Learning Objectives

A student will be able to:

- Summarize the properties of function including intercepts, domain, range, continuity, asymptotes, relative extreme, concavity, points of inflection, limits at infinity.
- Apply the First and Second Derivative Tests to sketch graphs.

Introduction

In this lesson we summarize what we have learned about using derivatives to analyze the graphs of functions. We will demonstrate how these various methods can be applied to help us examine a function's behavior and sketch its graph. Since we have already discussed the various techniques, this lesson will provide examples of using the techniques to analyze the examples of representative functions we introduced in the Lesson on Relations and Functions, particularly rational, polynomial, radical, and trigonometric functions. Before we begin our work on these examples, it may be useful to summarize the kind of information about functions we now can generate based on our previous discussions. Let's summarize our results in a table like the one shown because it provides a useful template with which to organize our findings.

Table Summary

$f(x)$	**Analysis**
Domain and Range	
Intercepts and Zeros	
Asymptotes and limits at infinity	
Differentiability	
Intervals where f ***is increasing***	
Intervals where f ***is decreasing***	
Relative extrema	

Table Summary	
$f(x)$	**Analysis**
Concavity	
Inflection points	

Example 1: ***Analyzing Rational Functions***

Consider the function $f(x)=x^2-4x^2-2x-8$.

General Properties: The function appears to have zeros at $x=\pm 2$.However, once we factor the expression we see
$x^2-4x^2-2x-8=(x+2)(x-2)(x-4)(x+2)=x-2x-4$.
Hence, the function has a zero at $x=2$, there is a hole in the graph at $x=-2$, the domain is $(-\infty,-2)\cup(-2,4)\cup(4,+\infty)$, and the $y-$intercept is at $(0,12)$.

Asymptotes and Limits at Infinity

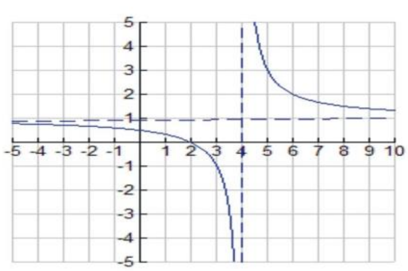
Given the domain, we note that there is a vertical asymptote at $x=4$. To determine other asymptotes, we examine the limit of f as $x\to\infty$ and $x\to-\infty$. We have
$\lim_{x\to\infty}x^2-4x^2-2x-8=\lim_{x\to\infty}x^2x^2-4x^2x^2x^2-2xx^2-8x^2=\lim_{x\to\infty}1-4x^21-2x-8x^2=1$.
Similarly, we see that $\lim_{x\to-\infty}x^2-4x^2-2x-8=1$. We also note that $y\neq 23$since $x\neq-2$.
Hence we have a horizontal asymptote at $y=1$.

Differentiability

$f'(x)=-2x^2-8x-8(x^2-2x-8)=-2(x-4)^2<0$. Hence the function is differentiable at every point of its domain, and since $f'(x)<0$ on its domain, then f is decreasing on its domain, $(-\infty,-2)\cup(-2,4)\cup(4,+\infty)$.
$f''(x)=4(x-4)^3$.
$f''(x)\neq 0$ in the domain of f. Hence there are no relative extrema and no inflection points.
So $f''(x)>0$ when $x>4$. Hence the graph is concave up for $x>4$.
Similarly, $f''(x)<0$ when $x<4$. Hence the graph is concave down for $x<4$, $x\neq-2$.
Let’s summarize our results in the table before we sketch the graph.

Table Summary	
$f(x)=x2-4x2-2x-8$	***Analysis***
Domain and Range	$D=(-\infty,-2)\cup(-2,4)\cup(4,+\infty)$ $R=\{\text{all reals}\neq 1 \text{ or } 23\}$
Intercepts and Zeros	zero at $x=2$, y–intercept at $(0,12)$
Asymptotes and limits at infinity	VA at $x=4$, HA at $y=1$, hole in the graph at $x=-2$
Differentiability	differentiable at every point of its domain
Intervals where f ***is increasing***	nowhere
Intervals where f ***is decreasing***	$(-\infty,-2)\cup(-2,4)\cup(4,+\infty)$
Relative extrema	none
Concavity	concave up in $(4,+\infty)$, concave down in $(-\infty,-2)\cup(-2,4)$
Inflection points	none

Finally, we sketch the graph as follows:



Let's look at examples of the other representative functions we introduced in Lesson 1.2.

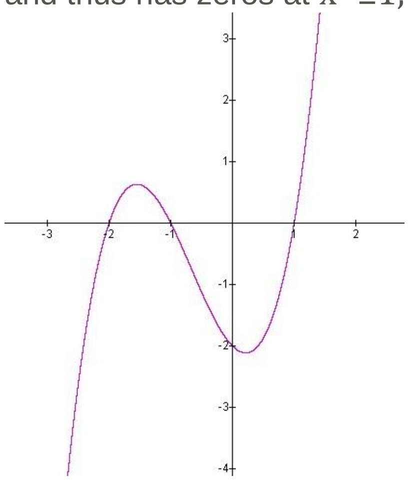
Example 2:

Analyzing Polynomial Functions

Consider the function $f(x)=x^3+2x^2-x-2$.
General Properties

The domain of f is $(-\infty,+\infty)$ and the $y-$intercept at $(0,-2)$.
The function can be factored

$f(x)=x^3+2x^2-x-2=x^2(x+2)-1(x+2)=(x^2-1)(x+2)=(x-1)(x+1)(x+2)$
and thus has zeros at $x=\pm 1,-2$.



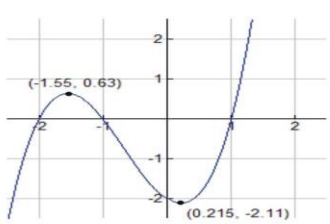
Asymptotes and limits at infinity

Given the domain, we note that there are no vertical asymptotes. We note that $\lim_{x\to\infty} f(x)=+\infty$ and $\lim_{x\to-\infty} f(x)=-\infty$.
Differentiability

$f'(x)=3x^2+4x-1=0$ if $x=-4\pm 28--\sqrt{6}=-2\pm 7-\sqrt{3}$. These are the critical values. We note that the function is differentiable at every point of its domain.
$f'(x)>0$ on $(-\infty,-2-7-\sqrt{3})$ and $(-2+7-\sqrt{3},+\infty)$; hence the function is increasing in these intervals.
Similarly, $f'(x)<0$ on $(-2-7-\sqrt{3},-2+7-\sqrt{3})$ and thus is fdecreasing there.
$f''(x)=6x+4=0$ if $x=-23$, where there is an inflection point.
In addition, $f''(-2-7-\sqrt{3})<0$. Hence the graph has a relative maximum at $x=-2-7-\sqrt{3}$ and located at the point $(-1.55,0.63)$.
We note that $f''(x)<0$ for $x<-23$. The graph is concave down in $(-\infty,-23)$.
And we have $f''(-2+7-\sqrt{3})>0$; hence the graph has a relative minimum at $x=-2+7-\sqrt{3}$ and located at the point $(0.22,-2.11)$.
We note that $f''(x)>0$ for $x>-23$. The graph is concave up in $(-23,+\infty)$.

Table Summary	
$f(x)=x^3+2x^2-x-2$	***Analysis***
Domain and Range	$D=(-\infty,+\infty), R=\{\text{all reals}\}$
Intercepts and Zeros	zeros at $x=\pm1,-2$, y, intercept at $(0,-2)$
Asymptotes and limits at infinity	no asymptotes
Differentiability	differentiable at every point of it's domain
Intervals where f ***is increasing***	$(-\infty,-2-7-\sqrt{3})$ and $(-2+7-\sqrt{3},+\infty)$
Intervals where f ***is decreasing***	$(-2-7-\sqrt{3},-2+7-\sqrt{3})$
Relative extrema	relative maximum at $x=-2-7-\sqrt{3}$ and located at the point $(-1.55,0.63)$; relative minimum at $x=-2+7-\sqrt{3}$ and located at the point $(0.22,-2.11)$.
Concavity	concave up in $(-23,+\infty)$. concave down in $(-\infty,-23)$.
Inflection points	$x=-23$, located at the point $(-23,-.74)$

Here is a sketch of the graph:



Example 3: ***Analyzing Radical Functions***

Consider the function $f(x)=2x-1-----\surd$.

General Properties

The domain of f is $(12,+\infty)$, and it has a zero at $x=12$.

Asymptotes and Limits at Infinity

Given the domain, we note that there are no vertical asymptotes. We note that $\lim_{x\to\infty} f(x)=+\infty$.

Differentiability

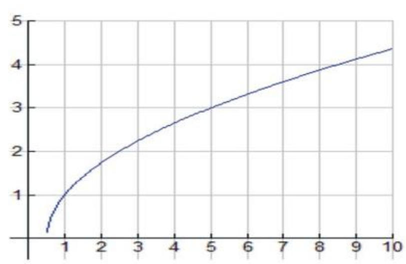
$f'(x)=12x-1-----\surd>0$ for the entire domain of f. Hence f is increasing everywhere in its domain. $f'(x)$ is not defined at $x=12$, so $x=12$ is a critical value.
$f''(x)=-1(2x-1)3--------\surd<0$ everywhere in $(12,+\infty)$. Hence f is concave down in $(12,+\infty)$. $f'(x)$ is not defined at $x=12$, so $x=12$is an absolute minimum.

Table Summary

$f(x)=2x-1-----\surd$	***Analysis***
Domain and Range	$D=(12,+\infty),R=\{y\geq 0\}$
Intercepts and Zeros	zeros at $x=12$, no $y-$intercept
Asymptotes and limits at infinity	no asymptotes
Differentiability	differentiable in $(12,+\infty)$
Intervals where f is increasing	everywhere in $D=(12,+\infty)$
Intervals where f is decreasing	nowhere
Relative extrema	none absolute minimum at $x=12$, located at $(12,0)$
Concavity	concave down in $(12,+\infty)$

Table Summary	
$f(x)=2x-1-----\surd$	***Analysis***
Inflection points	none

Here is a sketch of the graph:



Example 4: ***Analyzing Trigonometric Functions***

We will see that while trigonometric functions can be analyzed using what we know about derivatives, they will provide some interesting challenges that we will need to address. Consider the function $f(x)=x-2\sin x$on the interval $[-\pi,\pi]$.

General Properties

We note that f is a continuous function and thus attains an absolute maximum and minimum in $[-\pi,\pi]$. Its domain is $[-\pi,\pi]$ and its range is $R=\{-\pi\le y\le\pi\}$.

Differentiability

$f'(x)=1-2\cos x=0$ at $x=-\pi3,\pi3$.

Note that $f'(x)>0$ on $(\pi3,\pi)$ and $(-\pi,-\pi3)$; therefore the function is increasing in $(\pi3,\pi)$ and $(-\pi,-\pi3)$.

Note that $f'(x)<0$ on $(-\pi3,\pi3)$; therefore the function is decreasing in $(-\pi3,\pi3)$.

$f''(x)=2\sin x=0$ if $x=0,\pi,-\pi$. Hence the critical values are at $x=-\pi,-\pi3,\pi3,$ and π.

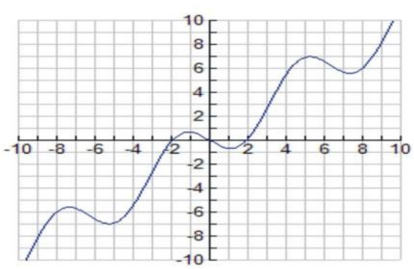
$f''(\pi3)>0$; hence there is a relative minimum at $x=\pi3$.

$f''(-\pi3)<0$; hence there is a relative maximum at $x=-\pi3$.

$f''(x)>0$ on $(0,\pi)$ and $f''(x)<0$ on $(-\pi,0)$. Hence the graph is concave up and decreasing on $(0,\pi)$ and concave down on $(-\pi,0)$.There is an inflection point at $x=0$, located at the point $(0,0)$.

Finally, there is absolute minimum at $x=-\pi$, located at $(-\pi,-\pi)$, and an absolute maximum at $x=\pi$, located at (π,π).

Table Summary	
$f(x)=x-2\sin x$	***Analysis***
Domain and Range	$D=[-\pi,\pi]$, $R=\{-\pi\leq y\leq\pi\}$
Intercepts and Zeros	$x=-\pi 3,\pi 3$
Asymptotes and limits at infinity	no asymptotes
Differentiability	differentiable in $D=[-\pi,\pi]$
Intervals where f ***is increasing***	$(\pi 3,\pi)$ and $(-\pi,-\pi 3)$
Intervals where f ***is decreasing***	$(-\pi 3,\pi 3)$
Relative extrema	relative maximum at $x=-\pi/3$ relative minimum at $x=\pi/3$ absolute maximum at $x=\pi$, located at (π,π) absolute minimum at $x=-\pi$, located at $(-\pi,-\pi)$
Concavity	concave up in $(0,\pi)$
Inflection points	$x=0$, located at the point $(0,0)$



Lesson Summary

1. We summarized the properties of functions, including intercepts, domain, range, continuity, asymptotes, relative extreme, concavity, points of inflection, and limits at infinity.

2. We applied the First and Second Derivative Tests to sketch graphs.

Review Questions

Summarize each of the following functions by filling out the table. Use the information to sketch a graph of the function.

1. $f(x)=x^3+3x^2-x-3$

$f(x)=x^3+3x^2-x-3$	***Analysis***
Domain and Range	
Intercepts and Zeros	
Asymptotes and limits at infinity	
Differentiability	
Intervals where f ***is increasing***	
Intervals where f ***is decreasing***	
Relative extrema	
Concavity	
Inflection points	

2. $f(x)=-x^4+4x^3-4x^2$

$f(x)=-x^4+4x^3-4x^2$	***Analysis***
Domain and Range	
Intercepts and Zeros	
Asymptotes and limits at infinity	

$f(x)=-x^4+4x^3-4x^2$	***Analysis***
Differentiability	
Intervals where f ***is increasing***	
Intervals where f ***is decreasing***	
Relative extrema	
Concavity	
Inflection points	

3. $f(x)=2x-2x^2$

$f(x)=2x-2x^2$	***Analysis***
Domain and Range	
Intercepts and Zeros	
Asymptotes and limits at infinity	
Differentiability	
Intervals where f ***is increasing***	
Intervals where f ***is decreasing***	
Relative extrema	
Concavity	
Inflection points	

4. $f(x)=x-x^{13}$

$f(x)=x-x_{13}$	***Analysis***
Domain and Range	
Intercepts and Zeros	
Asymptotes and limits at infinity	
Differentiability	
Intervals where f ***is increasing***	
Intervals where f ***is decreasing***	
Relative extrema	
Concavity	
Inflection points	

5. f(x)=−2x−6−−−−−√+3

f(x)=−2x−6−−−−−√+3	***Analysis***
Domain and Range	
Intercepts and Zeros	
Asymptotes and limits at infinity	
Differentiability	
Intervals where f ***is increasing***	
Intervals where f ***is decreasing***	

$f(x)=-\sqrt{2x-6}+3$	***Analysis***
Relative extrema	
Concavity	
Inflection points	

6. $f(x)=\sqrt{x^2-2x}$

$f(x)=\sqrt{x^2-2x}$	***Analysis***
Domain and Range	
Intercepts and Zeros	
Asymptotes and limits at infinity	
Differentiability	
Intervals where f ***is increasing***	
Intervals where f ***is decreasing***	
Relative extrema	
Concavity	
Inflection points	

7. $f(x)=1+\cos x$ *on the interval* $[-\pi,\pi]$

$f(x)=1+\cos x$	***Analysis***
Domain and Range	
Intercepts and Zeros	

$f(x)=1+\cos x$	***Analysis***
Asymptotes and limits at infinity	
Differentiability	
Intervals where f ***is increasing***	
Intervals where f ***is decreasing***	
Relative extrema	
Concavity	
Inflection points	