Learning Objectives

A student will be able to:

- Compute by hand the integrals of a wide variety of functions by using technique of Integration by Partial Fractions.
- Combine the technique of partial fractions with $u-$substitution to solve various integrals.

This is the third technique that we will study. This technique involves decomposing a rational function into a sum of two or more simple rational functions. For example, the rational function

$$x+4x^2+x-2$$

can be decomposed into

$$x+4x^2+x-2=2x+2+3x-1.$$

The two partial sums on the right are called *partial factions.* Suppose that we wish to integrate the rational function above. By decomposing it into two partial fractions, the integral becomes manageable:

$$\int x+4x^2+x-2dx=\int(2x+2+3x-1)=2\int 1x+1dx+3\int 1x-1dx=2\ln|x+1|+3\ln|x-1|+C.$$

To use this method, we must be able to factor the denominator of the original function and then decompose the rational function into two or more partial fractions. The examples below illustrate the method.

Example 1:

Find the partial fraction decomposition of

$$2x-19x^2+x-6.$$

Solution:

We begin by factoring the denominator as $x^2+x-6=(x+3)(x-2)$.Then write the partial fraction decomposition as

$$2x-19x^2+x-6=Ax+3+Bx-2.$$

Our goal at this point is to find the values of A and B. To solve this equation, multiply both sides of the equation by the factored denominator $(x+3)(x-2)$. This process will produce the *basic equation.*

$$2x-19=A(x-2)+B(x+3).$$

This equation is true for all values of x. The most convenient values are the ones that make a factor equal to zero, namely, $x=2$ and $x=-3$.Substituting $x=2$,

$$2(2)-19-15-3=A(2-2)+B(2+3)=0+5B=B$$

Similarly, substituting for $x=-3$ into the basic equation we get

$$2(-3)-19-255=A(-3-2)+B(-3+3)=-5A+0=A$$

We have solved the basic equation by finding the values of A and B.Therefore, the partial fraction decomposition is
$2x-19x^2+x-6=5x+3-3x-2$.

General Description of the Method

To be able to write a rational function $f(x)/g(x)$ as a sum of partial fractions, we must apply two conditions:

- The degree of $f(x)$ must be less than the degree of $g(x)$. If so, the rational function is called *proper*. If it is not, divide $f(x)$ by $g(x)$ (use long division) and work with the remainder term.
- The factors of $g(x)$ are known. If not, you need to find a way to find them. The guide below shows how you can write $f(x)/g(x)$ as a sum of partial fractions if the factors of $g(x)$ are known.

A Guide to Finding Partial Fractions Decomposition of a Rational Function

1. To find the partial fraction decomposition of a proper rational function, $f(x)/g(x)$, factor the denominator $g(x)$ and write an equation that has the form
$f(x)g(x)=(\text{sum of partial fractions.})$
2. For each distinct factor $ax+b$, the right side must include a term of the form
$Aax+b$.
3. For each repeated factor $(ax+b)^n$, the right side must include n terms of the form
$A_1(ax+b)+A_2(ax+b)^2+A_3(ax+b)^3+\ldots+A_n(ax+b)^n$.

Example 2:

Use the method of partial fractions to evaluate $\int x+1(x+2)^2dx$.

Solution:

According to the guide above (item #3), we must assign the sum of $n=2$partial sums:
$x+1(x+2)^2=A(x+2)+B(x+2)^2$.
Multiply both sides by $(x+2)^2$:
$x+1x+1=A(x+2)+B=Ax+(2A+B)$.
Equating the coefficients of like terms from both sides,

$11=A=2A+B$.
Thus

$AB=1.=-1$.
Therefore the partial fraction decomposition is

$x+1(x+2)^2=1x+2-1(x+2)^2$.
The integral will become

$\int x+1(x+2)^2dx=\int(1x+2-1(x+2)^2)=\int 1x+2dx-\int 1(x+2)^2dx=\ln|x+2|+1x+2+C,$
where we have used $u-$substitution for the second integral.
Example 3:

Evaluate $\int 3x^2+3x+1x^3+2x^2+xdx$.
Solution:

We begin by factoring the denominator as $x(x+1)^2$. Then the partial fraction decomposition is
$3x^2+3x+1x^3+2x^2+x=Ax+Bx+1+C(x+1)^2.$
Multiplying each side of the equation by $x(x+1)^2$ we get the basic equation
$3x^2+3x+1=A(x+1)^2+Bx(x+1)+Cx.$
This equation is true for all values of x. The most convenient values are the ones that make a factor equal to zero, namely, $x=-1$ and $x=0$.
Substituting $x=-1$,
$3(-1)^2+3(-1)+11-1=A(-1+1)^2+B(-1)(-1+1)+C(-1)=0+0-C=C.$
Substituting $x=0$,
$3(0)^2+3(0)+111=A(0+1)^2+B(0)(0+1)+C(0)=A+0+0=A.$
To find B we can simply substitute any value of x along with the values of A and C obtained.
Choose $x=1$:
$3(1)^2+3(1)+172=A(1+1)^2+B(1)(1+1)+C(1)=4+2B-1=B.$
Now we have solved for A,B, and C. We use the partial fraction decomposition to integrate.
$\int 3x^2+3x+1x^3+2x^2+xdx.=\int(1x+2x+1-1(x+1)^2)dx=\ln|x|+2\ln|x+1|+1x+1+C.$
Example 4:

This problem is an example of an improper rational function. Evaluate the definite integral

$\int_1^2 x^3-4x^2-3x+3x^2-3xdx.$
Solution:

This rational function is improper because its numerator has a degree that is higher than its denominator. The first step is to divide the denominator into the numerator by long division and obtain

$x^3-4x^2-3x+3x^2-3x=(x-1)+-6x+3x^2-3x.$
Now apply partial function decomposition only on the remainder,

$-6x+3x^2-3x=-6x+3x(x-3)=Ax+Bx-3.$

As we did in the previous examples, multiply both sides by $x(x-3)$ and then set $x=0$ and $x=3$ to obtain the basic equation

$-6x+3=A(x-3)+Bx$

For $x=0$,

$3-1=-3A+0=A.$

For $x=3$,

$-18+3-15-5=0+3B=3B=B.$

Thus our integral becomes

$$\int_1^2 x^3-4x^2-3x+3x^2-3x\,dx=\int_1^2[(x-1)+-6x+3x^2-3x]dx=\int_1^2[x-1-1x-5x-3]dx.$$

Integrating and substituting the limits,

$$=[x^2 2-x-\ln|x|-5\ln|x-3|]_1^2=(42-2-\ln2-5\ln1)-(12-1-\ln1-5\ln2)=4\ln2+12.$$

Review Questions

Evaluate the following integrals.

1. $\int 1x^2-1dx$
2. $\int xx^2-2x-3dx$
3. $\int 1x^3+x^2-2xdx$
4. $\int x^3x^2+4dx$
5. $\int_0^1 \phi 1+\phi d\phi$
6. $\int_1^5 x-1x^2(x+1)dx$
7. Evaluate the integral by making the proper $u-$substitution to convert to a rational function: $\int \cos\theta\sin^2\theta+4\sin\theta-5d\theta.$
8. Evaluate the integral by making the proper $u-$substitution to convert to a rational function: $\int 3e^\theta e^{2\theta}-1d\theta.$
9. Find the area under the curve $y=1/(2+e^x)$, over the interval $[-\ln3,\ln4]$. (*Hint*: make a $u-$substitution to convert the integrand into a rational function.)
10. Show that $\int 1a^2-x^2dx=12a\ln|||a+xa-x|||+C.$