Learning Objectives

A student will be able to:

- Find the derivative of variety of functions by using the technique of implicit differentiation.

Consider the equation

$2xy=1.$

We want to obtain the derivative dy/dx. One way to do it is to first solve for y,

$y=12x,$

and then project the derivative on both sides,

$dydx=ddx[12x]=-12x^2.$

There is another way of finding dy/dx. We can directly differentiate both sides:

$ddx[2xy]=ddx[1].$

Using the Product Rule on the left-hand side,

$yddx[2x]+2xddx[y]y[2]+2xdydx=0=0.$

Solving for dy/dx,

$dydx=-2y2x=-yx.$

But since $y=12x$, substitution gives

$dydx=-1x(2x)=-12x^2.$

which agrees with the previous calculations. This second method is called the **implicit differentiation** method. You may wonder and say that the first method is easier and faster and there is no reason for the second method. That's probably true, but consider this function:

$3y^2-\cos y=x^3.$

How would you solve for y? That would be a difficult task. So the method of implicit differentiation sometimes is very useful, especially when it is inconvenient or impossible to solve for y in terms of x. Explicitly defined functions may be written with a direct relationship between two variables with clear independent and dependent variables. Implicitly defined functions or relations connect the variables in a way that makes it impossible to separate the variables into a simple input output relationship. More notes on explicit and implicit functions can be found at http://en.wikipedia.org/wiki/Implicit_function.

Example 1:

Find dy/dx if $3y^2-\cos y=x^3.$

Solution:

Differentiating both sides with respect to x and then solving for dy/dx,

$ddx[3y^2-cosy]3ddx[y^2]-ddx[cosy]3(2ydydx)-(-siny)dydx6ydydx+sinydydx[6y+siny]dydx=ddx[x^3]=3x^2=3x^2=3x^2=3x^2$.

Solving for dy/dx, we finally obtain

$dydx=3x^2 6y+siny$.

Implicit differentiation can be used to calculate the slope of the tangent line as the example below shows.

Example 2:

Find the equation of the tangent line that passes through point $(1,2)$ to the graph of $8y^3+x^2y-x=3$.

Solution:

First we need to use implicit differentiation to find dy/dx and then substitute the point $(1,2)$ into the derivative to find slope. Then we will use the equation of the line (either the slope-intercept form or the point-intercept form) to find the equation of the tangent line. Using implicit differentiation,

$ddx[8y^3+x^2y-x]24y^2dydx+[(x^2)(1)dydx+y(2x)]-124y^2dydx+x^2dydx+2xy-1[24y^2+x^2]dydxdydx=ddx[3]=0=0=1-2xy=1-2xy24y^2+x^2$.

Now, substituting point $(1,2)$ into the derivative to find the slope,

$dydx=1-2(1)(2)24(2)^2+(1)^2=-397$.

So the slope of the tangent line is $-3/97$, which is a very small value. (What does this tell us about the orientation of the tangent line?)

Next we need to find the equation of the tangent line. The slope-intercept form is

$y=mx+b$,

where $m=-3/97$ and b is the $y-$intercept. To find it, simply substitute point $(1,2)$ into the line equation and solve for b to find the $y-$intercept.

$2b=(-397)(1)+b=19797$.

Thus the equation of the tangent line is

$y=-397x+19797$.

Remark: we could have used the point-slope form $y-y_1=m(x-x_1)$and obtained the same equation.

Example 3:

Use implicit differentiation to find d^2y/dx^2 if $5x^2-4y^2=9$. Also find $d^2ydx^2||_{(x,y)=(2,3)}$. What does the second derivative represent?

Solution:

$ddx[5x^2-4y^2]10x-8ydydx=ddx[9]=0$.

Solving for dy/dx,

$\frac{dy}{dx}=\frac{5x}{4y}$.
Differentiating both sides implicitly again (and using the quotient rule),

$$\frac{d^2y}{dx^2}=\frac{(4y)(5)-(5x)(4dy/dx)}{(4y)^2}=\frac{20y}{16y^2}-\frac{20x}{16y^2}\frac{dy}{dx}=\frac{5}{4y}-\frac{5x}{4y^2}\frac{dy}{dx}.$$

But since $dy/dx=5x/4y$, we substitute it into the second derivative:

$$\frac{d^2y}{dx^2}\frac{d^2y}{dx^2}=\frac{5}{4y}-\frac{5x}{4y^2}.\frac{5x}{4y}=\frac{5}{4y}-\frac{25x^2}{16y^3}.$$

This is the second derivative of y.

The next step is to find: $\left.\frac{d^2y}{dx^2}\right|_{(x,\,y)=(2,\,3)}$

$$\left.\frac{d^2y}{dx^2}\right|_{(2,\,3)}=\frac{5}{4(3)}-\frac{25(2)^2}{16(3)^3}=\frac{5}{27}.$$

Since the first derivative of a function represents the rate of change of the function $y=f(x)$ with respect to x, the second derivative represents the rate of change of the rate of change of the function. For example, in kinematics (the study of motion), the speed of an object (y') signifies the change of position with respect to time but acceleration (y'') signifies the rate of change of the speed with respect to time.

Review Questions

Find dy/dx by implicit differentiation.

1. $x^2+y^2=500$
2. $x^2y+3xy-2=1$
3. $\frac{1}{x}+\frac{1}{y}=\frac{1}{2}$
4. $\sqrt{x}-\sqrt{y}=\sqrt{3}$
5. $\sin(25xy^2)=x$
6. $\tan^3(x^2-y^2)=\tan(\pi/4)$

In problems #7 and 8, use implicit differentiation to find the slope of the tangent line to the given curve at the specified point.

7. $x^2y-y^2x=0$ at $(1,1)$
8. $\sin(xy)=y$ at $(\frac{\pi}{2},1)$
9. Find y'' by implicit differentiation for $x^3y^3=5$.
10. Use implicit differentiation to show that the tangent line to the curve $y^2=kx$ at (x_0,y_0) is given by $yy_0=\frac{1}{2}k(x+x_0)$, where k is a constant.

$$\frac{d(y^2)}{dx}\quad 2y\frac{dy}{dx}\quad \frac{dy}{dx}=\frac{d(kx)}{dx}=k=\frac{k}{2y}$$

Second, we substitute y_0 for y, and that gives us the slope m of our tangent line at (x_0,y_0):

$$m=\frac{k}{2y_0}$$

Third, we set up the equation for our tangent line using point-slope form:

$$y-y_0=\frac{k}{2y_0}(x-x_0)$$

Fourth, and finally, we manipulate this linear equation to get the term yy_0 isolated on the left hand side:

$y-y_0yyy_0yy_0yy_0=k2y_0(x-x_0)=k2y_0(x-x_0)+y_0=k2(x-x_0)+(y_0)_2=k2(x-x_0)+kx_0$ (Using the fact that $y_2=kx)=k2(x+x_0)$