1.7 Continuity

Difficulty Level: **Basic** | Created by: CK-12

Learning Objectives

A student will be able to:

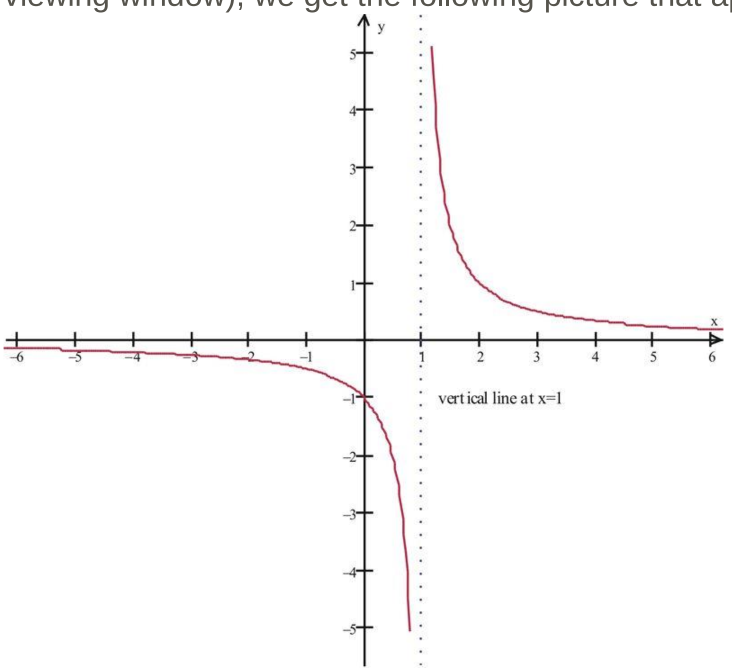
- Learn to examine continuity of functions.
- Find one-sided limits.
- Understand properties of continuous functions.
- Solve problems using the Min-Max theorem.
- Solve problems using the Intermediate Value Theorem.

Introduction

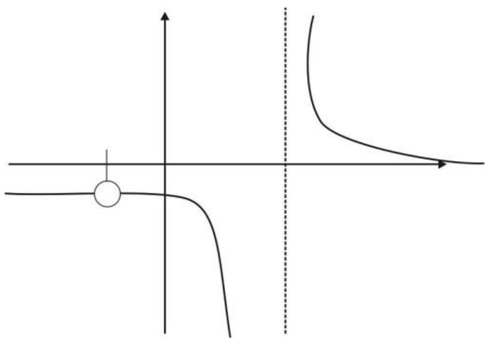
In this lesson we will discuss the property of continuity of functions and examine some very important implications. Let's start with an example of a rational function and observe its graph. Consider the following function:

$f(x)=(x+1)/(x^2-1)$.

We know from our study of domains that in order for the function to be defined, we must use $x\neq-1,1$. Yet when we generate the graph of the function (using the standard viewing window), we get the following picture that appears to be defined at $x=-1$:



The seeming contradiction is due to the fact that our original function had a common factor in the numerator and denominator, $x+1$, that cancelled out and gave us a picture that appears to be the graph of $f(x)=1/(x-1)$.
But what we actually have is the original function, $f(x)=(x+1)/(x^2-1)$, that we know is not defined at $x=-1$. At $x=-1$, we have a hole in the graph, or a discontinuity of the function at $x=-1$. That is, the function is defined for all other x−values close to $x=-1$. Loosely speaking, if we were to hand-draw the graph, we would need to take our pencil off the page when we got to this hole, leaving a gap in the graph as indicated:



Now we will formalize the property of continuity of a function and provide a test for determining when we have continuous functions.

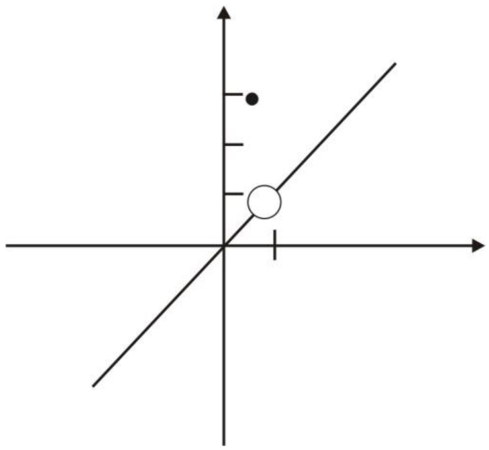
Continuity of a Function

Definition:

The function $f(x)$ is ***continuous at*** $x=a$ if the following conditions all hold:

1. a is in the domain of $f(x)$;
2. $\lim_{x\to a} f(x)$ exists;
3. $\lim_{x\to a} f(x)=f(a)$

Note that it is possible to have functions where two of these conditions are satisfied but the third is not. Consider the piecewise function

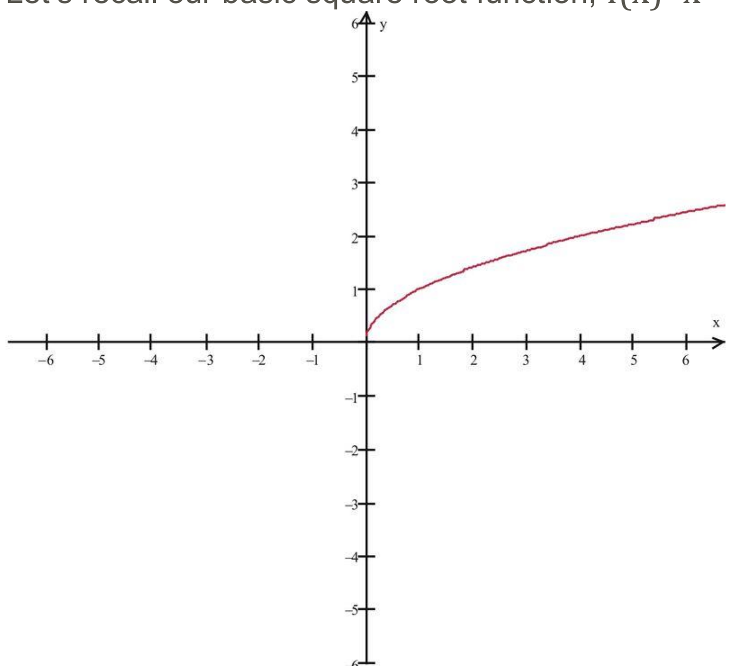


$f(x)=\begin{cases} x, & \text{if } x\neq 1 \\ 3, & \text{if } x=1 \end{cases}$

In this example we have $\lim_{x\to 1} f(x)$ exists, $x=1$ is in the domain of $f(x)$, but $\lim_{x\to 1} f(x)\neq f(1)$.

One-Sided Limits and Closed Intervals

Let's recall our basic square root function, $f(x)=\sqrt{x}$.

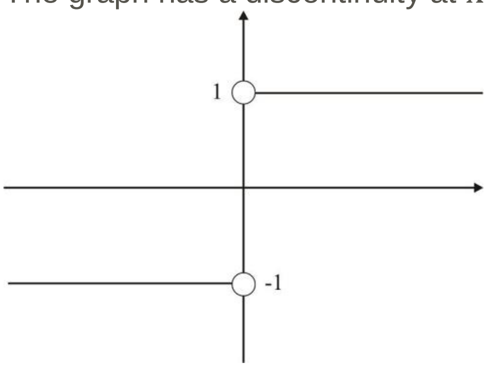


Since the domain of $f(x)=\sqrt{x}$ is $x\geq 0$, we see that that $\lim_{x\to 0}\sqrt{x}$ does not exist. Specifically, we cannot find open intervals around $x=0$ that satisfy the limit definition. However we do note that as we approach $x=0$ from the right-hand side, we see the successive values tending towards $x=0$. This example provides some rationale for how we can define ***one-sided limits***.

Definition:

We say that the ***right-hand*** *limit* of a function $f(x)$ at a is b, written as $\lim_{x\to a^+} f(x)=b$, if for every open interval N of b, there exists an open interval $(a,a+\delta)$ contained in the domain of $f(x)$, such that $f(x)$ is in N for every x in $(a,a+\delta)$.
For the example above, we write $\lim_{x\to 0^+} \sqrt{x}=0$.
Similarly, we say that the ***left-hand limit*** of $f(x)$ at a is b, written as $\lim_{x\to a^-} f(x)=b$, if for every open interval N of b there exists an open interval $(a-\delta,a)$ contained in the domain of $f(x)$, such that $f(x)$ is in N for every x in $(a-\delta,a)$.
Example 1:

Find $\lim_{x\to 0^+} x|x|$.
The graph has a discontinuity at $x=0$ as indicated:



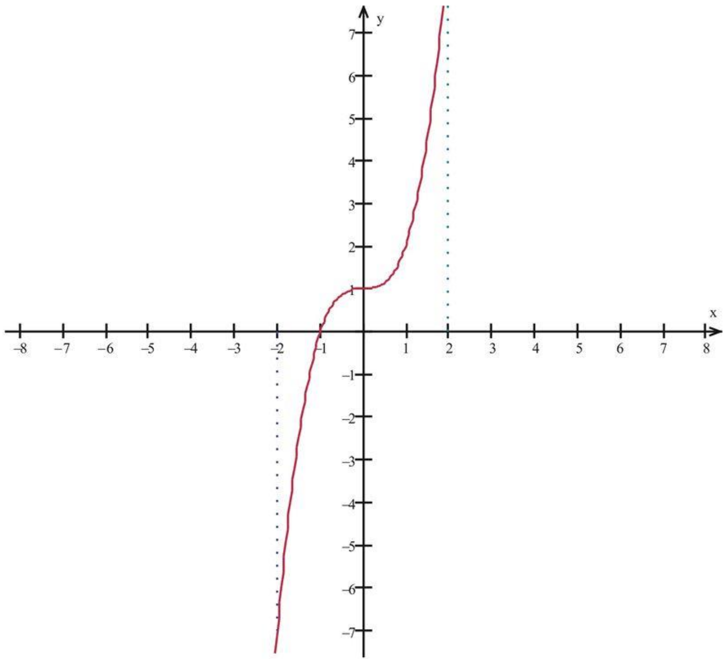
We see that $\lim_{x\to 0^+} x|x|=1$ and also that $\lim_{x\to 0^-} x|x|=-1$.
Properties of Continuous Functions

Let’s recall our example of the limit of composite functions:

$f(x)=1/(x+1)$, $g(x)=-1$.
We saw that $f(g(x))$ is undefined and has the indeterminate form of $1/0$.
Hence $\lim_{x\to -1}(f\circ g)(x)$ does not exist.
In general, we will require that f be continuous at $x=g(a)$ and $x=g(a)$ must be in the domain of $(f\circ g)$ in order for $\lim_{x\to a}(f\circ g)(x)$ to exist.
We will state the following theorem and delay its proof until Chapter 3 when we have learned more about real numbers.

Min-Max Theorem: If a function $f(x)$ is continuous in a closed interval I, then $f(x)$ has both a maximum value and a minimum value in I.
Example 2:

Consider $f(x)=x^3+1$ and interval $I=[-2,2]$.
The function has a minimum value at value at $x=-2$, $f(-2)=-7$,and a maximum value at $x=2$, where $f(2)=9$



We will conclude this lesson with a theorem that will enable us to solve many practical problems such as finding zeros of functions and roots of equations.

Intermediate Value Theorem

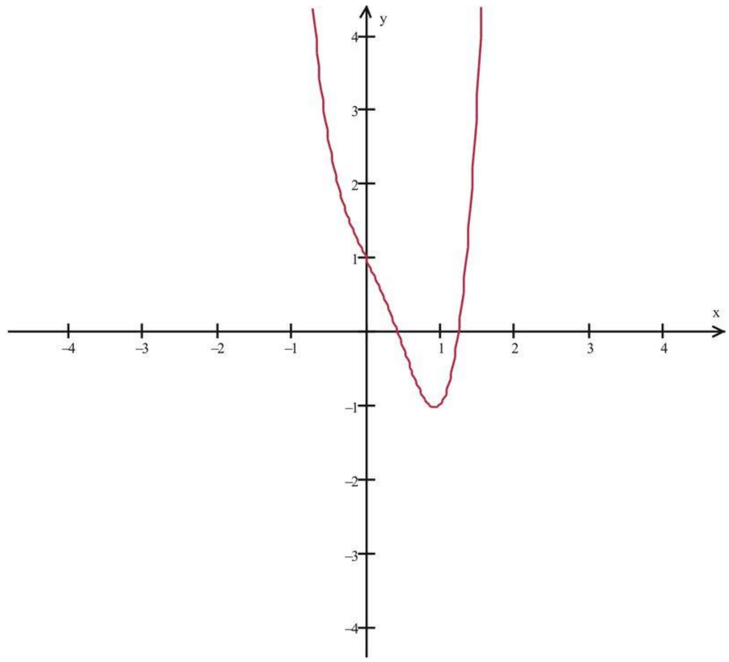
If a function is continuous on a closed interval $[a,b]$, then the function assumes every value between $f(a)$ and $f(b)$.
The proof is left as an exercise with some hints provided. (Homework #10).

We can use the Intermediate Value Theorem to analyze and approximate zeros of functions.

Example 3:

Use the Intermediate Value Function to show that there is at least one zero of the function in the indicated interval.

$f(x)=3x^4-3x^3-2x+1, (1,2)$
We recall that the graph of this function is shaped somewhat like a parabola; viewing the graph in the standard window, we get the following graph:



Of course we could zoom in on the graph to see that the lowest point on the graph lies within the fourth quadrant, but let's use the **[CALC VALUE]**function of the calculator to verify that there is a zero in the interval $(1,2)$.In order to apply the Intermediate Value Theorem, we need to find a pair of $x-$values that have function values with different signs. Let's try some in the table below.

$x1.11.21.3f(x)-.80-.36.37$

We see that the sign of the function values changes from negative to positive somewhere between 1.2 and 1.3. Hence, by the Intermediate Value theorem, there is some value c in the interval $(1.2,1.3)$ such that $f(c)=0$.

Lesson Summary

1. We learned to examine continuity of functions.
2. We learned to find one-sided limits.
3. We observed properties of continuous functions.
4. We solved problems using the Min-Max theorem.
5. We solved problems using the Intermediate Value Theorem.

Review Questions

1. Generate the graph of $f(x)=(|x+1|)/(x+1)$ using your calculator and discuss the continuity of the function.

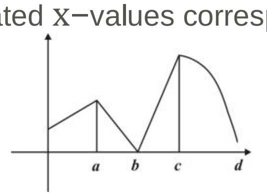
2. Generate the graph of $f(x)=(3x-6)/(x^2-4)$ using your calculator and discuss the continuity of the function.

Compute the limits in #3 - 6.

3. $\lim_{x\to 0^+} x--\surd 1+x--\surd ------\surd -1$
4. $\lim_{x\to 2^-} x^3-8|x-2|(x-2)$
5. $\lim_{x\to 1^+} 2x|x-1|x-1$
6. $\lim_{x\to -2^-} |x+2|+x+2|x+2|-x-2$

In problems 7 and 8, explain how you know that the function has a root in the given interval. (Hint: Use the Intermediate Value Function to show that there is at least one zero of the function in the indicated interval.):

7. $f(x)=x^3+2x^2-x+1$, in the interval $(-2,-3)$
8. $f(x)=x--\surd -x--\surd_3 -1$, in the interval $(9,10)$
9. State whether the indicated $x-$values correspond to maximum or minimum values of the function depicted below.



10. Prove the Intermediate Value Theorem: If a function is continuous on a closed interval $[a,b]$, then the function assumes every value between $f(a)$ and $f(b)$.