Learning Objectives

A student will be able to:

- Find the limit of a function numerically.
- Find the limit of a function using a graph.
- Identify cases when limits do not exist.
- Use the formal definition of a limit to solve limit problems.

Introduction

In this lesson we will continue our discussion of the limiting process we introduced in Lesson 1.4. We will examine numerical and graphical techniques to find limits where they exist and also to examine examples where limits do not exist. We will conclude the lesson with a more precise definition of limits.

Let's start with the notation that we will use to denote limits. We indicate the limit of a function as the x values approach a particular value of x,say a, as

$\lim_{x\to a} f(x)$.

So, in the example from Lesson 1.3 concerning the function $f(x)=x^2$,we took points that got closer to the point on the graph $(1,1)$ and observed the sequence of slope values of the corresponding secant lines. Using our limit notation here, we would write

$\lim_{x\to 1} x^2-1x-1$.

Recall also that we found that the slope values tended to the value $x=2$; hence using our notation we can write

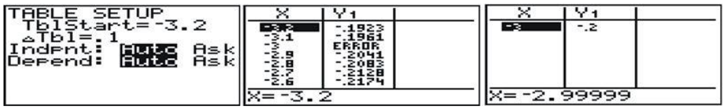
$\lim_{x\to 1} x^2-1x-1=2$.

Finding Limits Numerically

In our example in Lesson 1.3 we used this approach to find that $\lim_{x\to 1} x^2-1x-1=2$.
Let's apply this technique to a more complicated function.
Consider the rational function $f(x)=x+3x^2+x-6$. Let's find the following limit:

$\lim_{x\to -3} x+3x^2+x-6$.

Unlike our simple quadratic function, $f(x)=x^2$, it is tedious to compute the points manually. So let's use the **[TABLE]** function of our calculator. Enter the equation in your calculator and examine the table of points of the function. Do you notice anything unusual about the points? ***(Answer: There are error readings indicated for*** $x=-3,2$ ***because the function is not defined at these values.)***
Even though the function is not defined at $x=-3$, we can still use the calculator to read the $y-$values for x values very close to $x=-3$.Press **2ND [TBLSET]** and set **Tblstart** to -3.2 and Δ to 0.1 (see screen on left below). The resulting table appears in the middle below.



Can you guess the value of $\lim_{x \to -3} x+3x^2+x-6$? If you guessed $-.20=-(1/5)$ you would be correct. Before we finalize our answer, let's get even closer to $x=-3$ and determine its function value using the **[CALC VALUE]** tool.

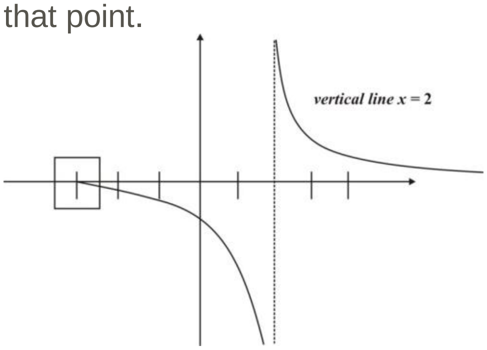
Press **2ND [TBLSET]** and change **Indpnt** from **Auto** to **Ask**. Now when you go to the table, enter $x=-2.99999$. and press **[ENTER]** and you will see the screen on the right above. Press **[ENTER]** and see that the function value is $x=-0.2$, which is the closest the calculator can display in the four decimal places allotted in the table. So our guess is correct and $\lim_{x \to -3} x+3x^2+x-6=-15$.

Finding Limits Graphically

Let's continue with the same problem but now let's focus on using the graph of the function to determine its limit.

$\lim_{x \to -3} x+3x^2+x-6$

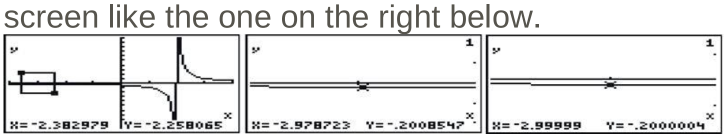
We enter the function in the $Y=$ menu and sketch the graph. Since we are interested in the value of the function for x close to $x=-3$, we will look to **[ZOOM]** in on the graph at that point.



Our graph above is set to the normal viewing window $[-10,10]$. Hence the values of the function appear to be very close to 0. But in our numerical example, we found that the function values approached $-.20=-(1/5)$. To see this graphically, we can use the **[ZOOM]** and **[TRACE]** function of our calculator. Begin by choosing **[ZOOM]** function and choose **[BOX]**. Using the directional arrows to move the cursor, make a box around the x value -3. (See the screen on the left below

Press **[ENTER]** and **[TRACE]** and you will see the screen in the middle below.)

In **[TRACE]** mode, type the number -2.99999 and press **[ENTER]**. You will see a screen like the one on the right below.

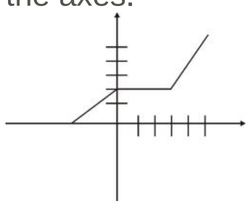


The graphing calculator will allow us to calculate limits graphically, provided that we have the function rule for the function so that we can enter its equation into the calculator. What if we have only a graph given to us and we are asked to find certain limits?

It turns out that we will need to have pretty accurate graphs that include sufficient detail about the location of data points. Consider the following example.

Example 1:

Find $\lim_{x\to 3} f(x)$ for the function pictured here. Assume units of value 1 for each unit on the axes.



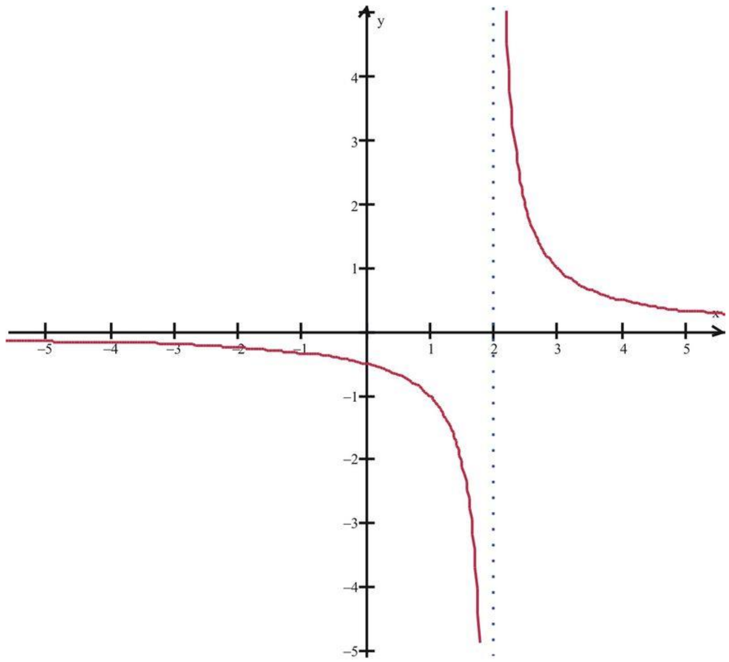
By inspection, we see that as we approach the value $x=3$ from the left, we do so along what appears to be a portion of the horizontal line $y=2$.We see that as we approach the value $x=3$ from the right, we do so along a line segment having positive slope. In either case, the y values of $f(x)$ approaches $y=2$.

Nonexistent Limits

We sometimes have functions where $\lim_{x\to a} f(x)$ does not exist. We have already seen an example of a function where a value was not in the domain of the function. In particular, the function was not defined for $x=-3,2,$ but we could still find the limit as $x\to -3$.

$\lim_{x\to -3} \frac{x+3}{x^2+x-6} = -\frac{1}{5}$

What do you think the limit will be as we let $x\to 2$?



$\lim_{x\to 2} \frac{x+3}{x^2+x-6}$

Our inspection of the graph suggests that the function around $x=2$does not appear to approach a particular value. For $x>2$, the points all lie in the first quadrant and appear to grow very quickly to large positive numbers as we get close to $x=2$. Alternatively, for $x<2$ we see that the points all lie in the fourth quadrant and decrease to large negative numbers. If we inspect actual values very close to $x=2$ we can see that the values of the function do not approach a particular value.

x	1.999	1.9999	2	2.001	2.0001
y	−1000	−10000	ERROR	1000	10000

For this example, we say that $\lim_{x\to 2} \frac{x+3}{x^2+x-6}$ does not exist.

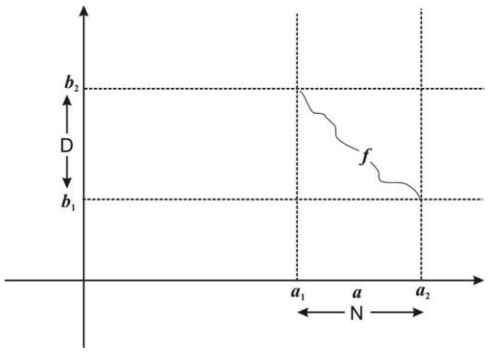
Formal Definition of a Limit

We conclude this lesson with a formal definition of a limit.

Definition:

> We say that the limit of a function $f(x)$ at a is L, written as $\lim_{x\to a} f(x)=L$, if for every open interval D of L, there exists an open interval N of a, that does not include a, such that $f(x)$ is in D for every x in N.
>
> This definition is somewhat intuitive to us given the examples we have covered. Geometrically, the definition means that for any lines $y=b_1, y=b_2$ below and above the line $y=L$, there exist vertical lines $x=a_1, x=a_2$ to the left and right of $x=a$ so that the graph of $f(x)$between $x=a_1$ and $x=a_2$ lies between the lines $y=b_1$ and $y=b_2$. The key phrase in the above statement is "for every open interval D", which means that even

if D is very, very small (that is, $f(x)$ is very, very close to L), it still is possible to find interval N where $f(x)$ is defined for all values except possibly $x=a$.



Example 2:

Use the definition of a limit to prove that

$\lim_{x\to 3}(2x+1)=7$.

We need to show that for each open interval of 7, we can find an open neighborhood of 3, that does not include 3, so that all x in the open neighborhood map into the open interval of 7.

Equivalently, we must show that for every interval of 7, say $(7-\varepsilon,7+\varepsilon)$, we can find an interval of 3, say $(3-\delta,3+\delta)$, such that $(7-\varepsilon<2x+1<7+\varepsilon)$ whenever $(3-\delta<x<3+\delta)$.

The first inequality is equivalent to $6-\varepsilon<2x<6+\varepsilon$ and solving for x, we have $3-\varepsilon 2<x<3+\varepsilon 2$.

Hence if we take $\delta=\varepsilon 2$, we will have $3-\delta<x<3+\delta \Rightarrow 7-\varepsilon<2x+1<7+\varepsilon$.

Fortunately, we do not have to do this to evaluate limits. In Lesson 1.6 we will learn several rules that will make the task manageable.

Lesson Summary

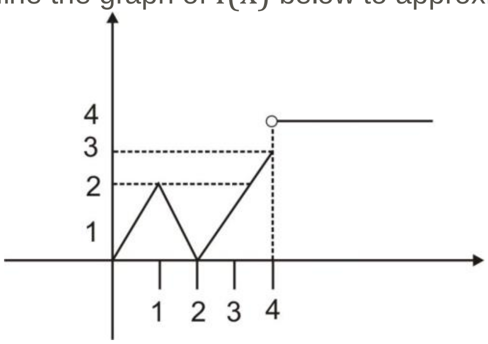
1. We learned to find the limit of a function numerically.
2. We learned to find the limit of a function using a graph.
3. We identified cases when limits do not exist.
4. We used the formal definition of a limit to solve limit problems.

Multimedia Links

For another look at the definition of a limit, the series of videos at Tutorials for the Calculus Phobe has a nice intuitive introduction to this fundamental concept (despite the whimsical name). If you want to experiment with limits yourself, follow the sequence of activities using a graphing applet at Informal Limits. Directions for using the graphing applets at this very useful site are also available at Applet Intro.

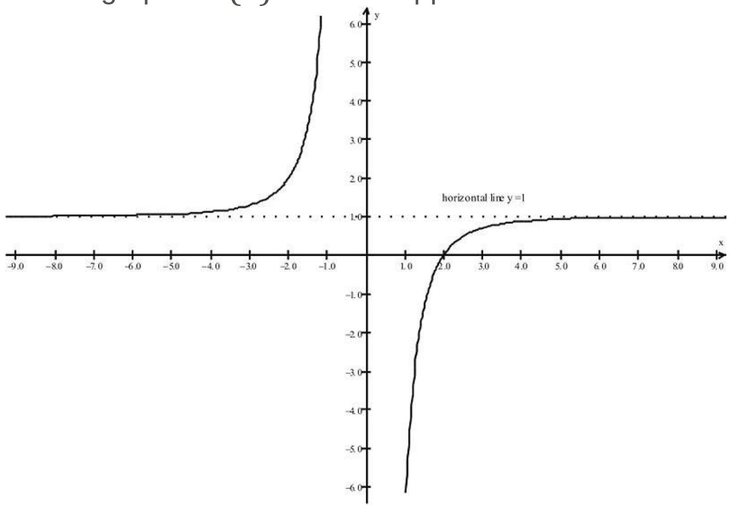
Review Questions

1. Use a table of values to find $\lim_{x\to -2} x^2-4x+2$.
   a. Use x−values of $x=-1.9, -1.99, -1.999, -2.1, -2.099, -2.0099$.
   b. What value does the sequence of values approach?
2. Use a table of values to find $\lim_{x\to 12} 2x-12x^2+3x-2$.
   a. Use x−values of $x=.49, .495, .49999, .51, .5099, .500001$.
   b. What value does the sequence of values approach?
3. Consider the function $p(x)=3x^3-3x$. Generate the graph of $p(x)$using your calculator. Find each of the following limits if they exist. Use tables with appropriate x values to determine the limits.
   a. $\lim_{x\to 4}(3x^3-3x)$
   b. $\lim_{x\to -4}(3x^3-3x)$
   c. $\lim_{x\to 0}(3x^3-3x)$
   d. Find the values of the function corresponding to $x=4,-4,0$. How do these function values compare to the limits you found in #a-c? Explain your answer.
4. Examine the graph of $f(x)$ below to approximate each of the following limits if they



exist.
   a. $\lim_{x\to 3} f(x)$
   b. $\lim_{x\to 2} f(x)$
   c. $\lim_{x\to 1} f(x)$
   d. $\lim_{x\to 4} f(x)$

5. Examine the graph of $f(x)$ below to approximate each of the following limits if they



exist.

a. $\lim_{x\to 2} f(x)$
b. $\lim_{x\to 0} f(x)$
c. $\lim_{x\to 4} f(x)$
d. $\lim_{x\to 50} f(x)$

In problems #6-8, determine if the indicated limit exists. Provide a numerical argument to justify your answer.

6. $\lim_{x\to 2}(x^2+3)$
7. $\lim_{x\to -1}\frac{x+1}{x^2-1}$
8. $\lim_{x\to 2}\sqrt{-2x+5}$

In problems #9-10, determine if the indicated limit exists. Provide a graphical argument to justify your answer. (Hint: Make use of the **[ZOOM]**and **[TABLE]** functions of your calculator to view functions values close to the indicated x value.

9. $\lim_{x\to 4}(x^2+3x)$
10. $\lim_{x\to -1}\frac{|x+1|}{x+1}$