Learning Objectives

A student will be able to:

- Find antiderivatives of functions.
- Represent antiderivatives.
- Interpret the constant of integration graphically.
- Solve differential equations.
- Use basic antidifferentiation techniques.
- Use basic integration rules.

Introduction

In this lesson we will introduce the idea of the ***antiderivative*** of a function and formalize as ***indefinite integrals.*** We will derive a set of rules that will aid our computations as we solve problems.

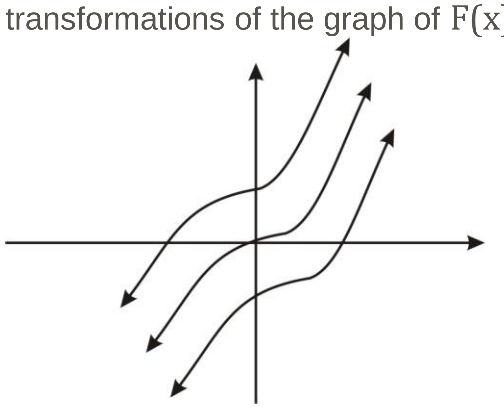
Antiderivatives

Definition

> A function $F(x)$ is called an ***antiderivative*** of a function f if $F'(x)=f(x)$ for all x in the domain of f.

Example 1:

Consider the function $f(x)=3x^2$. Can you think of a function $F(x)$ such that $F'(x)=f(x)$? ***(Answer:*** $F(x)=x^3$, $F(x)=x^3-6$***, many other examples.)***
Since we differentiate $F(x)$ to get $f(x)$, we see that $F(x)=x^3+C$ will work for any constant C. Graphically, we can think the set of all antiderivatives as vertical transformations of the graph of $F(x)=x^3$. The figure shows two such transformations.



With our definition and initial example, we now look to formalize the definition and develop some useful rules for computational purposes, and begin to see some applications.

Notation and Introduction to Indefinite Integrals

The process of finding antiderivatives is called ***antidifferentiation***, more commonly referred to as ***integration.*** We have a particular sign and set of symbols we use to indicate integration:

$$\int f(x)\,dx = F(x) + C.$$

We refer to the left side of the equation as "the indefinite integral of $f(x)$ with respect to x." The function $f(x)$ is called the ***integrand*** and the constant C is called the ***constant of integration.*** Finally the symbol dx indicates that we are to integrate with respect to x.

Using this notation, we would summarize the last example as follows:

$$\int 3x^2\,dx = x^3 + C$$

Using Derivatives to Derive Basic Rules of Integration

As with differentiation, there are several useful rules that we can derive to aid our computations as we solve problems. The first of these is a rule for integrating power functions, $f(x) = x^n \; [n \neq -1]$, and is stated as follows:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C.$$

We can easily prove this rule. Let $F(x) = \frac{1}{n+1}x^{n+1} + C, n \neq -1$. We differentiate with respect to x and we have:

$$F'(x) = \frac{d}{dx}\left(\frac{1}{n+1}x^{n+1} + C\right) = \frac{d}{dx}\left(\frac{1}{n+1}x^{n+1}\right) + \frac{d}{dx}(C) = \left(\frac{1}{n+1}\right)\frac{d}{dx}\left(x^{n+1}\right) + \frac{d}{dx}(C) = \left(\frac{n+1}{n+1}\right)x^n + 0 = x^n.$$

The rule holds for $f(x) = x^n \; [n \neq -1]$. What happens in the case where we have a power function to integrate with $n = -1$, say $\int x^{-1}\,dx = \int \frac{1}{x}\,dx$. We can see that the rule does not work since it would result in division by 0. However, if we pose the problem as finding $F(x)$ such that $F'(x) = \frac{1}{x}$, we recall that the derivative of logarithm functions had this form. In particular, $\frac{d}{dx}\ln x = \frac{1}{x}$. Hence

$$\int \frac{1}{x}\,dx = \ln x + C.$$

In addition to logarithm functions, we recall that the basic exponentional function, $f(x) = e^x$, was special in that its derivative was equal to itself. Hence we have

$$\int e^x\,dx = e^x + C.$$

Again we could easily prove this result by differentiating the right side of the equation above. The actual proof is left as an exercise to the student.

As with differentiation, we can develop several rules for dealing with a finite number of integrable functions. They are stated as follows:

If f and g are integrable functions, and C is a constant, then
$\int[f(x)+g(x)]dx\int[f(x)-g(x)]dx\int[Cf(x)]dx=\int f(x)dx+\int g(x)dx,=\int f(x)dx-\int g(x)dx,=C\int f(x)dx.$

Example 2:

Compute the following indefinite integral.

$\int[2x^3+3x^2-1x]dx.$

Solution:

Using our rules we have

$\int[2x^3+3x^2--1x]dx=2\int x^3dx+3\int 1x^2dx-\int 1xdx=2(x^4 4)+3(x^{-1}-1)-\ln x+C=x^4 2-3x-\ln x+C.$

Sometimes our rules need to be modified slightly due to operations with constants as is the case in the following example.

Example 3:

Compute the following indefinite integral:

$\int e^{3x}dx.$

Solution:

We first note that our rule for integrating exponential functions does not work here since $ddx e^{3x}=3e^{3x}$. However, if we remember to divide the original function by the constant then we get the correct antiderivative and have
$\int e^{3x}dx=e^{3x}3+C.$
We can now re-state the rule in a more general form as

$\int e^{kx}dx=e^{kx}k+C.$

Differential Equations

We conclude this lesson with some observations about integration of functions. First, recall that the integration process allows us to start with function f from which we find another function $F(x)$ such that $F'(x)=f(x)$. This latter equation is called a ***differential equation.*** This characterization of the basic situation for which integration applies gives rise to a set of equations that will be the focus of the Lesson on The Initial Value Problem.

Example 4:

Solve the general differential equation $f'(x)=x^{23}+x--\sqrt{}.$

Solution:

We solve the equation by integrating the right side of the equation and have

$f(x)=\int f'(x)dx=\int x^{23}dx+\int x--\sqrt{}dx.$

We can integrate both terms using the power rule, first noting that $x--\sqrt{}=x^{12}$, and have

$f(x)=\int x^{23}dx+\int x^{12}dx=35x^{53}+23x^{32}+C.$

Lesson Summary

1. We learned to find antiderivatives of functions.
2. We learned to represent antiderivatives.
3. We interpreted constant of integration graphically.
4. We solved general differential equations.
5. We used basic antidifferentiation techniques to find integration rules.
6. We used basic integration rules to solve problems.

Review Questions

In problems #1–3, find an antiderivative of the function

1. $f(x)=1-3x^2-6x$
2. $f(x)=x-x^{23}$
3. $f(x)=2x+1-----\sqrt{}_5$

In #4–7, find the indefinite integral

4. $\int(2+5-\sqrt{})dx$
5. $\int 2(x-3)^3dx$
6. $\int(x^2\cdot x--\sqrt{}_3)dx$
7. $\int(x+1x^4x--\sqrt{})dx$
8. Solve the differential equation $f'(x)=4x^3-3x^2+x-3$
9. Find the antiderivative $F(x)$ of the function $f(x)=2e^{2x}+x-2$ that satisfies $F(0)=5.$
10. Evaluate the indefinite integral $\int|x|dx$ (Hint: Examine the graph of $f(x)=|x|.$)