Learning Objectives

A student will be able to:

- Compute by hand the integrals of a wide variety of functions by using the technique of Improper Integration.
- Combine this technique with other integration techniques to integrate.
- Distinguish between proper and improper integrals.

The concept of ***improper integrals*** is an extension to the concept of definite integrals. The reason for the term *improper* is because those integrals either

- include integration over infinite limits or
- the integrand may become infinite within the limits of integration.

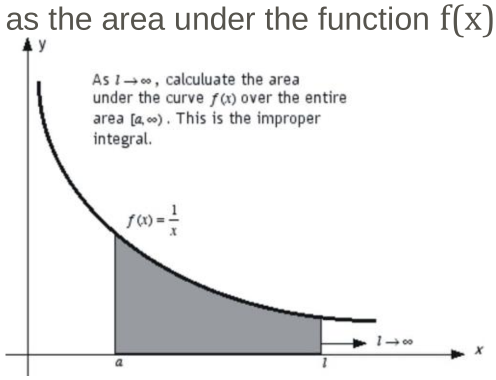
We will take each case separately. Recall that in the definition of definite integral $\int_a^b f(x)dx$ we assume that the interval of integration $[a,b]$ is finite and the function f is continuous on this interval.

Integration Over Infinite Limits

If the integrand f is continuous over the interval $[a,\infty)$, then the improper integral in this case is defined as

$$\int_a^\infty f(x)dx=\lim_{l\to\infty}\int_a^l f(x)dx.$$

If the integration of the improper integral exists, then we say that it ***converges***. But if the limit of integration fails to exist, then the improper integral is said to ***diverge.*** The integral above has an important geometric interpretation that you need to keep in mind. Recall that, geometrically, the definite integral $\int_a^b f(x)dx$ represents the area under the curve. Similarly, the integral $\int_a^l f(x)dx$ is a definite integral that represents the area under the curve $f(x)$ over the interval $[a,l]$, as the figure below shows. However, as l approaches ∞, this area will expand to the area under the curve of $f(x)$ and over the entire interval $[a,\infty)$. Therefore, the improper integral $\int_a^\infty f(x)dx$ can be thought of as the area under the function $f(x)$ over the interval $[a,\infty)$.



Example 1:

Evaluate $\int_1^\infty \frac{dx}{x}$.
Solution:

We notice immediately that the integral is an improper integral because the upper limit of integration approaches infinity. First, replace the infinite upper limit by the finite limit l and take the limit of l to approach infinity:
$$\int_1^\infty \frac{dx}{x} = \lim_{l\to\infty}\int_1^l \frac{dx}{x} = \lim_{l\to\infty}[\ln x]_1^l = \lim_{l\to\infty}(\ln l - \ln 1) = \lim_{l\to\infty}\ln l = \infty.$$
Thus the integral diverges.

Example 2:

Evaluate $\int_2^\infty \frac{dx}{x^2}$.
Solution:

$$\int_2^\infty \frac{dx}{x^2} = \lim_{l\to\infty}\int_2^l \frac{dx}{x^2} = \lim_{l\to\infty}\left[-\frac{1}{x}\right]_2^l = \lim_{l\to\infty}\left(-\frac{1}{l}+\frac{1}{2}\right) = \frac{1}{2}.$$
Thus the integration converges to $\frac{1}{2}$.
Example 3:

Evaluate $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$.
Solution:

What we need to do first is to split the integral into two intervals $(-\infty,0]$and $[0,+\infty)$. So the integral becomes
$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{+\infty} \frac{dx}{1+x^2}.$$
Next, evaluate each improper integral separately. Evaluating the first integral on the right,

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{l\to-\infty}\int_l^0 \frac{dx}{1+x^2} = \lim_{l\to-\infty}[\tan^{-1}x]_l^0 = \lim_{l\to-\infty}[\tan^{-1}0 - \tan^{-1}l] = \lim_{l\to-\infty}\left[0-\left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{2}.$$
Evaluating the second integral on the right,

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{l\to\infty}\int_0^l \frac{dx}{1+x^2} = \lim_{l\to\infty}[\tan^{-1}x]_0^l = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$
Adding the two results,

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$
Remark: In the previous example, we split the integral at $x=0$.However, we could have split the integral at any value of $x=c$ without affecting the convergence or divergence of the integral. The choice is completely arbitrary. This is a famous theorem that we will not prove here. That is,
$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{+\infty} f(x)\,dx.$$

Integrands with Infinite Discontinuities

This is another type of integral that arises when the integrand has a vertical asymptote (an infinite discontinuity) at the limit of integration or at some point in the interval of integration. Recall from Chapter 5 in the Lesson on Definite Integrals that in order for the function f to be integrable, it must be bounded on the interval $[a,b]$. Otherwise, the function is not integrable and thus does not exist. For example, the integral

$$\int_0^4 \frac{dx}{x-1}$$

develops an infinite discontinuity at $x=1$ because the integrand approaches infinity at this point. However, it is continuous on the two intervals $[0,1)$ and $(1,4]$. Looking at the integral more carefully, we may split the interval $[0,4]\to[0,1)\cup(1,4]$ and integrate between those two intervals to see if the integral converges.

$$\int_0^4 \frac{dx}{x-1}=\int_0^1 \frac{dx}{x-1}+\int_1^4 \frac{dx}{x-1}.$$

We next evaluate each improper integral. Integrating the first integral on the right hand side,

$$\int_0^1 \frac{dx}{x-1}=\lim_{l\to 1^-}\int_0^l \frac{dx}{x-1}=\lim_{l\to 1^-}\left[\ln|x-1|\right]_0^l=\lim_{l\to 1^-}\left[\ln|l-1|-\ln|-1|\right]=-\infty.$$

The integral diverges because $\ln(0)$ is undefined, and thus there is no reason to evaluate the second integral. We conclude that the original integral diverges and has no finite value.

Example 4:

Evaluate $\int_1^3 \frac{dx}{\sqrt{x-1}}$.

Solution:

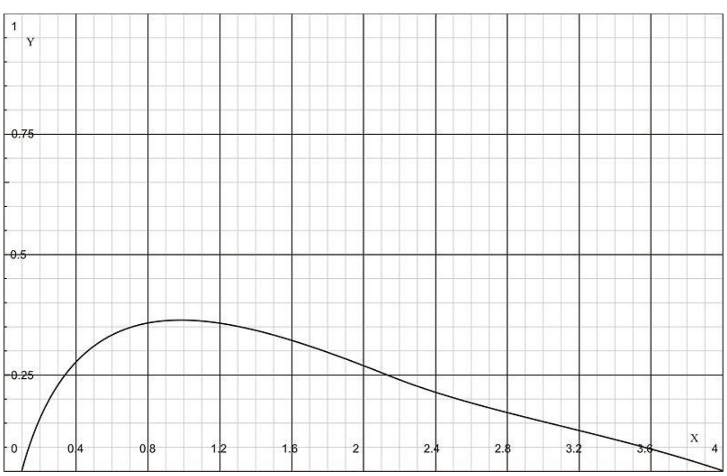
$$\int_1^3 \frac{dx}{\sqrt{x-1}}=\lim_{l\to 1^+}\int_l^3 \frac{dx}{\sqrt{x-1}}=\lim_{l\to 1^+}\left[2\sqrt{x-1}\right]_l^3=\lim_{l\to 1^+}\left[2\sqrt{2}-2\sqrt{l-1}\right]=2\sqrt{2}.$$

So the integral converges to $2\sqrt{2}$.

Example 5:

In Chapter 5 you learned to find the volume of a solid by revolving a curve. Let the curve be $y=xe^{-x}, 0\le x\le\infty$ and revolving about the $x-$axis. What is the volume of revolution?

Solution:



From the figure above, the area of the region to be revolved is given by $A=\pi y^2=\pi x^2 e^{-2x}$. Thus the volume of the solid is

$$V=\pi\int_0^{\infty} x^2 e^{-2x}\,dx=\pi\lim_{l\to\infty}\int_0^{l} x^2 e^{-2x}\,dx.$$

As you can see, we need to integrate by parts twice:

$$\int x^2 e^{-2x}\,dx=-\frac{x^2}{2}e^{-2x}+\int x e^{-2x}\,dx=-\frac{x^2}{2}e^{-2x}-\frac{x}{2}e^{-2x}-\frac{1}{4}e^{-2x}+C.$$

Thus

$$V=\pi\lim_{l\to\infty}\left[-\frac{x^2}{2}e^{-2x}-\frac{x}{2}e^{-2x}-\frac{1}{4}e^{-2x}\right]_0^{l}=\pi\lim_{l\to\infty}\left[\frac{2x^2+2x+1}{-4e^{2x}}\right]_0^{l}=\pi\lim_{l\to\infty}\left[\frac{2l^2+2l+1}{-4e^{2l}}-\frac{1}{-4e^0}\right]=\pi\lim_{l\to\infty}\left[\frac{2l^2+2l+1}{4e^{2l}}+\frac{1}{4}\right].$$

At this stage, we take the limit as l approaches infinity. Notice that the when you substitute infinity into the function, the denominator of the expression $\frac{2l^2+2l+1}{-4e^{2l}}$, being an exponential function, will approach infinity at a much faster rate than will the numerator. Thus this expression will approach zero at infinity. Hence

$$V=\pi\left[0+\frac{1}{4}\right]=\frac{\pi}{4},$$

So the volume of the solid is $\pi/4$.

Example 6:

Evaluate $\int_{-\infty}^{+\infty}\frac{dx}{e^x+e^{-x}}$.

Solution:

This can be a tough integral! To simplify, rewrite the integrand as

$$\frac{1}{e^x+e^{-x}}=\frac{1}{e^{-x}(e^{2x}+1)}=\frac{e^x}{e^{2x}+1}=\frac{e^x}{1+(e^x)^2}.$$

Substitute into the integral:

$$\int\frac{dx}{e^x+e^{-x}}=\int\frac{e^x}{1+(e^x)^2}\,dx.$$

Using $u-$substitution, let $u=e^x, du=e^x dx$.

$$\int\frac{dx}{e^x+e^{-x}}=\int\frac{du}{1+u^2}=\tan^{-1}u+C=\tan^{-1}e^x+C.$$

Returning to our integral with infinite limits, we split it into two regions. Choose as the split point the convenient $x=0$.

$$\int_{-\infty}^{+\infty}\frac{dx}{e^x+e^{-x}}=\int_{-\infty}^{0}\frac{dx}{e^x+e^{-x}}+\int_{0}^{+\infty}\frac{dx}{e^x+e^{-x}}.$$

Taking each integral separately,

$$\int_{-\infty}^{0}\frac{dx}{e^x+e^{-x}}=\lim_{l\to-\infty}\int_{l}^{0}\frac{dx}{e^x+e^{-x}}=\lim_{l\to-\infty}\left[\tan^{-1}e^x\right]_l^0=\lim_{l\to-\infty}\left[\tan^{-1}e^0-\tan^{-1}e^l\right]=\frac{\pi}{4}-0=\frac{\pi}{4}.$$

Similarly,

$$\int_{0}^{+\infty}\frac{dx}{e^x+e^{-x}}=\lim_{l\to\infty}\int_{0}^{l}\frac{dx}{e^x+e^{-x}}=\lim_{l\to\infty}\left[\tan^{-1}e^x\right]_0^l=\lim_{l\to\infty}\left[\tan^{-1}e^l-\tan^{-1}1\right]=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}.$$

Thus the integral converges to

$$\int_{-\infty}^{+\infty}\frac{dx}{e^{x}+e^{-x}}=\frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}.$$

For a video presentation of Improper Integrals with Infinity in the Upper and Lower Limits **(22.0)**, see [Improper Integrals, www.justmathtutoring.com] (7:55).

Review Questions

1. Determine whether the following integrals are improper. If so, explain why.
   a. $\int_{1}^{7}\frac{x+2}{x-3}dx$
   b. $\int_{1}^{7}\frac{x+2}{x+3}dx$
   c. $\int_{0}^{1}\ln x\,dx$
   d. $\int_{0}^{\infty}\frac{1}{\sqrt{x-2}}dx$
   e. $\int_{0}^{\pi/4}\tan x\,dx$

Evaluate the integral or state that it diverges.

2. $\int_{1}^{\infty}\frac{1}{x^{2.001}}dx$
3. $\int_{-\infty}^{-2}\left[\frac{1}{x-1}-\frac{1}{x+1}\right]dx$
4. $\int_{-\infty}^{0}e^{5x}dx$
5. $\int_{3}^{5}\frac{1}{(x-3)^{4}}dx$
6. $\int_{-\pi/2}^{\pi/2}\tan x\,dx$
7. $\int_{0}^{1}\frac{1}{\sqrt{1-x^{2}}}dx$
8. The region between the $x-$axis and the curve $y=e^{-x}$ for $x\geq 0$ is revolved about the $x-$axis.
   a. Find the volume of revolution, V.
   b. Find the surface area of the volume generated, S.