DERIVATIVE OF THE SINE FUNCTION

Extending a tradition of complex interrelations among the trigonometric functions, it turns out that the derivative of the sine function is the cosine function. (Algebraic relationships known as **trigonometric identities** are often a topic of much discussion in courses on trigonometry.)

> Trigonometry review available in the Field Guide to Functions.

Let us first attempt to simplify the difference quotient for the sine function. Let x and h be real numbers. The very first expression that we encounter in the difference quotient is the quantity sin(x+h). This expression can be rewritten using the **angle addition formula** for the sine function.

$$\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$$

We can now write down the difference quotient and follow our instincts, gathering like terms where they appear.

Simplifying the Difference Quotient

$$\frac{\sin(x+h) - \sin(x)}{h} = \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h}$$

$$= \sin(x)\left(\frac{\cos(h) - 1}{h}\right) + \cos(x)\left(\frac{\sin(h)}{h}\right)$$

In computing the derivative of the sine function, we must find the limit of this expression as h approaches zero. This means that we can treat x (and hence sin(x) and cos(x)) as constants for the purposes of computing this limit. We will require these two basic trigonometric limits.

Trigonometric Limits

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1 \qquad \lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0$$

The first of these limits was discussed in Stages 3 and 4. (See, for example "Limits That Matter" in Stage 3.) The proof that sin(h)/h approaches 1 as h approaches zero is a marvelous argument involving the geometry of the unit circle and the squeeze law for limits. See Stage 4. A variation of the old "conjugate trick" enables one to derive the second limit from the first. This appears as an exercise in the Practice area for this Stage.

With these limits in hand, we see that as h approaches 0, the expression (cos(h) - 1)/h tends toward 0 and sin(h)/h goes to 1. Using the limit laws, and remembering that sin(x) are cos(x) are constant as h approaches zero, we find the derivative of the sine function as follows.

$$\frac{d}{dx}\sin(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \sin(x)\left(\lim_{h \to 0} \frac{\cos(h) - 1}{h}\right) + \cos(x)\left(\lim_{h \to 0} \frac{\sin(h)}{h}\right)$$

$$= \sin(x)(0) + \cos(x)(1)$$

$$= \cos(x)$$

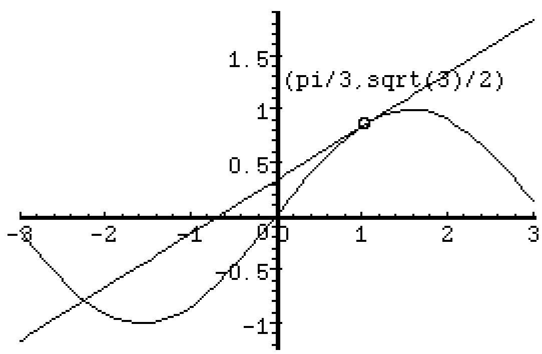
Example. The sine function is differentiable at the domain point x = π/3. We therefore know that the graph of the sine function has a tangent line at the point

(π/3, sin(π/3)) = (π/3, sqrt(3)/2).

Our second basic differentiation formula provides the slope of this tangent line; the slope is the value of the derivative at π/3, namely cos(π/3).

$$\left.\frac{d}{dx}\right|_{x = \pi/3} \sin(x) = \cos(\pi/3) = 1/2$$

The graph of the sine function and the tangent line to this graph at the point (π/3, sqrt(3)/2) are shown below.



A tangent line to y = sin(x)

Next, we verify that the exponential function y=e^x is its own derivative.

/Stage6/Lesson/sine.html

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