Learning Objectives

A student will be able to:

- Compute by hand the integrals of a wide variety of functions by using technique of Integration by Parts.
- Combine this technique with the $u-$substitution method to solve integrals.
- Learn to tabulate the technique when it is repeated.

In this section we will study a technique of integration that involves the product of algebraic and exponential or logarithmic functions, such as

$\int x \ln x dx$
and

$\int x e^x dx$.
Integration by parts is based on the product rule of differentiation that you have already studied:

$\frac{d}{dx}[uv]=u\frac{dv}{dx}+v\frac{du}{dx}$.
If we integrate each side,

$uv=\int u\frac{dv}{dx}dx+\int v\frac{du}{dx}dx=\int udv+\int vdu$.
Solving for $\int udv$,
$\int udv=uv-\int vdu$.
This is the formula for integration by parts. With the proper choice of $u$and dv, the second integral may be easier to integrate. The following examples will show you how to properly choose u and dv.
Example 1:

Evaluate $\int x \sin x dx$.
Solution:

We use the formula $\int udv=uv-\int vdu$.
Choose

$u=x$
and

$dv=\sin x dx$.
To complete the formula, we take the differential of u and the simplest antiderivative of $dv=\sin x dx$.
$duv=dx=-\cos x$.

The formula becomes

$$\int x\sin x\,dx = -x\cos x - \int(-\cos x)\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C.$$

A Guide to Integration by Parts

Which choices of u and dv lead to a successful evaluation of the original integral? In general, choose u to be something that simplifies when differentiated, and dv to be something that remains manageable when integrated. Looking at the example that we have just done, we chose $u=x$ and $dv=\sin x\,dx$. That led to a successful evaluation of our integral. However, let's assume that we made the following choice,

$u dv=\sin x = x dx.$

Then

$du v=\cos x dx = x^2/2.$

Substituting back into the formula to integrate, we get

$$\int u\,dv = uv - \int v\,du = \sin x \frac{x^2}{2} - \int \frac{x^2}{2}\cos x\,dx$$

As you can see, this integral is worse than what we started with! This tells us that we have made the wrong choice and we must change (in this case switch) our choices of u and dv.

Remember, the goal of the integration by parts is to start with an integral in the form $\int u\,dv$ that is hard to integrate directly and change it to an integral $\int v\,du$ that looks easier to evaluate. However, here is a general guide that you may find helpful:

1. Choose dv to be the more complicated portion of the integrand that fits a basic integration formula. Choose u to be the remaining term in the integrand.
2. Choose u to be the portion of the integrand whose derivative is simpler than u. Choose dv to be the remaining term.

Example 2:

Evaluate $\int xe^x\,dx$.

Solution:

Again, we use the formula $\int u\,dv = uv - \int v\,du$.

Let us choose

$u=x$

and

$dv=e^x dx.$

We take the differential of u and the simplest antiderivative of $dv=e^x dx$:

$du v=dx = e^x.$

Substituting back into the formula,

$\int udv=uv-\int vdu=xe^x-\int e^xdx.$
We have made the right choice because, as you can see, the new integral $\int vdu=\int e^xdx$ is definitely simpler than our original integral. Integrating, we finally obtain our solution
$\int xe^xdx=xe^x-e^x+C.$
Example 3:

Evaluate $\int \ln x dx$.
Solution:

Here, we only have one term, $\ln x$. We can always assume that this term is multiplied by 1:
$\int \ln x 1 dx.$
So let $u=\ln x$, and $dv=1dx$. Thus $du=1/xdx$ and $v=x$. Substituting,
$\int udv \int \ln x dx=uv-\int vdu=x\ln x-\int x 1 x dx=x\ln x-\int 1dx=x\ln x-x+C.$

Repeated Use of Integration by Parts

Oftentimes we use integration by parts more than once to evaluate the integral, as the example below shows.

Example 4:

Evaluate $\int x^2e^xdx$.
Solution:

With $u=x^2, dv=e^xdx, du=2xdx$, and $v=e^x$, our integral becomes
$\int x^2e^xdx=x^2e^x-2\int xe^xdx.$
As you can see, the integral has become less complicated than the original, $x^2e^x\to xe^x$. This tells us that we have made the right choice. However, to evaluate $\int xe^xdx$ we still need to integrate by parts with $u=x$ and $dv=e^xdx$. Then $du=dx$ and $v=e^x$, and
$\int x^2e^xdx=x^2e^x-2\int xe^xdx=x^2e^x-2[uv-\int udv]=x^2e^x-2[xe^x-\int e^xdx]=x^2e^x-2xe^x+2e^x+C.$
Actually, the method that we have just used works for any integral that has the form $\int x^ne^xdx$, where n is a positive integer. The following section illustrates a systematic way of solving repeated integrations by parts.

Tabular Integration by Parts

Sometimes, we need to integrate by parts several times. This leads to cumbersome calculations. In situations like these it is best to organize our calculations to save us a great deal of tedious work and to avoid making unpredictable mistakes. The example below illustrates the method of ***tabular integration.***

Example 5:

Evaluate $\int x^2 \sin 3x dx$.
Solution:

Begin as usual by letting $u=x^2$ and $dv=\sin 3x dx$. Next, create a table that consists of three columns, as shown below:

Alternate signs	u **and its derivatives**	dv **and its antiderivatives**
+	$x^2 \searrow$	$\sin 3x$
−	$2x \searrow$	$-13\cos 3x$
+	$2 \searrow$	$-19\sin 3x$
−	0	$127\cos 3x$

To find the solution for the integral, pick the sign from the first row $(+)$,multiply it by u of the first row (x^2) and then multiply by the dv of the second row, $-1/3\cos 3x$ (watch the direction of the arrows.) This is the first term in the solution. Do the same thing to obtain the second term: Pick the sign from the second row, multiply it by the u of the same row and then follow the arrow to multiply the product by the dv in the third row. Eventually we obtain the solution
$\int x^2 \sin 3x dx=-13x^2\cos 3x+29x\sin 3x+227\cos 3x+C$.

Solving for an Unknown Integral

There are some integrals that require us to evaluate two integrations by parts, followed by solving for the unknown integral. These kinds of integrals crop up often in electrical engineering and other disciplines.

Example 6:

Evaluate $\int e^x \cos x dx$.
Solution:

Let $u=e^x$, and $dv=\cos x dx$. Then $du=e^x dx, v=\sin x$, and
$\int e^x \cos x dx=e^x \sin x-\int e^x \sin x dx$.
Notice that the second integral looks the same as our original integral in form, except that it has a $\sin x$ instead of $\cos x$. To evaluate it, we again apply integration by parts to the second term with $u=e^x, dv=\sin x dx, v=-\cos x$, and $du=e^x dx$.
Then

$\int e^x \cos x dx=e^x \sin x-[-e^x \cos x-\int(-\cos x)(e^x dx)]=e^x \sin x-e^x \cos x-\int e^x \cos x dx$.

Notice that the unknown integral now appears on both sides of the equation. We can simply move the unknown integral on the right to the left side of the equation, thus adding it to our original integral:

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x + C.$$

Dividing both sides by 2, we obtain

$$\int e^x \cos x\,dx = \frac{1}{2}e^x \sin x + \frac{1}{2}e^x \cos x + \frac{1}{2}C.$$

Since the constant of integration is just a "dummy" constant, let $\frac{C}{2} \to C$.

Finally, our solution is

$$\int e^x \cos x\,dx = \frac{1}{2}e^x \sin x + \frac{1}{2}e^x \cos x + C.$$

Review Questions

Evaluate the following integrals. (*Remark:* Integration by parts is not necessarily a requirement to solve the integrals. In some, you may need to use $u-$substitution along with integration by parts.)

1. $\int 3xe^x\,dx$
2. $\int x^2 e^{-x}\,dx$
3. $\int \ln(3x+2)\,dx$
4. $\int \sin^{-1} x\,dx$
5. $\int \sec^3 x\,dx$
6. $\int 2x\ln(3x)\,dx$
7. $\int (\ln x)^2 x\,dx$
8. Use both the method of $u-$substitution and the method of integration by parts to integrate the integral below. Both methods will produce equivalent answers.

$$\int x\sqrt{5x-2}\,dx$$

9. Use the method of tabular integration by parts to solve $\int x^2 e^{5x}\,dx$.
10. Evaluate the definite integral $\int_0^1 x^2 e^x\,dx$.
11. Evaluate the definite integral $\int_1^3 \ln(x+1)\,dx$