Learning Objectives

A student will be able to:

- Compute by hand the integrals of a wide variety of functions by using the Trigonometric Integrals.
- Combine this technique with $u-$substitution.

Integrating Powers of Sines and Cosines

In this section we will study methods of integrating functions of the form

$\int \sin^m x \cos^n x dx$,

where m and n are nonnegative integers. The method that we will describe uses the famous trigonometric identities

$\sin^2 x=12(1-\cos 2x)$ and

$\cos^2 x=12(1+\cos 2x)$.

Example 1:

Evaluate $\int \sin^2 x dx$ and $\int \cos^2 x dx$.

Solution:

Using the identities above, the first integral can be written as

$$\int \sin^2 x dx=\int 12(1-\cos 2x)dx=12\int (1-\cos 2x)dx=12(x-12\sin 2x)+C=x2-14\sin 2x+C.$$

Similarly, the second integral can be written as

$$\int \cos^2 x dx=\int 12(1+\cos 2x)dx=12\int (1+\cos 2x)dx=12(x+12\sin 2x)+C=x2+14\sin 2x+C.$$

Example 2:

Evaluate $\int \cos^4 x dx$.

Solution:

$$\int \cos^4 x dx=\int (\cos^2 x)^2 dx=\int (12(1+\cos 2x))^2 dx=14\int (1+2\cos 2x+\cos^2 2x)dx=14\int (1+2\cos 2x+12+12\cos 4x)dx=14\int (32+2\cos 2x+12\cos 4x)dx.$$

Integrating term by term,

$$=14[32x+\sin 2x+18\sin 4x]+C=38x+14\sin 2x+132\sin 4x+C.$$

Example 3:

Evaluate $\int \sin^3 x dx$.

Solution:

$$\int \sin^3 x dx=\int \sin^2 x \sin x dx$$

Recall that $\sin^2 x+\cos^2 x=1$, so by substitution,
$=\int(1-\cos^2 x)\sin x\,dx=\int\sin x\,dx-\int\cos^2 x\sin x\,dx$.
The first integral should be straightforward. The second can be done by the method of u−substitution by letting $u=\cos x$, so $du=-\sin x\,dx$.The integral becomes
$=-\cos x-\int\left[-u^2\sin x\frac{du}{\sin x}\right]=-\cos x+\int u^2\,du=-\cos x+\frac{u^3}{3}+C=-\cos x+\frac{1}{3}\cos^3 x+C$.
If m and n are both positive integers, then an integral of the form
$\int\sin^m x\cos^n x\,dx$
can be evaluated by one of the procedures shown in the table below, depending on whether m and n are odd or even.

$\int\sin^m x\cos^n x\,dx$	**Procedure**	**Identities**
n odd	Let $u=\sin x$	$\cos^2 x=1-\sin^2 x$
m odd	Let $u=\cos x$	$\sin^2 x=1-\cos^2 x$
n and m even	Use identities to reduce powers	$\sin^2 x=(1/2)(1-\cos 2x)$ $\cos^2 x=(1/2)(1+\cos 2x)$

Example 4:

Evaluate $\int\sin^3 x\cos^4 x\,dx$.
Solution:

Here, m is odd. So according to the second procedure in the table above,
let $u=\cos x$, so $du=-\sin x$. Substituting,
$\int\sin^3 x\cos^4 x\,dx=\int u^4\sin^3 x\frac{-1}{\sin x}du=-\int u^4\sin^2 x\,du$.
Referring to the table again, we can now substitute $\sin^2 x=1-\cos^2 x$in the integral:
$=-\int u^4(1-\cos^2 x)du=-\int u^4(1-u^2)du=\int(-u^4+u^6)du=-\frac{1}{5}u^5+\frac{1}{7}u^7+C=-\frac{1}{5}\cos^5 x+\frac{1}{7}\cos^7 x+C$.
Example 5:

Evaluate $\int\sin^4 x\cos^4 x\,dx$.
Solution:

Here, $m=n=4$. We follow the third procedure in the table above:
$\int\sin^4 x\cos^4 x\,dx=\int(\sin^2 x)^2(\cos^2 x)^2dx=\int\left[\frac{1}{2}(1-\cos 2x)\right]^2\left[\frac{1}{2}(1+\cos 2x)\right]^2dx=\frac{1}{16}\int(1-\cos^2 2x)^2dx=\frac{1}{16}\int\sin^4 2x\,dx$.
At this stage, it is best to use u−substitution to integrate. Let $u=2x$, so $du=2dx$.
$\int\sin^4 x\cos^4 x\,dx=\frac{1}{32}\int\sin^4 u\,du=\frac{1}{32}\int(\sin^2 u)^2du=\frac{1}{32}\int\left[\frac{1}{2}(1-\cos 2u)\right]^2du=\frac{1}{32}\left(\frac{3}{8}u-\frac{1}{4}\sin 2u+\frac{1}{32}\sin 4u\right)+C=\frac{3}{256}x-\frac{1}{128}\sin 4x+\frac{1}{1024}\sin 8x+C$.

Integrating Powers of Secants and Tangents

In this section we will study methods of integrating functions of the form

$$\int \tan^m x \sec^n x\,dx,$$

where m and n are nonnegative integers. However, we will begin with the integrals

$$\int \tan x\,dx$$

and

$$\int \sec x\,dx.$$

The first integral can be evaluated by writing

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx.$$

Using $u-$substitution, let $u=\cos x$, so $du=-\sin x\,dx$. The integral becomes

$$\int \tan x\,dx\,du = \int \frac{\sin x}{u}\,\frac{-1}{\sin x} = -\int \frac{1}{u}\,du = -\ln|u|+C = -\ln|\cos x|+C = \ln(1/|\cos x|)+C = \ln|\sec x|+C.$$

The second integral $\int \sec x\,dx$, however, is not straightforward—it requires a trick. Let

$$\int \sec x\,dx = \int \sec x\,\frac{\sec x+\tan x}{\sec x+\tan x}\,dx = \int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}\,dx.$$

Use $u-$substitution. Let $u=\sec x+\tan x$, then $du=(\sec^2 x+\sec x\tan x)dx$, the integral becomes,

$$\int \sec x\,dx = \int \frac{du}{u} = \ln|u|+C = \ln|\sec x+\tan x|+C.$$

There are two reduction formulas that help evaluate higher powers of tangent and secant:

$$\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx, \qquad \int \tan^m x\,dx = \frac{\tan^{m-1}x}{m-1} - \int \tan^{m-2}x\,dx.$$

Example 6:

Evaluate $\int \sec^3 x\,dx$.

Solution:

We use the formula above by substituting for $n=3$.

$$\int \sec^3 x\,dx = \frac{\sec x\tan x}{3-1} + \frac{3-2}{3-1}\int \sec x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\int \sec x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x+\tan x|+C.$$

Example 7:

Evaluate $\int \tan^5 x\,dx$.

Solution:

We use the formula above by substituting for $m=5$.

$$\int \tan^5 x\,dx = \frac{\tan^4 x}{4} - \int \tan^3 x\,dx.$$

We need to use the formula again to solve the integral $\int \tan^3 x\,dx$:

$$\int \tan^5 x\,dx = \frac{\tan^4 x}{4} - \int \tan^3 x\,dx = \frac{\tan^4 x}{4} - \left[\frac{\tan^2 x}{2} - \int \tan x\,dx\right] = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x|+C.$$

If m and n are both positive integers, then an integral of the form

$\int \tan^m x \sec^n x dx$
can be evaluated by one of the procedures shown in the table below, depending on whether m and n are odd or even.

$\int \tan^m x \sec^n x dx$	**Procedure**	**Identities**
n even	Let $u=\tan x$	$\sec^2 x=\tan^2 x+1$
m odd	Let $u=\sec x$	$\tan^2 x=\sec^2 x-1$
m even n odd	Reduce powers of $\sec x$	$\tan^2 x=\sec^2 x-1$

Example 8:

Evaluate $\int \tan^2 x \sec^4 x dx$.
Solution:

Here $n=4$ is even, and so we will follow the first procedure in the table above.
Let $u=\tan x$, so $du=\sec^2 x dx$. Before we substitute, split off a factor of $\sec^2 x$.
$\int \tan^2 x \sec^4 x dx=\int \tan^2 x \sec^2 x \sec^2 x dx$.
Since $\sec^2 x=\tan^2 x+1$,
$=\int \tan^2 x(\tan^2 x+1)\sec^2 x dx$.
Now we make the $u-$substitution:
$=\int u^2(u^2+1)du=15u^5+13u^3+C=15\tan^5 x+13\tan^3 x+C$.
Example 9:

Evaluate $\int \tan^3 x \sec^3 x dx$.
Solution:

Here $m=3$ is odd. We follow the second procedure in the table. Make the substitution, $u=\sec x$ and $du=\sec x \tan x dx$. Our integral becomes
$\int \tan^3 x \sec^3 x dx=\int \tan^2 x \sec^2 x(\sec x \tan x)dx=\int(\sec^2 x-1)\sec^2 x(\sec x \tan x)dx=\int(u^2-1)u^2 du=15u^5-13u^3+C=15\sec^5 x-13\sec^3 x+C$.

Review Questions

Evaluate the integrals.

1. $\int \cos^4 x \sin x dx$
2. $\int \sin^2 5\phi d\phi$
3. $\int \sin^2 2z \cos^3 2z dz$

4. $\int \sin x \cos(x/2)dx$
5. $\int \sec^4 x \tan^3 x dx$
6. $\int \tan^4 x \sec x dx$
7. $\int \sqrt{\tan x} \sec^4 x dx$
8. $\int_0^{\pi/2} \tan^5 \frac{x}{2} dx$
9. Graph and then find the volume of the solid that results when the region enclosed by $y=\sin x, y=\cos x, x=0$, and $x=\pi/4$ is revolved around the $x-$axis.
   a. Prove that $\int \csc x dx = -\ln|\csc x + \cot x| + C$
   b. Show that it can also be written in the following two forms: $\int \csc x dx = \ln|\tan \frac{1}{2}x| + C = \ln|\csc x - \cot x| + C$.