2}-\frac{2}{x^5}+\frac{1}{x^6}}$.
We observe that the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ both exist. In particular, $\lim_{x\to\infty} f(x)=0$ and $\lim_{x\to\infty} g(x)=1$. Hence Property #5 now applies and we have $\lim_{x\to\infty} \frac{2x^3-x^2+x-1}{x^6-x^5+3x^4-2x+1}=\frac{0}{1}=0$.

Lesson Summary

1. We learned to find infinite limits of functions.
2. We analyzed properties of infinite limits.

3. We identified asymptotes of functions.
4. We analyzed end behavior of functions.

Review Questions

In problems 1 - 7, find the limits if they exist.

1. $\lim_{x\to 3^+}(x+2)^2(x-2)^2-1$
2. $\lim_{x\to\infty}(x+2)^2(x-2)^2-1$
3. $\lim_{x\to 1^+}(x+2)^2(x-2)^2-1$
4. $\lim_{x\to\infty}2x-1x+1$
5. $\lim_{x\to-\infty}x^5+3x^4+1x^3-1$
6. $\lim_{x\to\infty}3x^4-2x^2+3x+12x^4-2x^2+x-3$
7. $\lim_{x\to\infty}2x^2-x+3x^5-2x^3+2x-3$

In problems 8 - 10, analyze the given function and identify all asymptotes and the end behavior of the graph.

8. $f(x)=(x+4)^2(x-4)^2-1$
9. $f(x)=-3x^3-x^2+2x+2$
10. $f(x)=2x^2-8x+2$
11. Consider $f(x)=1x+1, g(x)=x^2$. We previously found $\lim_{x\to-1}(f\circ g)(x)=12$. Find $\lim_{x\to-1}(g\circ f)(x)$.