Calculus/Derivatives of Exponential and Logarithm Functions

Logarithm Function

We shall first look at the value of e :

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Now we find the derivative of $\ln(x)$ using the formal definition of the derivative:

$$\frac{d}{dx}\ln(x) = \lim_{h\to 0}\frac{\ln(x+h)-\ln(x)}{h} = \lim_{h\to 0}\frac{1}{h}\ln\left(\frac{x+h}{x}\right) = \lim_{h\to 0}\ln\left(\frac{x+h}{x}\right)^{\frac{1}{h}}$$

Let $n = \frac{x}{h}$. Note that as $n \to \infty$, we get $h \to 0$. So we can redefine our limit as:

$$\lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^{\frac{n}{x}} = \frac{1}{x}\ln\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right) = \frac{1}{x}\ln(e) = \frac{1}{x}$$

Here we could take the natural logarithm outside the limit because it doesn't have anything to do with the limit (we could have chosen not to). We then substituted the value of e .

Derivative of the Natural Logarithm

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

If we wanted, we could go through that same process again for a generalized base, but it is easier just to use properties of logs and realize that:

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

Since $\frac{1}{\ln(a)}$ is a constant, we can just take it outside of the derivative:

$$\frac{d}{dx}\log_a(x) = \frac{1}{\ln(a)}\cdot\frac{d}{dx}\ln(x)$$

Which leaves us with the generalized form of:

Derivative of the Logarithm

$$\frac{d}{dx}\log_a(x) = \frac{1}{\ln(a)x}$$

Exponential Function

We shall take two different approaches to finding the derivative of $\ln(e^x)$. The first approach:

$$\frac{d}{dx}\ln(e^x) = \frac{d}{dx}x = 1$$

The second approach:

$$\frac{d}{dx}\ln(e^x) = \frac{1}{e^x}\left(\frac{d}{dx}e^x\right)$$

Note that in the second approach we made some use of the chain rule. Thus:

$$\frac{1}{e^x}\left(\frac{d}{dx}e^x\right) = 1$$

$$\frac{d}{dx}e^x = e^x$$

so that we have proved the following rule:

Derivative of the exponential function

$$\frac{d}{dx}e^x = e^x$$

Now that we have derived a specific case, let us extend things to the general case. Assuming that a is a positive real constant, we wish to calculate:

$$\frac{d}{dx}a^x$$

One of the oldest tricks in mathematics is to break a problem down into a form that we already know we can handle. Since we have already determined the derivative of e^x , we will attempt to rewrite a^x in that form.

Using that $e^{\ln(c)} = c$ and that $\ln(a^b) = b \cdot \ln(a)$, we find that:

$$a^x = e^{\ln(a)x}$$

Thus, we simply apply the chain rule:

$$\frac{d}{dx}e^{\ln(a)x} = e^{\ln(a)x} \cdot \frac{d}{dx}[\ln(a)x] = \ln(a)a^x$$

Derivative of the exponential function

$$\frac{d}{dx}a^x = \ln(a)a^x$$

Logarithmic Differentiation

We can use the properties of the logarithm, particularly the natural log, to differentiate more difficult functions, such a products with many terms, quotients of composed functions, or functions with variable or function exponents. We do this by taking the natural logarithm of both sides, re-arranging terms using the logarithm laws below, and then differentiating both sides implicitly, before multiplying through by y .

$$\log(a) + \log(b) = \log(ab) \qquad \log\left(\frac{a}{b}\right) = \log(a) - \log(b) \qquad \log(a^n) = n\log(a)$$

See the examples below.

Example 1

We shall now prove the validity of the power rule using logarithmic differentiation.

$$\frac{d}{dx}\ln(x^n) = n\frac{d}{dx}\ln(x) = nx^{-1}$$

$$\frac{d}{dx}\ln(x^n) = \frac{1}{x^n} \cdot \frac{d}{dx}x^n$$

Thus:

$$\frac{1}{x^n} \cdot \frac{d}{dx}x^n = nx^{-1}$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

Example 2

Suppose we wished to differentiate

$$y = \frac{(6x^2+9)^2}{\sqrt{3x^3-2}}$$

We take the natural logarithm of both sides

$$\begin{aligned}\ln(y) &= \ln\left(\frac{(6x^2+9)^2}{\sqrt{3x^3-2}}\right) \\ &= \ln\left((6x^2+9)^2\right) - \ln\left((3x^3-2)^{\frac{1}{2}}\right) \\ &= 2\ln(6x^2+9) - \frac{\ln(3x^3-2)}{2}\end{aligned}$$

Differentiating implicitly, recalling the chain rule

$$\begin{aligned}\frac{1}{y}\cdot\frac{dy}{dx} &= 2\cdot\frac{12x}{6x^2+9} - \frac{1}{2}\cdot\frac{9x^2}{3x^3-2} \\ &= \frac{24x}{6x^2+9} - \frac{\frac{9}{2}x^2}{3x^3-2} \\ &= \frac{24x(3x^3-2) - \frac{9}{2}x^2(6x^2+9)}{(6x^2+9)(3x^3-2)}\end{aligned}$$

Multiplying by y , the original function

$$\frac{dy}{dx} = \frac{(6x^2+9)^2}{\sqrt{3x^3-2}}\cdot\frac{24x(3x^3-2) - \frac{9}{2}x^2(6x^2+9)}{(6x^2+9)(3x^3-2)}$$

Example 3

Let us differentiate a function

$$y = x^x$$

Taking the natural logarithm of left and right

$$\begin{aligned}\ln(y) &= \ln(x^x) \\ &= x\ln(x)\end{aligned}$$

We then differentiate both sides, recalling the product and chain rules

$$\begin{aligned}\frac{1}{y}\cdot\frac{dy}{dx} &= \ln(x) + x\frac{1}{x} \\ &= \ln(x) + 1\end{aligned}$$

Multiplying by the original function y

$$\frac{dy}{dx} = x^x(\ln(x)+1)$$

Example 4

Take a function

$$y = x^{6\cos(x)}$$

Then

$$\begin{aligned}\ln(y) &= \ln(x^{6\cos(x)})\\ &= 6\cos(x)\ln(x)\end{aligned}$$

We then differentiate

$$\frac{1}{y}\cdot\frac{dy}{dx} = -6\sin(x)\ln(x) + \frac{6\cos(x)}{x}$$

And finally multiply by y

$$\begin{aligned}\frac{dy}{dx} &= y\left(-6\sin(x)\ln(x) + \frac{6\cos(x)}{x}\right)\\ &= 6x^{6\cos(x)}\left(\frac{\cos(x)}{x} - \sin(x)\ln(x)\right)\end{aligned}$$