Learning Objectives

A student will be able to:

- Solve problems that involve extrema.
- Study Rolle's Theorem.
- Use the Mean Value Theorem to solve problems.

Introduction

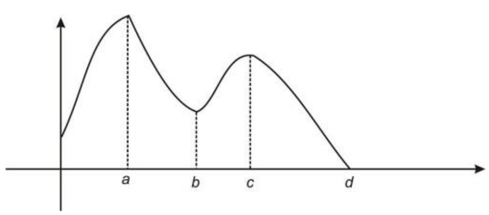
In this lesson we will discuss a second application of derivatives, as a means to study extreme (maximum and minimum) values of functions. We will learn how the maximum and minimum values of functions relate to derivatives.

Let's start our discussion with some formal working definitions of the maximum and minimum values of a function.

Definition

A function f has a **maximum** at $x=a$ if $f(a)\geq f(x)$ for all x in the domain of f. Similarly, f has a ***minimum*** at $x=a$ if $f(a)\leq f(x)$ for all x in the domain of f. The values of the function for these $x-$values are called ***extreme*** values or ***extrema.***

Here is an example of a function that has a maximum at $x=a$ and a minimum at $x=d$:



Observe the graph at $x=b$. While we do not have a minimum at $x=b$, we note that $f(b)\leq f(x)$ for all x near b. We say that the function has a **local minimum** at $x=b$. Similarly, we say that the function has a ***local maximum*** at $x=c$ since $f(c)\geq f(x)$ for some x contained in open intervals of c.

Let's recall the Min-Max Theorem that we discussed in lesson on Continuity.

Min-Max Theorem: If a function $f(x)$ is continuous in a closed interval I,then $f(x)$ has both a maximum value and a minimum value in I. In order to understand the proof for the Min-Max Theorem conceptually, attempt to draw a function on a closed interval (including the endpoints) so that no point is at the highest part of the graph. No matter how the function is sketched, there will be at least one point that is highest.

We can now relate extreme values to derivatives in the following Theorem by the French mathematician Fermat.

Theorem: If $f(c)$ is an extreme value of f for some open interval of c,and if $f'(c)$ exists, then $f'(c)=0$.
Proof: The theorem states that if we have a local max or local min, and if $f'(c)$ exists, then we must have $f'(c)=0$.
Suppose that f has a local max at $x=c$. Then we have $f(c)\geq f(x)$for some open interval $(c-h,c+h)$ with $h>0$.
So $f(c+h)-f(c)\leq 0$.
Consider $\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$.
Since $f(c+h)-f(c)\leq 0$, we have $\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}\leq\lim_{h\to 0^+}0=0$.
Since $f'(c)$ exists, we have $f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$, and so $f'(c)\leq 0$.
If we take the left-hand limit, we get $f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}\geq 0$.
Hence $f'(c)\geq 0$ and $f'(c)\leq 0$ it must be that $f'(c)=0$.
If $x=c$ is a local minimum, the same argument follows.

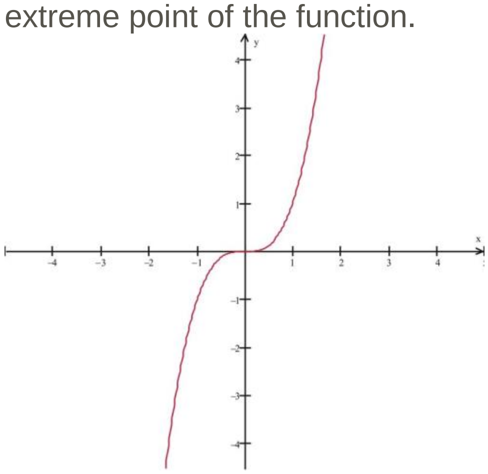
Definition

> We will call $x=c$ a ***critical value*** in $[a,b]$ if $f'(c)=0$ or $f'(c)$ does not exist, or if $x=c$ is an endpoint of the interval.
> We can now state the Extreme Value Theorem.

Extreme Value Theorem: If a function $f(x)$ is continuous in a closed interval $[a,b]$, with the maximum of f at $x=c_1$ and the minimum of f at $x=c_2$, then c_1 and c_2 are critical values of f.
Proof: The proof follows from Fermat's theorem and is left as an exercise for the student.

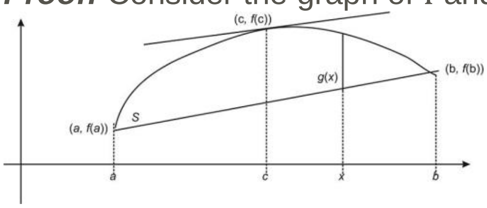
Example 1:

Let's observe that the converse of the last theorem is not necessarily true: If we consider $f(x)=x^3$ and its graph, then we see that while $f'(0)=0$at $x=0$, $x=0$ is not an extreme point of the function.



Rolle's Theorem: If f is continuous and differentiable on a closed interval $[a,b]$ and if $f(a)=f(b)$, then f has at least one value c in the open interval (a,b) such that $f'(c)=0$. The proof of Rolle's Theorem can be found at http://en.wikipedia.org/wiki/Rolle's_theorem.

Mean Value Theorem: If f is a continuous function on a closed interval $[a,b]$ and if f' contains the open interval (a,b) in its domain, then there exists a number c in the interval (a,b) such that $f(b)-f(a)=(b-a)f'(c)$.
Proof: Consider the graph of f and secant line s as indicated in the figure.



By the Point-Slope form of line s we have
$y-f(a)=m(x-a)$ and $y=m(x-a)+f(a)$.
For each x in the interval (a,b), let $g(x)$ be the vertical distance from line S to the graph of f. Then we have
$g(x)=f(x)-[m(x-a)+f(a)]$ for every x in (a,b).
Note that $g(a)=g(b)=0$. Since g is continuous in $[a,b]$ and g' exists in (a,b), then Rolle's Theorem applies. Hence there exists c in $(a,b)$with $g'(c)=0$.
So $g'(x)=f'(x)-m$ for every x in (a,b).
In particular,

$g'(c)=f'(c)-m=0$ and
$f'(c)f'(c)f(b)-f(a)=m=f(b)-f(a)b-a=(b-a)f'(c)$.
The proof is complete.

Example 2:

Verify that the Mean Value Theorem applies for the function $f(x)=x^3+3x^2-24x$ on the interval $[1,4]$.
Solution:

We need to find c in the interval $(1,4)$ such that $f(4)-f(1)=(4-1)f'(c)$.
Note that $f'(x)=3x^2+6x-24$, and $f(4)=16$, $f(1)=-20$.Hence, we must solve the following equation:
$3612=3f'(c)=f'(c)$.
By substitution, we have

$123c^2+6c-36c^2+2c-12c=3c^2+6c-24=0=0=-2\pm 52--\sqrt{2}\approx-4.61,2.61.$

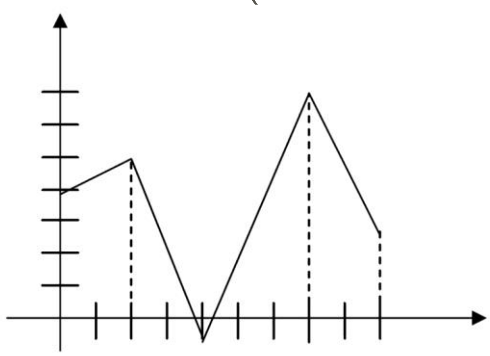
Since we need to have c in the interval $(1,4)$, the positive root is the solution, $c=-2+52--\sqrt{2}\approx 2.61$.

Lesson Summary

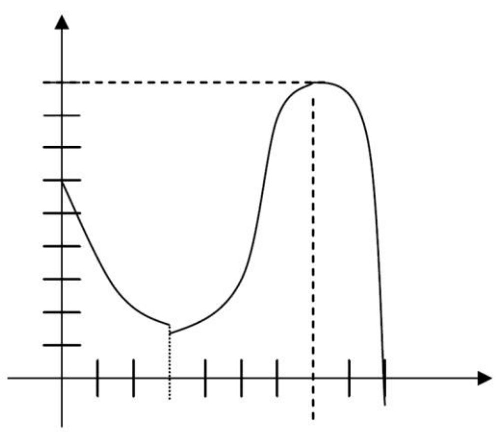
1. We learned to solve problems that involve extrema.
2. We learned about Rolle’s Theorem.
3. We used the Mean Value Theorem to solve problems.

Review Questions

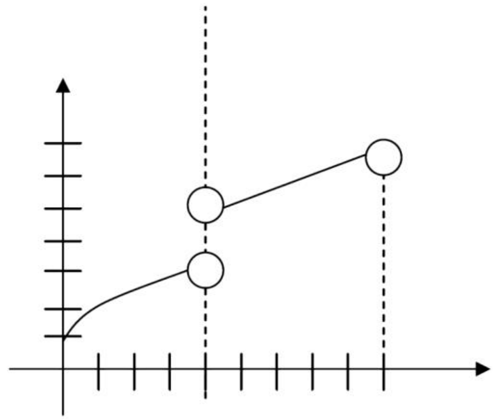
In problems #1–3, identify the absolute and local minimum and maximum values of the function (if they exist); find the extrema. (Units on the axes indicate 1unit).



1. Continuous on $[0,9]$



2. Continuous on $[0,3]\cup(3,9]$



3. Continuous on $[0,4]\cup(4,9)$

In problems #4–6, find the extrema and sketch the graph.

4. $f(x)=-x^2-6x+4$, $[-4,1]$
5. $f(x)=x^3-x^4$, $[0,2]$
6. $f(x)=-x^2+4x^2$, $[-2,0]$
7. Verify Rolle’s Theorem by finding values of x for which $f(x)=0$ and $f'(x)=0$. $f(x)=3x^3-12x$
8. Verify Rolle’s Theorem for $f(x)=x^2-2x-1$.
9. Verify that the Mean Value Theorem works for $f(x)=(x+2)x$, $[1,2]$.
10. Prove that the equation $x^3+a_1x^2+a_2x=0$ has a positive root at $x=r$, and that the equation $3x^2+2a_1x+a_2=0$ has a positive root less than r.